Mathematical Methods 2
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Complex Variables
Lecture - 39
Evaluation of integrals-II

So, we have looked at how the residue theorem can be massaged to help us evaluate some integrals of real variables. So, in this lecture, we will look at some more examples you know and where some slightly more settled arguments are involved and so, of course, we have to use the Jordan lemma and you know arguments of a similar nature and you know come up with a suitable contour integral and solve it using the residue theorem and extract integrals of a real variable ok.

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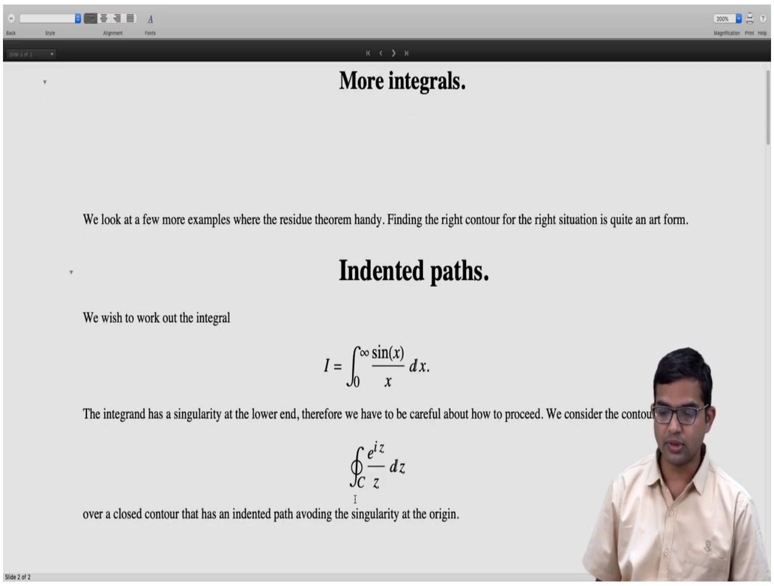

So, there are a couple of techniques we will discuss in this lecture and so, we will avoid getting into some more fancy techniques available, but there are these two you know kinds of integrals which we should discuss right. So, that is the subject matter for this lecture. So, one of them has to do this thing called an indented path right.

So, we saw, how you know the basic idea is when you given an integral over a real variable, your thinking of you know the axis, the real axis, but then you come up with some contour and you complete it, make it an a simple closed loop and so, typically it is taken to be some kind of a some semicircular arc which could be either in the upper half plane or in the lower half plane of the complex plane.

And then by some means you will try to argue that the value of the integral on this curved region will actually just vanish right. So, there are examples where you know the entire x axis is not free of singularity. So, we look at you know one such example where it is possible to sort of go around the singular point and you know it will work with what is called an indented path and still you know use somewhat similar arguments, but with some extra care around this special singularity.

So, this is best illustrated with the help of an example. So, we will consider this integral from 0 to infinity sin x by x dx. So, now, both the limits of this integral are somewhat problematic, one of them is infinity so, one has to be careful, but even at the point 0, there is this you know weirdness which comes in because you have sin x over x so, this is singularity so, you need to be careful.

And so, when we consider you know this contour integral so, because it is sin x so, I will think of e to the i z right and so, later on may be if necessary, we can think of taking the imaginary part of anything. So, we will see that you know that the integral that we care about will automatically come out. So, we will start with this you know contour integral with the integrand be e to the i z divided by z dz.

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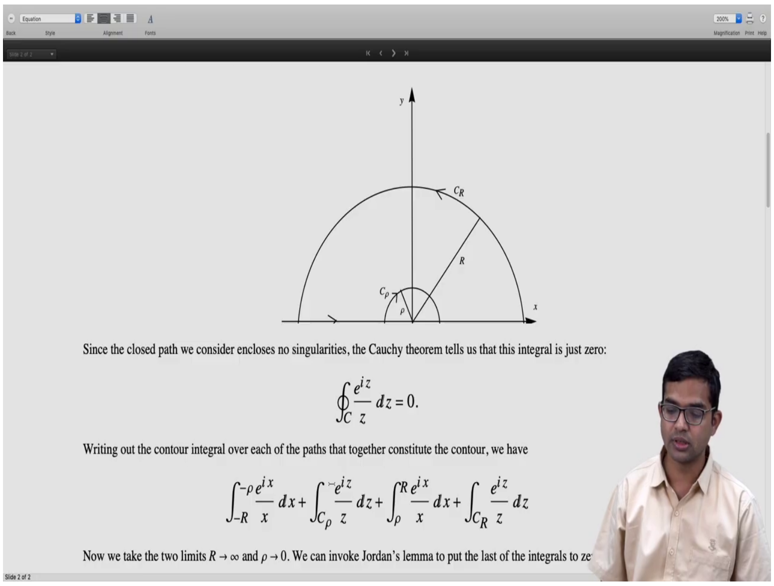

And then, now, we have to come up with a suitable contour. So, we will come up with a contour similar to what we did last time except that we also have this indented path. So, this is what it looks like. So, you know we start at some large negative value, you know keep on increasing along the real axis and go all the way up to this you know this point rho.

So, then, you enter this small circle, go semi-circle, complete this, this loop and then again go along the positive real axis all the way up to plus R and then, come go left or go upwards and complete this semi-circular arc of radius R in the other direction right. So, if you look at you know this region, then you see that this is going to be our simple closed loop.

So, there is mess sitting here at the origin, but for this function in fact, there is no other singularity anywhere else so, in fact, we can immediately apply the Cauchy theorem and we have this result that contour integral of this function e to the i z divided by z dz over this entire contour is just 0 because on the contour and within the region enclosed by this closed contour, there are no singularities right, it is completely a an analytic function.

So, by itself, this is not going to get us so far, so we must actually be able to rewrite this you know as contributions coming from each of the different parts that constitute this contour right so, that is the next step right. So, we infer this has four parts. So, there is this part from minus capital R to minus rho.

So, where it is just e to the i x divided by x dx and then, there is this contour which I am calling C rho and in this direction, the in the negative direction or in the clockwise direction if you wish and so, that is e to the i z divided by z dz, I just write it as it is and then, there is this integral along the x axis, the positive axis, it goes from small rho to capital R e to the i x by x dx.

And then, there is this contribution coming from this big semicircular arc which is carried out in the other direction, in the anti-clockwise direction right. So, that is. So, these are the four parts which together constitute this integral. So, let us argue you know piece by piece.

So, I mean we imagine that eventually, we want to take the limit capital R going to infinity so, make this as large as you wish, you know take it away to infinity and then, this small inner semi-circle we want to make it as small as possible. In fact, we are going to consider the limit of rho becoming arbitrary so, small rho going to 0 and so, that is the eventual you know prescription that we are going to use.

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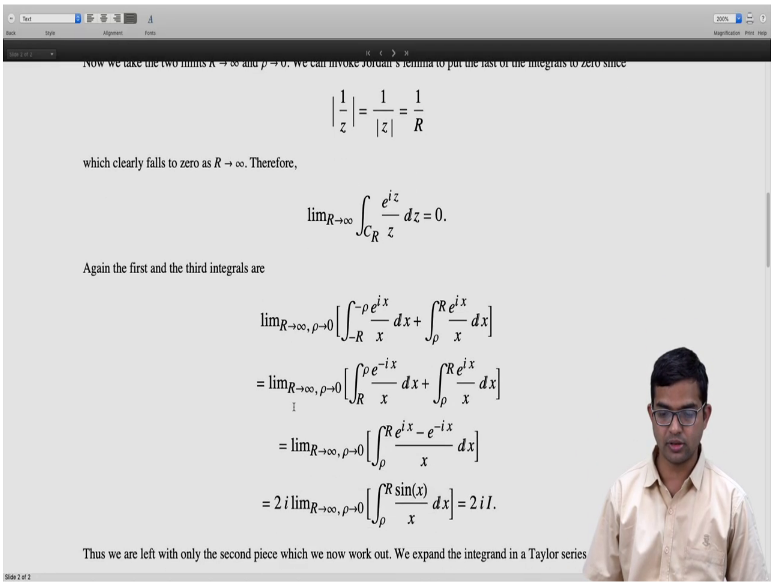

So, now, we can invoke Jordan's lemma as far as this outer integral is concerned e to the i z divided by z dz over C R is actually going to be 0 because we can invoke Jordan's lemma since basically we have this 1 over z, mod of 1 over z will go as 1 over R and which means it is going to diagonal sufficiently fast basically, that is what Jordan lemma says your if you are integrand right.

I mean other than this e to the i z is going to fall of sufficiently rapidly for large R, then we have seen that it is possible to use this Jordan's inequality and argue that if the contribution from this integral along this path C R will just go to 0 as R becomes very large and so, that is so, we will directly invoke Jordan's lemma and then we simply put this last term to 0.

So, then we are left with these three terms and of these three terms, let us work out the first and the third integrals together. So, basically, we are interested in taking the limit R tending to infinity and rho tending to 0 so, we will write this as minus R to minus rho e to the i x over x dx plus rho to R e to the i x by x dx, but this is nothing, but I can do you know change a variable where I put a take x to minus x so, then the first of these integrals will go from capital R to rho e to the minus i x over x dx right. So, the signs have been taken care of properly.

The second integral remains as it is, but I might as well change these limits instead of going from R to rho, I will make it rho to R and put an overall minus sign. So, then I bring that minus sign inside. So, then I get just one integral where the limits are from rho to R and I have the integrand which is e to the i x minus e to the minus i x over x dx.

And then, finally, I write this as I mean instead of e to the i x minus e to the i x by x, I can write this as 2i times sin x over x, 2i comes out and then, I see that basically this is nothing, but the integral that we initially set out to work out. So, it is actually 2i times i, this whole stuff. Therefore, I just leave it as it is. So, I have now managed to work out three out of these four terms right. So, is there a way to also work out this other term? So, that is what we will do next.

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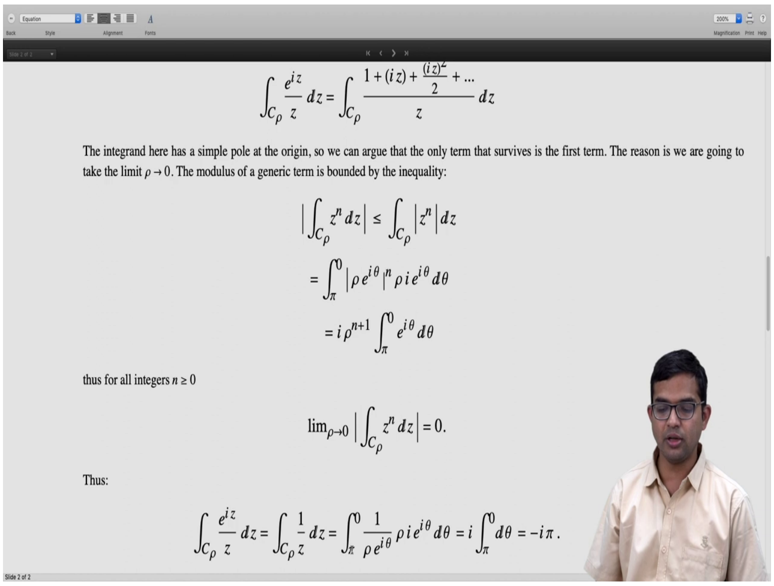

So, let us look at the second term. So, the way to evaluate these integrals where the radius is eventually going to be shrunk to 0 is to do a Taylor expansion right. So, we will argue that in fact, all of these terms except one over them is going to vanish because of you know because of this radius becoming arbitrarily small right. So, this has a simple pole.

So, this works when you have a simple pole, you could have any function here instead of e to the i z, you if you had some other function such that overall, basically you should have a simple pole at this point. And then, you write down this expansion and so, you have a Taylor expansion as far as your numerator is concerned, then you divide by z. So, you have a Laurent expansion, but there is only 1 over z term or this can, this argument can be applied at some other point z naught as well right.

So, then it would be 1 over z minus z naught that is a matter of some detail, but essentially, there is a; there is only the residue in the Laurent expansion. So, if this happens so, then we can immediately argue that actually all these higher order terms including the constant for this small curve are going to go to 0 and so, the reason is simply this right.

So, if you are doing this integral over 0, z to the n dz right n could be even 0 right for any non-negative integer n z to the n. So, mod of this quantity is less than or equal to the same integral, mod of z to n this is somewhat like a triangular inequality type of inequality you know generalize to integrals.

And then, we argue that mod of z to the n is actually nothing, but mod of rho e times i to the theta mod of this the whole power n and now, the limits of integration are from pi to 0 because you are going in this direction so, it goes from pi to 0 rho e rho times i times e to i theta, but then, you see that mod of rho times e to the i theta is the same as mod of is the same as rho so, then, rho to the n plus 1 will come out and then, we will just left with i times rho to the n plus 1.

And there is an integral involving theta which actually does not matter because we are going to take the limit of rho going to 0 so, there since there is a rho to the n plus 1, even if n equal to 0, there is the rho sitting there and higher values of n also will leave a rho there which since its going to go to 0 basically, we can argue that modulus of this quantity will also go to 0 and its bounded from above by a quantity which is going to 0. So, this limit is 0.

And so, what it means is when we are trying to evaluate this expression, all we have to do is work out just 1 over z right. In this case, the residue is just 1, if there are some other residue that is going to feature and then basically, the point is the 1 over zeta survives and then, what is the value of you have for this integral you basically do the same thing.

Now, you see that when you have 1 over z, the rho's will cancel and then, your i comes out and then, it is just an integral from pi to 0 d theta. So, even if rho becomes very small, this term is going to survive and that is the key point, and it is going to survive and it has an exact value of minus i pi

So, the contribution from an integral of this kind you could have another function also instead of minus i pi will have some residue times minus i pi right and if you were to go in the other direction, it would be plus i pi times residue right. So, this basically comes from the fact that you have a simple pole at this point when you are along this indented path.

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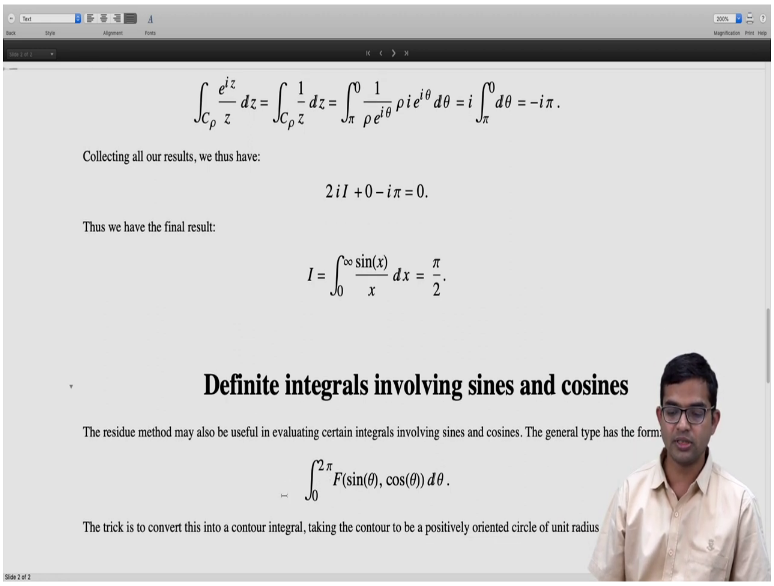

So, collecting all these results, now we are done right. So, what we have managed to show is 2 i I plus 0 minus i pi is equal to 0. So, immediately the result is equal to pi by 2 basically. So, we are done right. So, this is you know one you know another class of problems which can be solved with the help of; with the help of there is.

Well, I mean we are not even really using the residue theorem in this case, we just directly use the fact that the Cauchy theorem really. So, the Cauchy theorem and a clever set of arguments involving, clever use of contours and some clever arguments involving how some of these integrals will go to 0 and then, we are able to also compute this value along the indented path.

And then finally, I want to give you one example where you know the residue theorem can be used to compute integrals involving sines and cosines. Suppose we are interested in computing such an integral from 0 to 2 pi, some function of sin theta and cos theta d theta.

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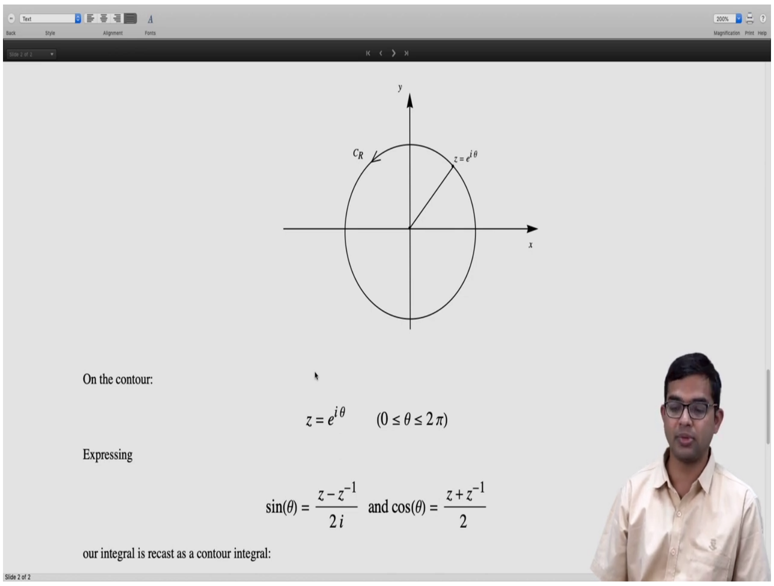

So, basically you can convert this into a contour integral. So, the trick is to take consider a unit circle so, since theta goes from 0 to 2 pi, you can think that it is happening on the complex plane and basically, it is z which is going from you know this part in a circle right and so, then you can convert this problem into a contour integral problem. So, sin theta can be written as z minus z inverse over 2 pi; 2i and cos theta can be written as to z plus z inverse over 2.

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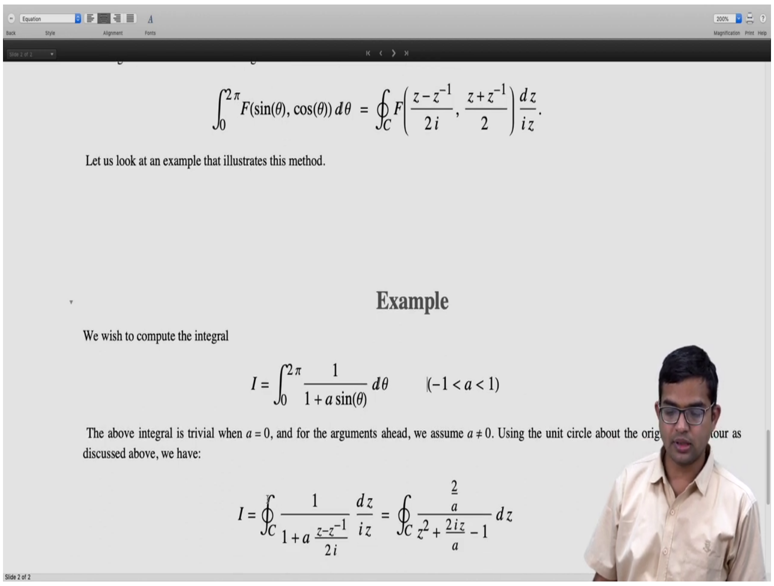

And then, our integral gets recast as a contour integral 0 to 2 pi F of sin theta, cos theta d theta becomes a contour integral where this contour is specified of you know F of this term which is written in terms of z's and z inverses and in place of d theta, we write down d z over i z because you can take the differential of d z and then, this comes out. So, now, and there it is a matter of doing the contour integral using the residue theorem.

So, let us look at an example. So, suppose we have our integral to be something like this 1 over 1 plus a sin theta where a is some number between minus 1 and plus 1 right. So, from the purposes of these arguments ahead, we will assume that a is not equal to 0 because if you put a equal to 0, you will have some you know operations we will do in terms of 1 over a and which will become absurd.

But I mean it is evident that if a is equal to 0, then this integral is directly you know just d theta so, it is just 2 pi so, there is no need to you know do any fancy calculation if a is equal to 0. So, let us assume that a is not equal to 0 and it lies between minus 1 and plus 1 and then, we have this you know trick that we just use above.

So, this integral can be written as the contour integral and then, some little bit simplification shows that you have a quadratic term in the denominator. So, like I said in the numerator, there is a 2 by a. So, since a is not equal to 0, it is not a problem.

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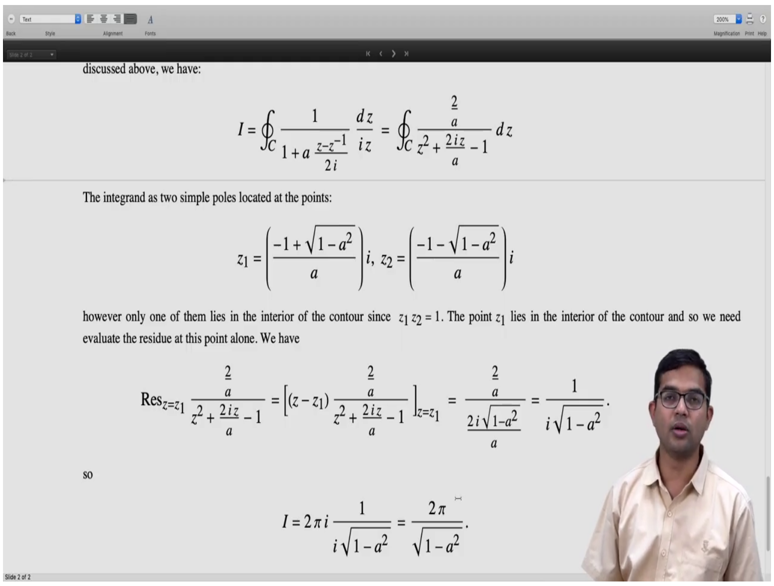

And then, we can quickly observe that the denominator has two 0's, it is a quadratic in you know quadratic term in z square so, it is going to have two rules which you can get by solving the corresponding quadratic equation. So, z 1 is given by this and z 2 is given by this, you know the stuff times i, then this other stuff times I, it comes from the quadratic formula.

And so, one of these routes lies within your unit circle right, in the interior of the unit circle and the other one is going to lie in the exterior which is evident if you see that z 1 times z 2 is actually 1. The modulus of one of them is greater than one, so the modulus of the other one has to be less than one.

Only one of these lies in the interior and it is also straightforward to see that you know it is z 1 that is in the interior right. So, if you know minus and minus, then it is going to go for away a outside and then, times i it does not matter where there is an i are not so, basically it lies sufficiently far away from the origin such that it lies in the exterior of this region defined by this by the contour mod z equal to 1.

So, all we have to do is work out the residue at this internal point. So, residue of at z equal to z 1 is simply given by you have to multiply by z minus z 1, it is a simple pole and the and take the value of this resulting function at z equal to z 1 that is the way to compute the residue.

So, a little bit of calculation and then, we can show that the residue is 1 over i times square root of 1 minus a square. So, the value of the integral is just 2 pi i times this value which turns out to be just 2 pi divided by square root of 1 minus a square. So, the final answer is so, this integral is just given by 2 pi divided by square root of 1 minus a square and now, we can also check that even if you put a equal to 0, this formula is correct right ok.

So, that is all for this lecture. We looked at a couple of you know different kinds of problems which can also be solved with the residue theorem or some tricks around this. There are more fancy integrals also which are possible to solve using the residue theorem, but for the purposes of our discussions, we are going to stop with this, we will you know look at another important idea, but as far as the residue theorem is concerned, we will close our discussion here.

Thank you.