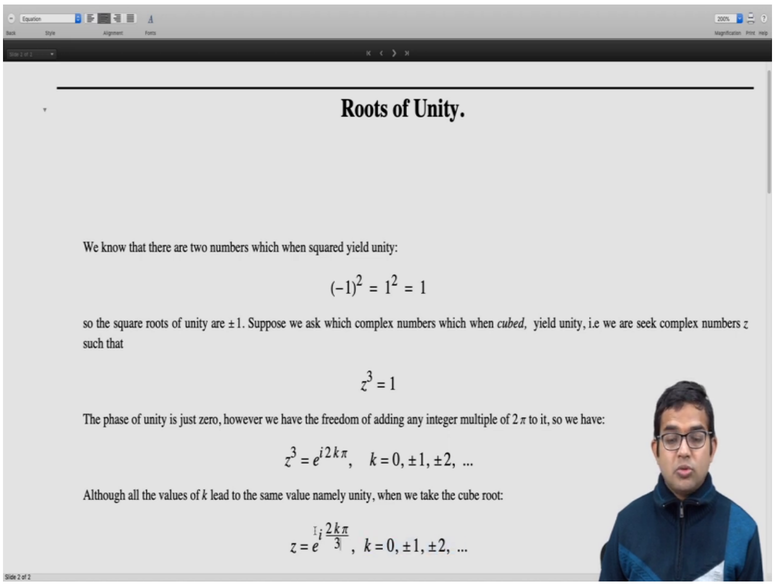
Mathematical Methods 2

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Numbers
Lecture - 04
Roots of Unity

So in this lecture, we look at the roots of unity and so this is also a means of exploiting some of the properties of complex numbers, which we have been talking about and which we have already looked at some of these properties and the roots of unity is a very instructive problem to consider - hat is the content for this lecture.

(Refer Slide Time: 00:47)



So, we know that if you square minus 1 or if you square 1, you are going to get 1, right. So, the answer to the question of what the square roots of unity are it is very simple, right. So, just minus 1 or plus 1, right. So, suppose we ask for the cube roots of unity.

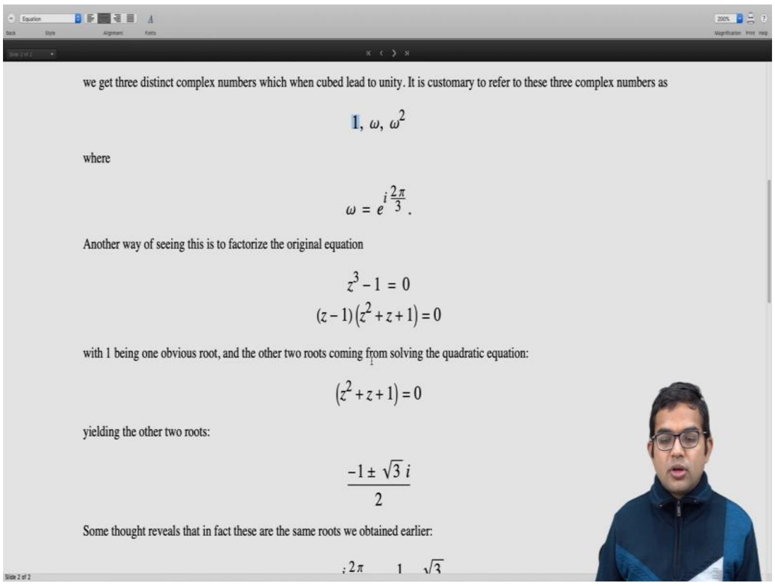
So, in other words, well we look for, in general, complex roots, right. So, if you ask for only real numbers which when cubed give you 1 of course; so the answer is just 1, there is only one such real number and that is 1. But if you allow for complex numbers, in fact there are three complex numbers which when cubed will give you 1, right.

So, we are solving for this equation z cube is equal to 1; but the right hand side can be written in this canonical form, where there is a modulus of a complex. So, 1 itself is a complex number, whose modulus is unity and whose phase is 0 right, phase or argument.

But we have seen that the argument of any complex number is not unique right; if it is 0, if 0 is an argument, then so is 2 pi, then so is 4 pi and so on. So, in fact you have this freedom to add as many 2 pi's as you want. So, if you write it in this form z cube is equal to e to the i 2 k pi right; so the right hand side is where k can be 0 plus or minus 1 plus or minus 2 so on right, it does not matter which value of k you choose on the right hand side, they will give you, all of them will give you just 1.

But when you take the cube root now of this quantity; so you get z is equal to e to the i 2 k pi by 3. And now once again you have all these different values of k available; but you know although there are not infinitely many different values, now it is also not just 1, so in fact you have three different, distinct complex numbers, which appear on the right hand side now.

(Refer Slide Time: 02:56)



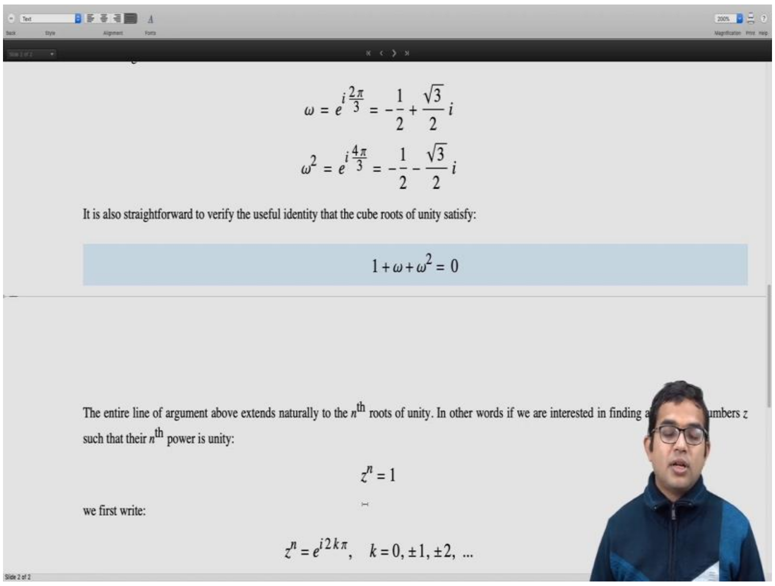
So, in other words there are these three distinct complex numbers, which when cubed will give you unity, right. So, it is in fact customary to call these three you know cube roots of unity in this manner. So, to refer to one of them is 1 of course; the other one is omega and the other one is omega squared. Where omega is you know e to the i 2 pi by 3, omega squared of course is e to the i 4 pi by 3 which is also you know which falls in this equation when you put k equal to 2, right.

So, another way of arriving at the same result is to start with this original algebraic equation z cube minus 1 is equal to 0 and factorize it. So, you have z cube minus 1 is the same as z minus 1 times z squared plus z plus 1 equal to. So, the equation is z minus 1 times z squared plus z plus 1 is equal to 0; now which means that either z must be equal to 1, so that is one root, which is anyway the obvious root which is the real root.

Or z squared plus z plus 1 equal to 0, that is a quadratic equation right, which you can solve and we know how to solve a quadratic equation, its roots are given by simply minus 1 plus or minus square root of b squared minus 4 a c in this case. 1 minus 4 which is, which is actually square root of 3 times I; so you have minus 1 plus or minus square root 3 times i the whole thing divided by 2, right.

So, some thought reveals that, in fact, these are indeed the same as these omegas, right. So, these omegas, this omega and omega squared.

(Refer Slide Time: 04:55)



So in fact omega is equal to e to the i 2 pi by 3 is equal to minus half plus square root 3 by 2 times i, and omega squared is e to the i 4 pi by 3 which you know which has the same real part minus a half, but the imaginary part is minus square root of 3 by 2, instead of plus square root of 3 by 2. So, this is omega and this is omega squared.

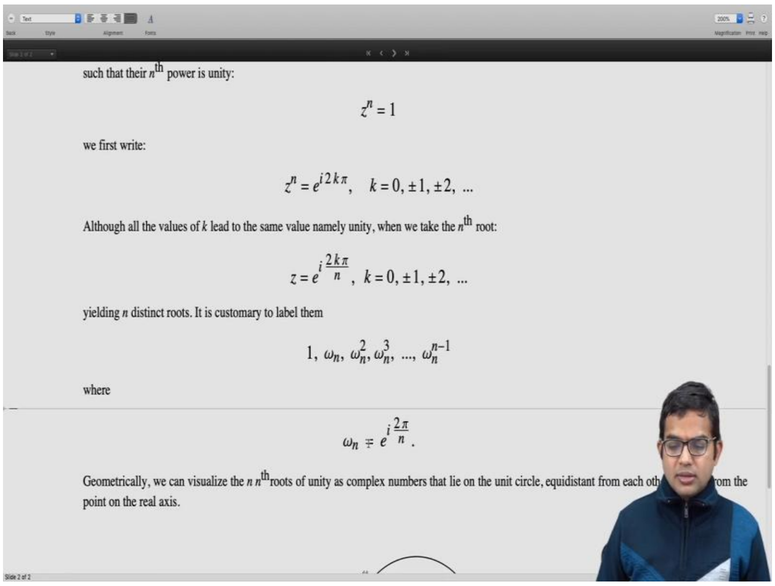
So, it is also straight forward to verify that, in fact the cube roots of unity satisfy this identity, 1 plus omega plus omega squared equal to 0, right. So, not only are the roots of an equation of this kind; you know if you solve for it, omega is a root, but omega squared is also a root, right.

So, each of them separately you know 1 plus omega plus omega squared is 0 and I mean you can also put omega square into this. So, you get 1 plus omega square plus omega to the four is 0; but that does not have any separate content, right. So, omega power 4 is the same as omega. So, that is why you get really the same you know identity.

So, the cube roots of unity satisfy this identity 1 plus omega plus omega squared equal to 0. Now, in fact all we have done so far can be extended to the you know to, can be generalized to the nth roots of unity, right.

So, so this whole argument follows through and so if you are looking for complex numbers z, such that z to the power n is equal to 1; we would you know like before write this equation of z to the n is equal to e to the i 2 k pi, where k can be any integer right, because we have this freedom of you know adding your phase by an arbitrary multiple of 2 pi.

(Refer Slide Time: 06:49)



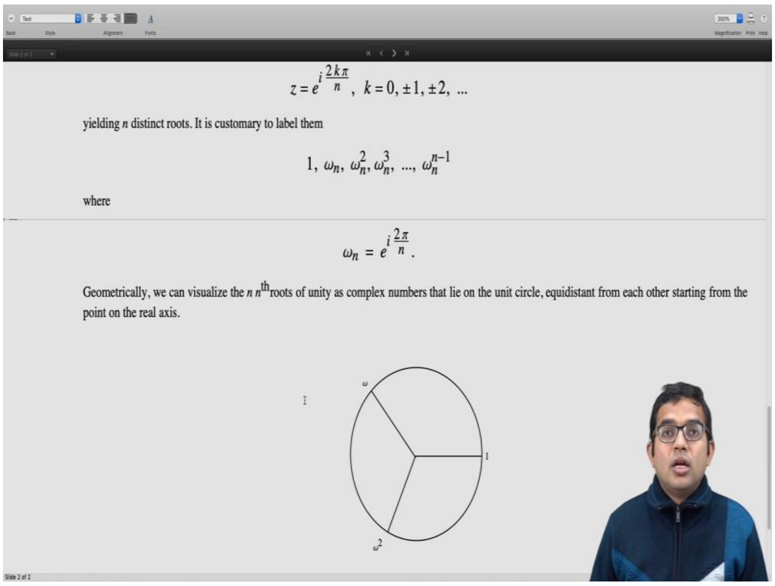
And then we take the nth root and so we have z equal e to the i 2 k pi by n. So, now, z itself has n distinct complex numbers; although changing k in this equation gave you no different values for the right hand side, they are all just 1. But now the right hand side here for z, there are actually n distinct complex numbers right.

Although k will take infinitely many values, there are n distinct complex numbers which come out of here, right. So, these are the n distinct roots. So, the nth as the nth root of unity, there are n complex numbers which are you know nth roots of unity.

So, again it is customary to label them as 1, omega n, omega n squared, omega n cube so on right; it is possible to verify that you know using this relation indeed, they are all of this kind, where omega n is just e to the i 2 pi by n, instead of 2 pi by 3 you have e to the i 2 pi by n, right.

And so, it is also possible to visualize all of this geometrically; so in fact the cube roots or the 4th roots or the nth roots of unity in general they all have modulus 1, but it is only in the phase that they all differ. And in fact, there can be thought of as you know points which are sitting on a circle of radius unity centered around the origin of the you know complex plane and these points are all equidistant from each other starting from the point of the real axis.

(Refer Slide Time: 08:33)



So, we will illustrate this for the cube roots of unity. So, if you take the complex plane and then draw a circle of radius unity. So, the first root is just this point on the x axis which is 1 and then you multiply by factors of omega_n. So, omega-n is nothing, but taking this vector and rotating it by angle of 2 pi by 3 and then to get omega squared, you rotate once again by another angle of 2 pi by 3 and then if you do once more, you return to where you started.

So, you know changing k to any higher values does not give you any new complex numbers; if you rotate around, then you will go back to omega. So, that is what I said omega, omega squared omega cube is just 1 and omega to the 4 is omega. So, that is why we saw how 1 plus omega plus omega squared equal to 0 is the same identity you would get, if you use the other root omega squared in that algebraic equation earlier.

So, in general if you are looking for the nth roots of unity, you are going to get n points on this circle which are all distinct; right you start with one and then there could be some point here, another point here, another point here so on. They are all equidistant on this circle, they all start with 1, because 1 is always a root no matter which root you are taking. So, they all appear here.

For example, if you if n equal to 4, you are going to get you know one point here, another point here, a third one here and the fourth one here that is it and then it keeps repeating after that. So, in general there are going to be n points equidistant on this circle.

So, this is a useful exercise, because you know we are looking at the nth roots of the simplest possible number, which is just a real number and also it is just unity. But in general, you can find the nth roots of an arbitrary complex number and so the way to do that, is to first of all write the complex number as you know r times e to the i theta.

So, it has a magnitude and a phase. So, the magnitude is a positive number. So, the nth root of a positive real number is going to be another real number, it is unique. So, you have something like r to the one over n, which you can pull out and then you are left with just the problem of finding the nth root of e to the i theta.

So, in place of e to the i 0 like it is the case here, you have e to the i theta right for an arbitrary complex number. And then to theta you can add you know an arbitrary 2 k pi and then of course so you are going to end up with some other you know sequence of numbers, which are also going to lie on some other circle, whose radius is not unity, because it has some other magnitude.

But the different complex numbers will all lie on a circle, which all get rotated by a constant amount, right. So, if we understand this problem well, then in general we can solve for nth roots of an arbitrary complex number. So, that is all for this lecture.

Thank you.