Mathematical Methods 2

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Orthogonal polynomials
Lecture - 43
The weight function

Ok. So, we have been looking at a general approach to construct a sequence of polynomials which are all orthogonal to each other with respect to some weight function in some interval right and we saw how it is possible to argue that all intervals really are of three types and so, we will you know follow these standard intervals possible.

And so, in this lecture we will look at how you know the prescription for a sequence of functions which are orthogonal to increasing powers of x right. So, when we demand that these functions become polynomials it effectively boils down to choosing the right kind of weight function, right? So, we will see how these weight functions can be obtained as solutions to some simple differential equations. That is what we will discuss in this lecture ok.

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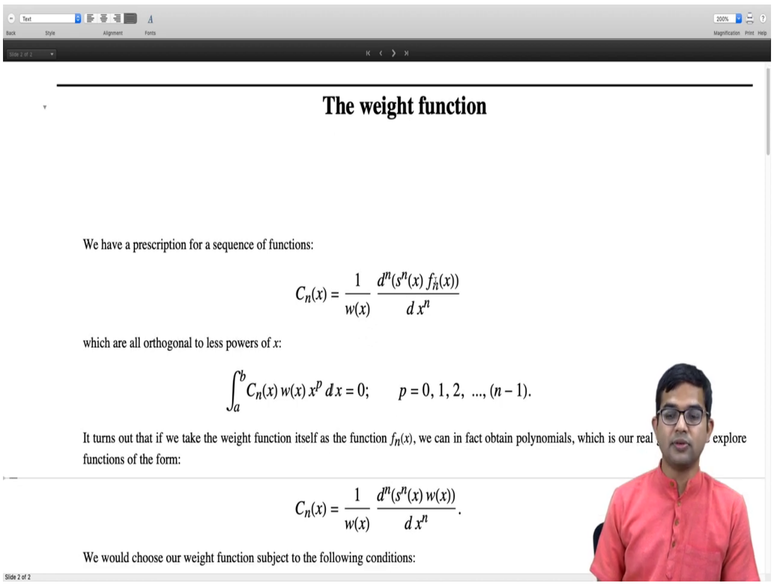

So, the prescription we have is for the sequence of functions right. So, we have this weight function 1 over w of x and then we saw how it is convenient to think of this as the nth derivative of some function and which has this padding which you know which is which I am writing it as s to the power n of x and so, this s of x itself is of three different kinds.

And then there is this additional function f n of x which is in there so that you know the function behaves nicely at the edges. So, we want all orders of derivatives to go to 0 at the end points right. So, f n of x will be the sole reason why this happens whenever you have these both the end points going to plus infinity and minus infinity.

But, if either or both of these end points are 0 then we have to rely on s of x to do the job for us. Now, you know if you just choose this kind of a set of functions we have explicitly seen how from integration by parts we can argue that they are all going to be orthogonal with respect to x to the p.

And if they are going to be orthogonal to x to the p basically they are orthogonal to all polynomials of lower degree. So, indeed the sequence of functions you would be constructing are going to be orthogonal to all polynomials of lower degree. And now what we want is of course, to ensure that these functions are themselves polynomial.

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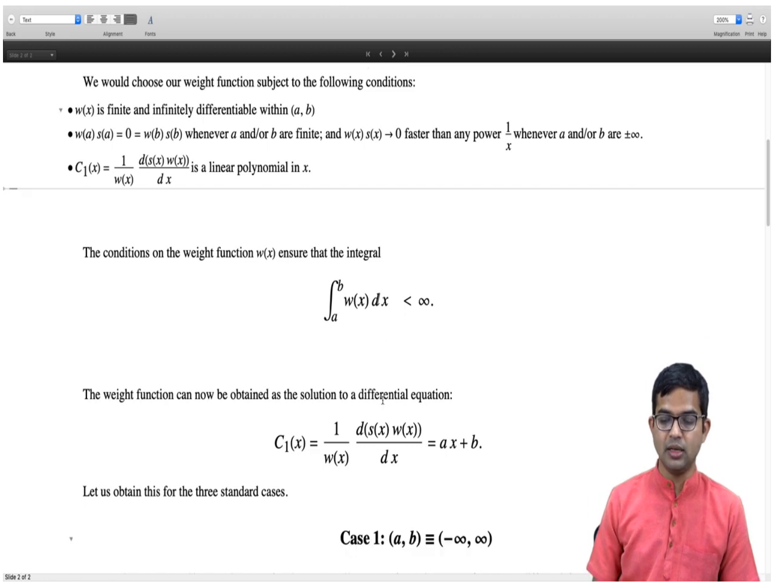

So, and then we stated that if you take this function f n itself to be this weight function and then impose certain very reasonable demands on these the properties of this weight function then in fact, you can actually get a sequence of polynomials. So, we will explicitly see this in this lecture.

So, you know the conditions on w are right w has to be finite and infinitely differentiable. So, that you know an expression like this will always give you polynomials inside your region of interest and both at both the boundaries it must fall off to 0 in a suitable way.

And which immediately implies basically it is a non-negative function which dies out you know in this fast enough at both ends. Therefore, indeed this integral is always going to be finite, right. So, you can actually think of this as specifying some measure right. So, w is a weight function which has this interpretation as a measure.

Now, yeah we will see if you can somehow force the first of these functions to be a polynomial. So, we will take this to be a linear polynomial and then we will see that all subsequent members of this sequence will all turn out to be polynomials right.

So, there are these three different cases. We will consider them separately and work out the differential equation involved. So, basically the idea is to write down C 1 of x you know define according to this prescription and force it to be some linear polynomial ax plus b. And then we will obtain these three standard cases.

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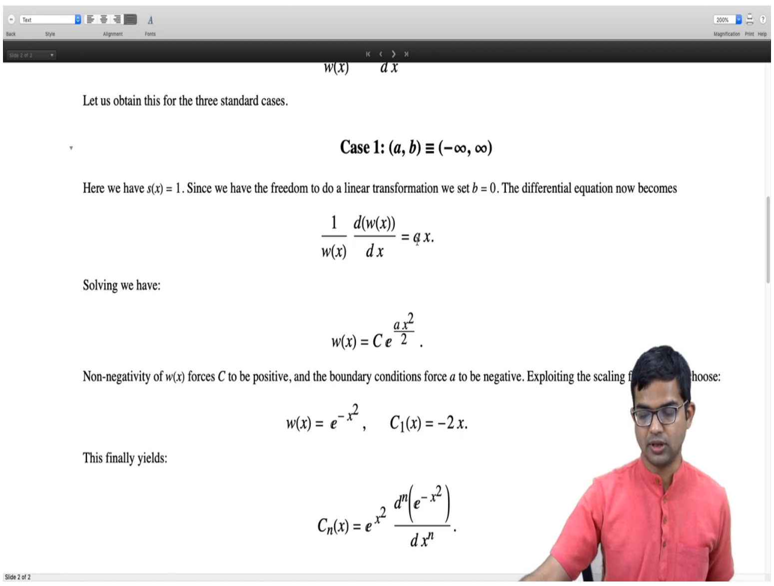

So, first let us look at the case when both a and b are you know minus infinity and plus infinity and here of course, s of x really has no role. s of x is just 1 and the entire burden for convergence is on f n which we have taken it to be the weight function itself.

So, the weight function must fall off sufficiently quickly both at plus infinity and at minus infinity. Now, the whole point of the weight function is of course, right like we said right at the beginning is to ensure convergence of these kinds of integrals which for which we can also give the interpretation of you know inner products of vectors of the space, vector space that we are considering.

So, the differential equation now is you know 1 over w times d w by dx must be a linear polynomial and here it is actually convenient to take v equal to 0 right. So, you have this freedom to choose your you know you can shift your first polynomial. So, in this particular case let us just take b to be 0 and demand that the first polynomial is actually just a x.

And then solving, we immediately get this straightforward differential equation to solve. So, you get log w of x on the left hand side and on the right hand side it becomes ax squared by 2 and then you have to take the exponents on both sides and then you get a free constant C.

So, the first weight function w of x is in this case. It turns out to be C times e to the power ax squared by 2. Now, we have to argue a little more and you know first of all we argue that w of x must fall off, therefore, a has to be negative. If a is positive then it is going to keep on its going to explode both at plus infinity and minus infinity, it is not a suitable weight function.

So, a has to be negative and scaling freedom we can you know choose this w of x to be just e to the minus x squared. C is some positive number. We can just set it to be 1, right. a is negative and it is just convenient to take it away minus 2 in this case right.

So, there is this freedom for normalization right. So, C 1 of x is just ax, but we might as well take it to be minus 2 x because the weight function looks nice in this form. So, it does not matter whether you take minus 2 times x or any other factor times x for the first function.

So, you will see that if you choose the first function and this weight function in this manner where we have use some physical intuition to argue that it should be e to the minus x squared and you cannot have a positive value and this overall constant of course, has to be some positive number which we take it to be 1.

So, if you do this then our sequence of functions which will be actually a sequence of polynomials is given by this. So, according to this prescription we have to choose C n of x is given by 1 over weight function. So, in this case 1 over weight function becomes e to the x squared.

Then the nth derivative of the nth derivative of s to the power n is nothing but just 1. So, it is just the nth derivative of w. So, the nth derivative of w here is e to the minus x squared right. So, that is simple enough. So, we can actually quickly verify that indeed this gives us a polynomial. So, let us look at just one example.

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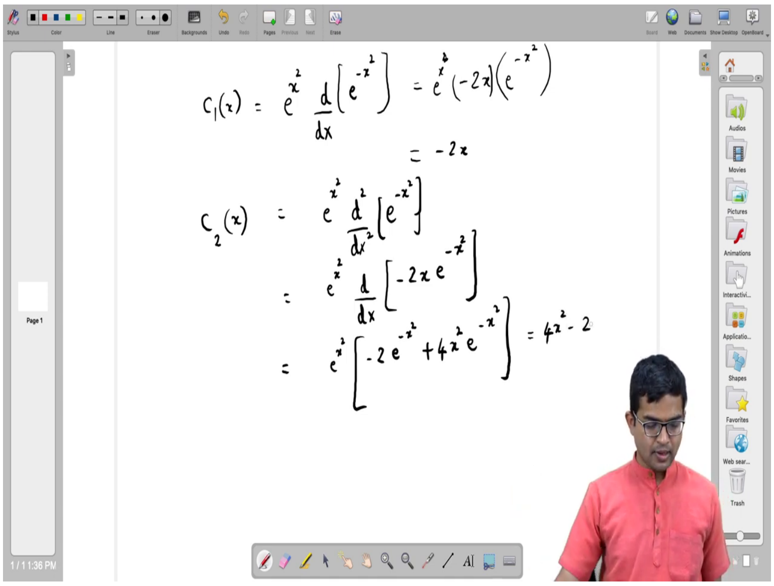

So, C 1 and then you can maybe try out a few more and convince yourself that this works out. So, suppose I take e to the minus x squared and then if I differentiate it. So, let us see if; well I mean if I do not do anything of course, it is just e to the x squared times e to the minus x squared. So, that is just going to give me 1, right.

So, that is C naught of x well and So, C 1 of x in this case is going to be minus 2x, but let us verify whether that actually happens. So, if I do this. So, it is minus 2x times e to the minus x squared and then I have to multiply by e to the x squared. So, I have e to the x squared times minus 2x times e to the minus x squared. So, which is indeed minus 2x.

So, C 1 of x is indeed a polynomial ok. Maybe let us see if I do this once more C 2 of x; in order to do this I have to do e to the x squared d squared by dx squared of e to the minus x squared yeah. So, that is nothing but e to the x squared d by dx of minus 2xe to the minus x squared which is nothing but e to the x squared times minus 2 e to the x squared.

And then I have a minus 2x then I get another minus 2x. So, it is a plus 4x squared e to the minus x squared. So, this e to the x squared will conveniently cancel with e to the x squared. So, all of these derivatives no matter to which order you take it will always have this e to the minus x squared and that will go away with this and will eventually get back for you a polynomial. So, you have 4x squared minus 2 right.

So, indeed it seems to work out. So, we have our prescription that seems to be following through. At least we are getting a bunch of volume I mean as you can also explicitly verify that they are going to be orthogonal right. Try out C 1, C 2, C 3, C 4 and so on and multiply these functions with respect to this weight function in the entire interval minus infinity to plus infinity. You can verify that indeed you are generating a sequence of orthogonal polynomials.

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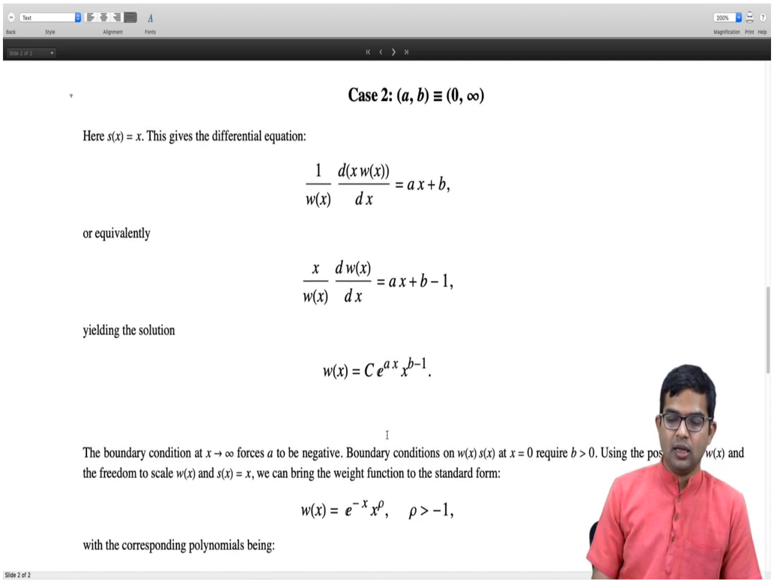

Let us look at case 2. So, case 2 is so, the argument is similar. So, here we have s of x is equal to x. So, x s also has a role to play. So, you know in order to get the first function to be a polynomial, C 1 of x must be a polynomial. So, we choose that to be a x plus b. So, 1 over w times d by dx of x times w of x is equal to x plus b.

So, equivalently, so, I mean I can take the derivative. So, dx by dx to the one of them is going to be 1 and then I have x you know I can pull out this x times d w by dx and only keep that on the left hand side. And the other term where you know w will cancel out in the numerator and denominator that just gives me a 1 and that I am going to send it to the other side.

So, this gives me ax plus b minus 1. So, that is just a shift in this constant which is, you know, basically insignificant. So, then I know how to integrate this on both sides. So, this is going to be I bring this x to the right hand side. So, I have a plus b over x b minus 1 over x then I integrate both sides.

So, the left hand side of course, will be log w and the right side right hand side is going to have you know ax plus b minus 1 over x. So, that is going to give me b minus 1 log x and then I take you know I have to take exponents on both sides. So, b minus 1 times log x is going to be log of x to the b minus 1 and so, that is going to just result in x to the b minus 1 when I take exponentials of both sides.

So, a to the x a times x will become e to the ax and then there is an overall free constant right. So, w of x is e to the a x times x to the b minus 1. So, the boundary condition is you know we have to argue with the help of boundary conditions now for how to fix these a's and b's.

So, the boundary condition at x going to infinity forces a to be negative right. So, we set a to minus 1 then boundary conditions on w of x s of x at x equal to 0 will force b to be positive and using the positivity of w of x. So, overall w of x is a weight function. So, it cannot be negative and the freedom to scale w of x also is present and s of and s of x is equal to x. We can bring this weight function to this standard form. So, we just set the weight function to be e to the minus x times x to the rho where rho is greater than minus 1 right.

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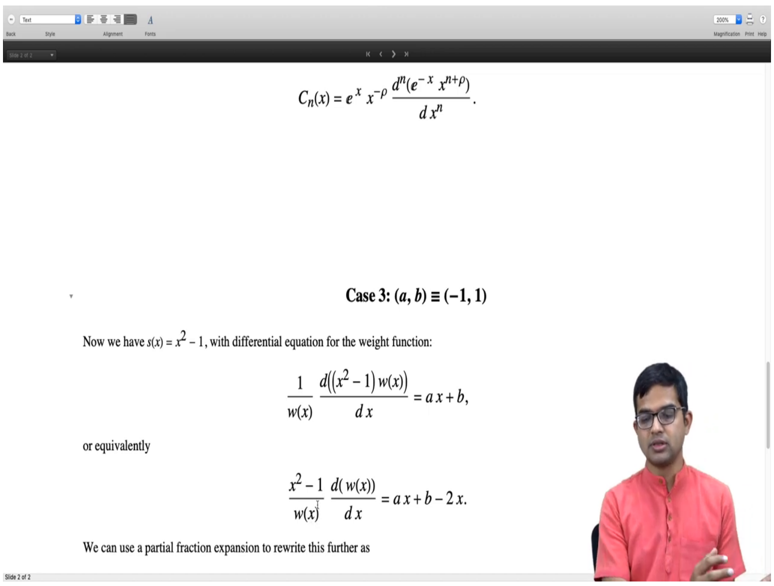

And then the corresponding polynomials become C n of x is this weight function e to the x times x to the minus rho times the nth derivative of this function x to e to the minus x to the n plus rho right. So, I invite you to explicitly check that this is going to generate for you your sequence of polynomials right. So, we did this with case 1. We looked at the first couple of them and indeed there were polynomials here also you will see that they are going to be polynomials.

So, I urge you to check this for yourself and see for yourself that it is true and also you might as well check that these polynomials are going to be orthogonal to each other in the correct range. So, in this case it is from 0 to infinity ok. Now, the third case is also similar. It is going to give us an expression which is a little more complicated, but really it is a wheel that we know how to crank now.

So, a and b are now minus 1 and 1 s of x is x squared minus 1 which is x minus 1 times x plus 1. So, we have to put this in here. The derivative of x squared minus 1 times w of x over all we have to divide by w and so, this we want it to be equal to ax plus b or equivalently x squared minus 1 divided by w of x. So, I can pull out this and then take the derivative only with respect to w and then if I do it the other way around that is going to give me a 2 x and the w is going to cancel that 2 x I can send it to the right hand side.

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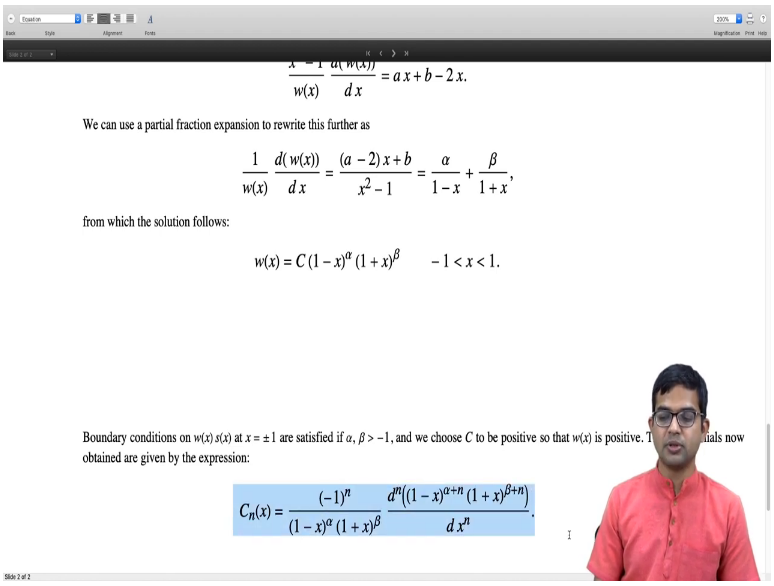

And then we can use partial fraction. So, I can divide throughout by x squared minus 1. And then I can use partial fractions to recast the right hand side as some alpha divided by 1 minus x plus beta divided by 1 plus x. So, you can work out these details right. So, I will allow you to check that indeed basically this effectively gives us this w.

So, the left hand side of course, is just log w and the right hand side you can integrate and then you can leave alphas and beta as it is right. So, see you have this freedom to choose some constant overall constant factor outside and of course, the region of interest is minus 1 to plus 1 right. So, that is because we have put a and b to be minus 1 and plus 1.

So, this a and b should not be confused with this a and b right. So, I think from the context it's clear what these two mean. And basically now we have to argue that the boundary conditions on w at x equal to plus or minus 1 are satisfied if both alpha and beta are greater than minus 1 right. You can check this and we choose C to be positive, so, that yeah.

So, C is a constant free constant, but it must be positive and the polynomials that are obtained - the expressions are a little more complicated than the first two cases. But again you can check that this expression is going to generate for you a sequence of polynomials and because of the way we have argued the whole thing for sure they are going to be orthogonal to each other with respect to this weight function ok.

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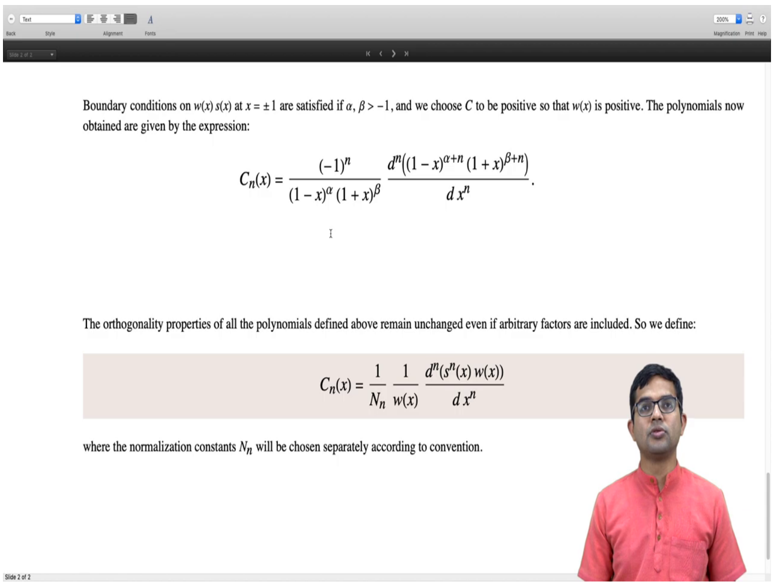

So, the orthogonality properties we have seen of these sequences of polynomials are unchanged now even if you put some arbitrary factors in front. So in fact, we will define these orthogonal polynomials with an overall normalization factor N 1 over N N. These normalization factors are set by some convention; many of these polynomials were first discovered in some other context.

And only and it made sense to fix certain normalization constants and. So, then there is this sort of more general theory which came up later. So, we are looking at the general theory and then we will see when we go to particular kinds of sets of polynomials these different normalization coefficients will become explicit.

So, you should once again check that this set of functions are also polynomials by putting out l N equal to 0, N equal to 1, N equal to 2. First four or five if you check then you will see that indeed it seems to work out fine and convince yourself that indeed this whole procedure is reasonable. That is all for this lecture.

Thank you.