Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Variables
Lecture - 34
Differentiation and integration of power series

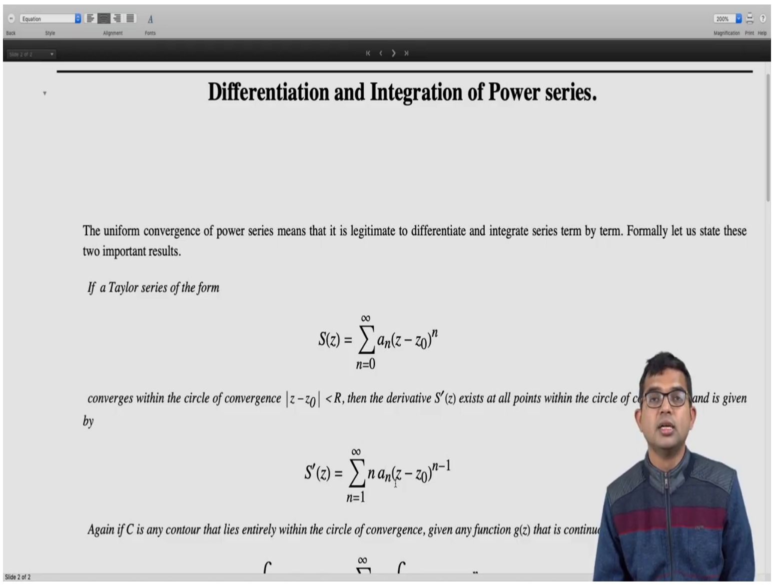
So, we have seen that a Taylor series you know for a function which is analytic about some point is convergent in some circle of convergences. So, when mod of z minus z naught is less than a certain radius, then it is going to be convergent and a Laurent series typically is convergent outside some circle and within another external circle.

So, it's in an annular region that typically a Laurent series is going to be convergent. So, in this lecture we will see that whenever a series of this kind is convergent in fact, we have seen already that there is uniform convergence if they are not only convergent, but they are actually uniformly convergent which means that they represent functions which are continuous.

And in fact, one can argue that they are analytic and therefore, you can take derivatives and you can do integrations of these functions represented by these series. And in this lecture we will show that in fact, we can change the order of integration and summation right.

So, uniform convergence which we have discussed earlier has very important consequences which we will be discussing in this lecture ok.

(Refer Slide Time: 01:34)



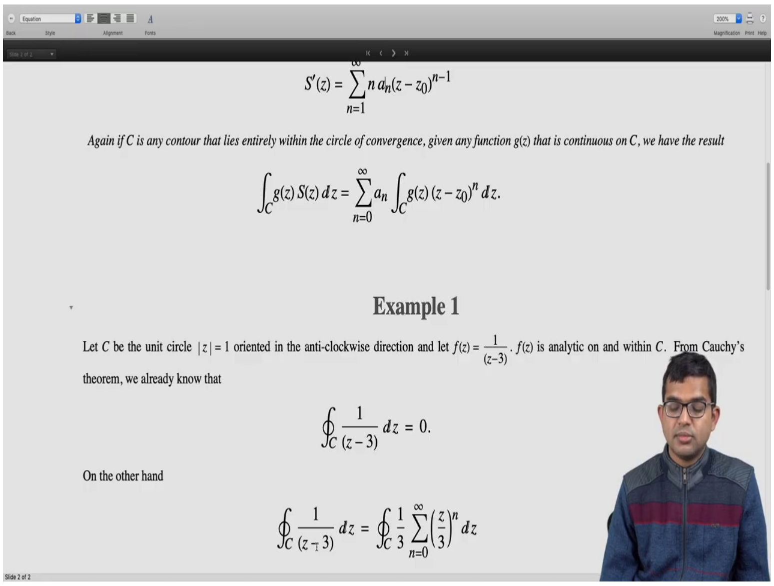
So, let us formally state these two important results, one is about differentiation and the other is about counter integration. So, you have a Taylor series of this form.

And so, you associate this series with some function S of z and you are given that it converges within the circle of convergence mod of z minus z naught is less than R then its completely legitimate to talk of the derivative of this function and in fact, the derivative of this function you know has exactly the same circle of convergence as the function S of z itself.

And its derivative is simply is also the c the series corresponding to this derivative is in fact obtained by taking a term by term differentiation of this series expansion. So, that is a consequence of uniform convergence right. So, this series not only converges inside the region of convergence which is given by the circle of convergence, but also converges uniformly.

Therefore, you can take the derivative and then you can do it term by term and therefore, you get S prime of z is simply summation over n, where n going from 1 to infinity n times a n times z minus z naught the whole power n minus 1.

(Refer Slide Time: 02:49)



And again if you have some contour that lies entirely within the circle of convergence and you are given some function g of z that is continuous you can multiply this S of z by this you know coefficient function g of z.

And on this contour you take this integral and you have the liberty to change these you know the order of the summation and the integration. So, you can actually do term by term contour integration and then take the summation and then that also is guaranteed to converge. Let us look at an example right.

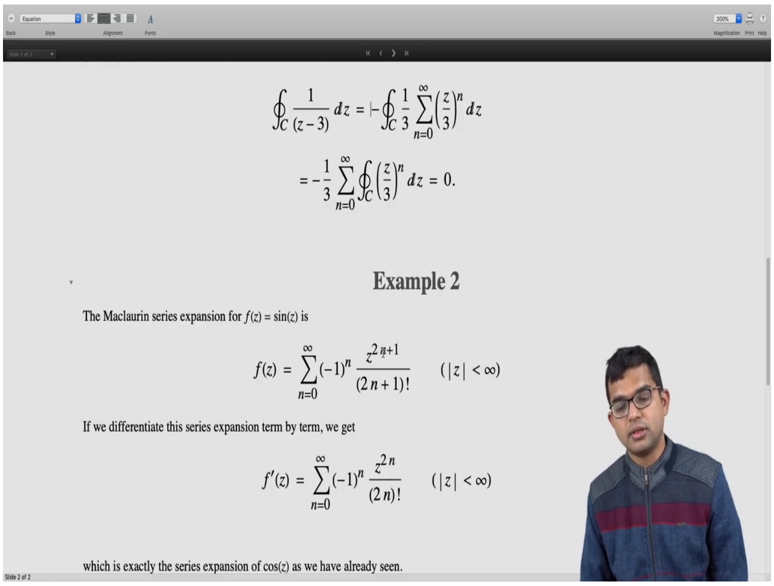
So, it's best understood with the help of a few examples. So, suppose we have we know we take the contour to be this unit circle mode z equal to 1 and oriented in the anticlockwise direction and suppose we take for simplicity as function like f of z is equal to 1 over z minus 3.

So, we know that z equal to 3 is a singularity. So, f of z is analytic everywhere other than at z equal to 3 and specifically f of z is analytic on and within the contour that we have chosen which is the unit circle mod z equal to 1. So, from Cauchy’s theorem we already know that this integral over this particular contour for this function 1 over z minus 3, z is actually 0.

But on the other hand we can actually expand this 1 over z minus 3 as you know you can pull out this 3 from the denominator. So, you have 1 over 3 times 1 minus 1 over 3 times z minus z by 3 minus 1. So, there is perhaps a sign that we should take care of.

So, I have said 1 over I am pulling out a 3. So, it should be z over 3 minus 1 so in fact, it should be minus it does not matter.

(Refer Slide Time: 04:42)



So, in the end it's just going to go to 0 in any case. So, there is a minus overall minus sign, but in the end. So, let me pull out this minus sign. So, we get minus of this and so, that minus sign in any case will come out.

And then you have a contour integral of this infinite series but which can actually be written as the summation over all these different contour integrals and then we use Cauchy's theorem which is evidently each of these terms has to be 0 right because the closed contour and you know these are very simple functions z to the n, they are all analytic within this region and for sure it is 0.

So, in some sense this is a trivial example because we could have directly argued that this counter integral must be 0, but this is just to illustrate how term by term contour integration is allowed. So, let us look at another example. Here it's about taking derivatives right.

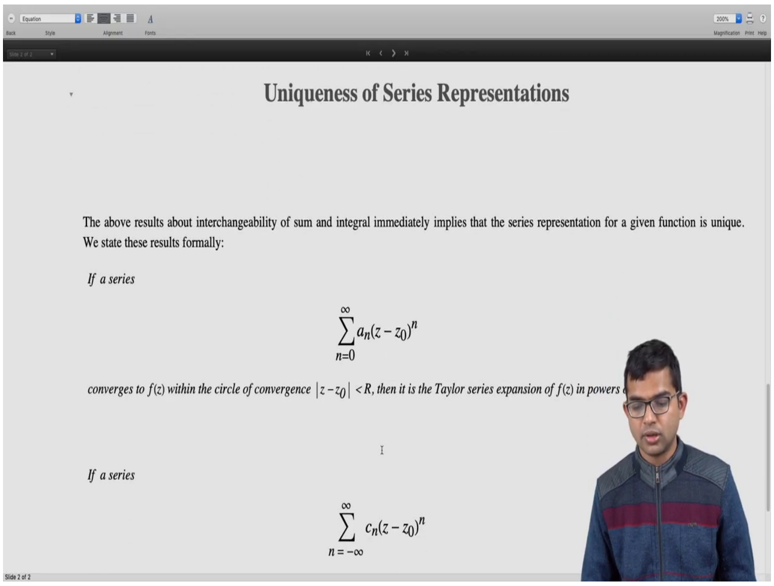
So, suppose you look at the function f of z equal to sin of z, we already worked out its Taylor series expansion its simply given by summation over n minus 1 to the whole power n, z to the 2 n plus 1 divided by 2 n plus 1 the whole factorial mod z less than infinity.

So, if we differentiate this term by term, you get a 2 n plus 1 z 2 2 n and that 2 n plus 1 will cancel with 2 n plus 1. In the denominator and then you get z to the 2 n divided by 2 n the whole factorial and of course, minus 1 to the n stays as it is and, but what is the derivative of this function? The derivative of sin of z we have seen is also this cos of z from first the definition of sin of z.

And we also obtain the Taylor expansion of cos of z from first principles and in fact, that is nothing, but this expansion. So, you see that indeed it is legitimate to differentiate a term by term as long as you are within the region of convergence.

So, if you have a Taylor series expansion for an analytic function and its convergent in some region of convergence, then you have the liberty to take the derivative of this function and then take the derivative of you know the series expansion term by term and that will correspond to the series expansion for derivative.

(Refer Slide Time: 06:59)

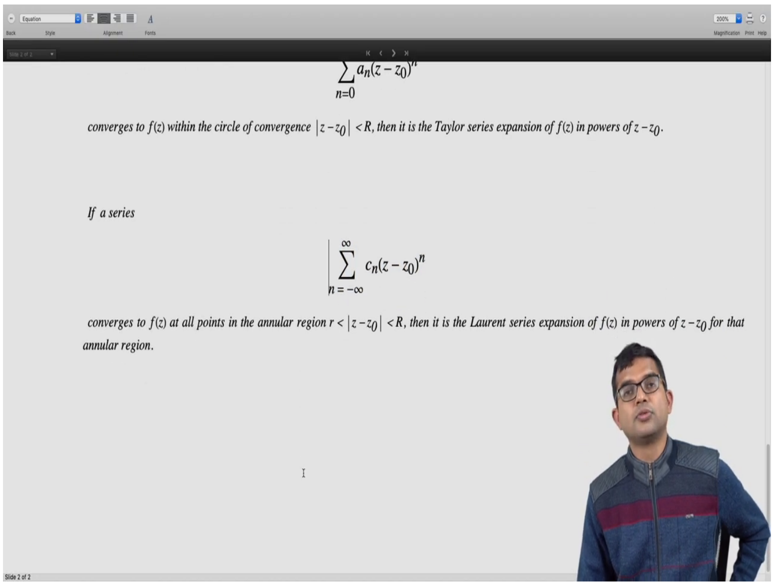


So, an immediate consequence of this you know freedom to take term by term integrals is the uniqueness of series representations right. So, if you find some series where you do not start with the function and then work out the series corresponding to it. So, if you find a series and then you find that it converges to f of z within some circle of convergence, then it is guaranteed to be the Taylor series expansion of f of z in powers of z minus z naught.

It cannot have two different series which converge to the same point right. So, the way to argue for this is to simply define this. We take this f of z to be this, and then you know divide by z minus z naught to a suitable power and then take a contour integral. So, and then on the right hand side you are going to get you know there is a contour integral over a summation.

But then the earlier theorem tells us that you can exchange the sum summation and the contour integral and then when you with the contour integral will go inside it's going to allow only one of these terms to survive and that is going to extract for you the coefficient and the expression for that is going to be exactly that which is you know which you would get from the Taylor series expansion.

(Refer Slide Time: 08:19)



And the same kind of argument will also hold for the Laurent expansion as well. So, the Laurent expansion we have seen you know these coefficients c n s can be written also in terms of the overall function. So, again think of this as representing a function f of z and then you have to take f of z and divide by z minus z naught to an appropriate power.

So, ultimately all of these results you know go back to the fact that the contour integral of 1 over z d z about the origin is 2 pi i, but the contour integral of any other function z to n any other power z to n dz does not matter whether n is a positive integer or a negative integer other than you know minus 1.

So, 1 over z alone is special and that will give you a constant 2 pi i. So, it's really the same result and the fact that you can exchange contour integral and summation.

So, because of this you can directly argue that for any function which is analytic if you are looking at a Taylor series expansion or if you are looking at a Laurent series expansion, the expansion is about a point of non analytics or centered around the region of non-analyticity and the expansion is unique for that particular point and for that region of interest ok. So, that is all for this lecture.

Thank you.