Mathematical Methods 2
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Complex Variables
Lecture - 30
Liouvilles Theorem and the Fundamental Theorem of Algebra

Ok, so in this lecture we look at something called Liouville's theorem and another theorem which is an immediate consequence of Liouville's theorem namely the fundamental theorem of algebra, ok.

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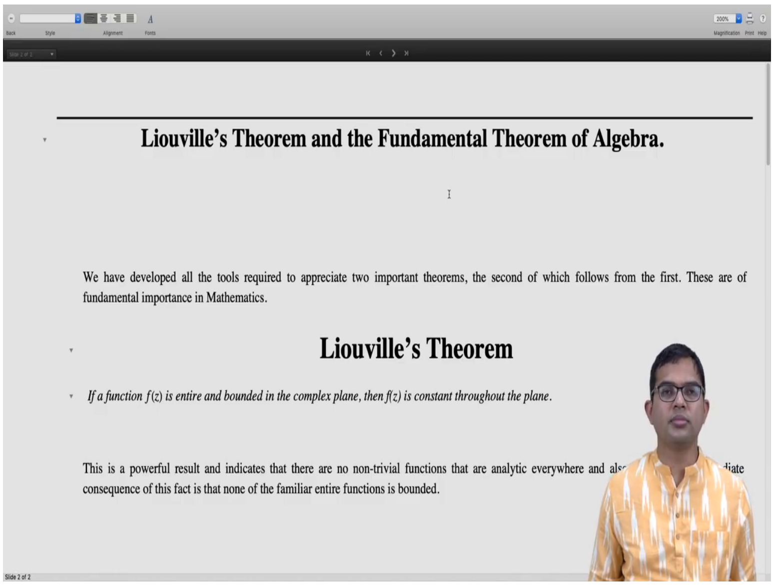

So, Liouville's theorem tells us that if a function is entire, it is analytic everywhere in the complex plane and it is bounded right. So, bounded means that the magnitude of this function is never greater than a certain constant then f of z is constant throughout the plane, right. So, that is the content of Liouville's theorem, right. So, this is a very powerful result.

And so what it means is that basically there are no nontrivial functions that are analytic everywhere and are also bounded, right. So, which also means that we know we are familiar with lots of functions which are entirely we looked at exponential of z, exponential of minus z, sin z, cos z. You know any of these trigonometric functions or you know other kinds of functions which are functions of e to the z try to some hyperbolic cosine function, hyperbolic sine function; many of these functions so we have you know we are familiar with and we know that to be entire functions.

But basically what Liouville's theorem tells us is that it cannot be bounded. So, if it is bounded and also entire then there is no way that this function can be anything other than a constant. It is a very dull boring sort of function which is stuff f of z is equal to constant, right. So, clearly sin of z is not a constant everywhere, but if we know that it is entire, so it cannot be bounded, right. So, that is the content of this theorem.

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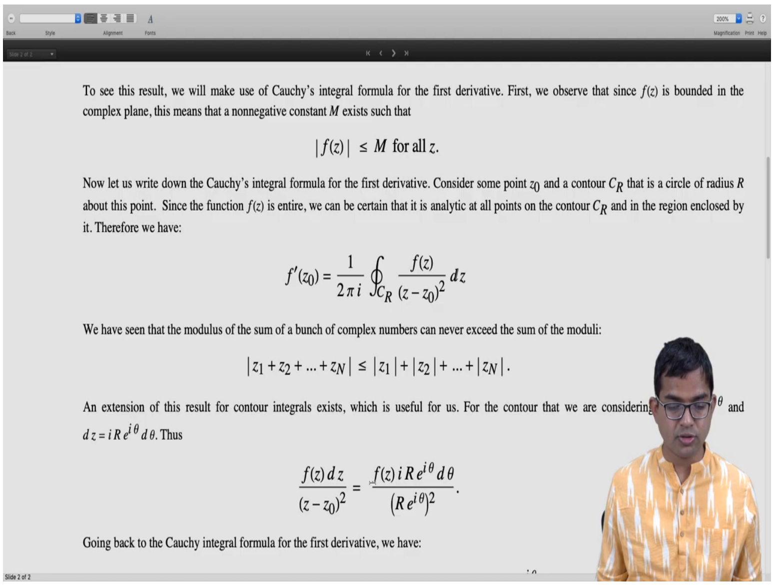

So, let us look at how to argue for this. This actually comes about from the Cauchy's integral formula itself, right. So, we will make use of the Cauchy's integral formula to show how Liouville's theorem comes about.

So, first we observe that f of z is bounded in the complex plane. So that means, there is a you know some positive constant M such that mod of f of z is less than or equal to M for all z, right. So, mod of f of z can never exceed this nonnegative constant M, right.

So, now let us write down Cauchy's integral formula for the first derivative. So, consider some point z naught which is the point of analyticity and consider a contour C R, we take it to be a circle of radius R centered about z naught, right.

So, since the function f of z is entire we can be certain that you know the conditions necessary for Cauchy's integral formula are indeed met. So, the function is analytic specifically in this entire region bounded by the contour including on the contour. And therefore, we can write down f prime of z naught is equal to 1 over 2 pi i contour integral C R f of z divided by z minus z naught the whole squared d z, right.

So, now comes this argument about you know which is somewhat like this triangle inequality, but you know applied to an integral, right. So, if we take a bunch of complex numbers and add them and take the modulus of the sum we have seen that this sum can never exceed the sum of the moduli of all of these complex numbers, right.

So, you know there is a result you know analogous to this also for integrals. So, basically the idea is if you take this kind of an integral here. Well, I mean we specifically choose the contour we have already chosen the contour to consist of a circle which can be written as z is equal to z naught plus R times e to the i theta. So, it is a circle of radius R centered about z naught. And so d z therefore immediately is seen to be i times R times e to the i theta d theta.

Thus, in place of f of z d z divided by z minus z naught square we can actually write f of z i times R times e to the i theta d theta divided by R e to the i theta the whole square. I mean, in place of f of z I could have also written f of z naught plus R times e to the i theta. But, for them you will see in a moment why I am just leaving it as f of z, right.

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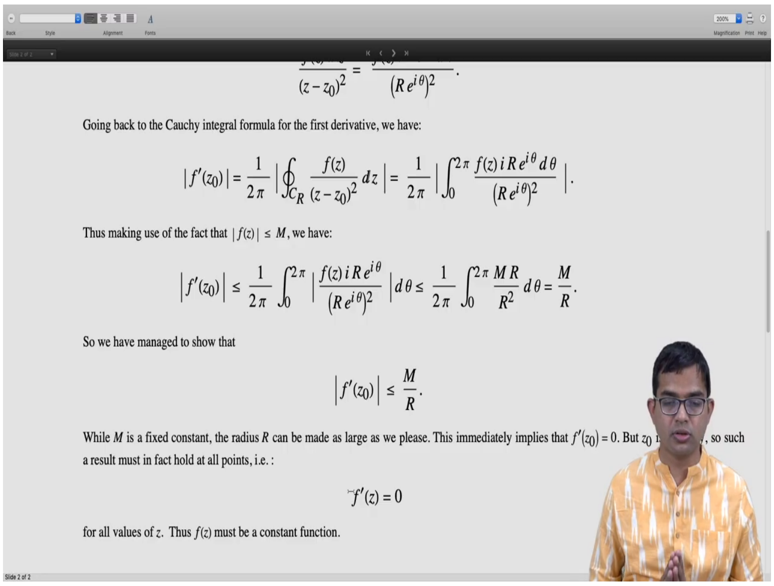

So, the idea is when I put this into this contour integral. So, I have and then I am interested in taking the mod of this, mod of f prime of z naught. Well, I mean mod of 1 on 1 over 2 pi i is just 1 over 2 pi so, i goes away. And then, I have this 1 over 2 pi times mod of this contour integral f of z divided by z minus z naught whole squared d z.

But we have just seen how this can be you know this integrand and d z can be replaced by this whole stuff. So, I write it as 1 over 2 pi mod of just a regular integral now, right. So, it is 0 to 2 pi f of z i times R times e to the i theta d theta divided by R times e to the i theta the whole squared, right. So, in place of z minus z naught, I am writing it as e to the R times z to the i theta.

Now, we make use of the fact that mod of f of z can never exceed M so, it is always less than or equal to M. So, I can actually take this modulus which is applied to the whole integral after the integral has been evaluated. But then I can argue that this mod will necessarily be less than or equal to 1 over 2 pi times you know this modulus being taken inside the integral for the integrand.

So, d theta of course is real so, I am taking it over all this stuff which is complex and then I just allow and then I go around this contour which is a circle. So, then immediately I see that this f of z; mod of f of z you know mod of all this stuff is actually you know product of the mods. So, I can separate this out as mod of f of z times R divided by R squared, but mod of f of z is necessarily less than or equal to M, right.

So, I you use that fact and I write this as less than or equal to 1 over 2 pi integral 0 to 2 pi M R divided by R square d theta and then d theta you know going around the circle once will just give me another 2 pi, that cancel and I am just left with M over R. So, what I managed to show is mod of f prime of z naught is less than or equal to some constant divided by R, right. So, this constant M is you know this is sort of part of the hypothesis we have said that this function is bounded.

Now this M is a fixed constant, but radius R can be made as large as we please. So, this immediately implies so, I can take this R to be you know very large. And so in fact, this implies that mod of f prime of z naught actually has to be 0, since R you can make R to be infinity basically, right. So, M is a constant whereas R can be as large as you want.

So, you can keep on increasing R. And the only way this can hold is if actually f prime of z naught must be 0; that is the only way this can hold for any R. But z naught is arbitrary, so in fact this result must hold for any point in the complex plane. So, we have the result f prime of z is equal to 0.

So, for all values of z; so, in other words we managed to show that f of z is equal to a constant. So, there is the only way that a function is analytic everywhere in the complex plane and it is bounded; if it is bounded and entire then that function can only be the constant function.

So, how do we reconcile with the fact that you know some functions like sin of z and you know seem to have not take very large values. You know all of these entire functions that we are familiar with you know have they acquire very large values at infinity. So, they have this, they have a singularity sitting at infinity.

So, some function like f of z is equal to z for example has a singularity sitting at z equal to infinity. It looks very nice and you know it is an entire function, but it is not bounded; it will mod of f of z you know cannot be less than or equal to M right; there is no constant M like that, it will keep. No matter what value you choose you will be able to find a z such that mod of f of z will be greater than such a value.

So, all of these entire functions that we are familiar with any interesting entire function is not going to be bounded, right. So, that is a very important result that is known as the Liouville's theorem.

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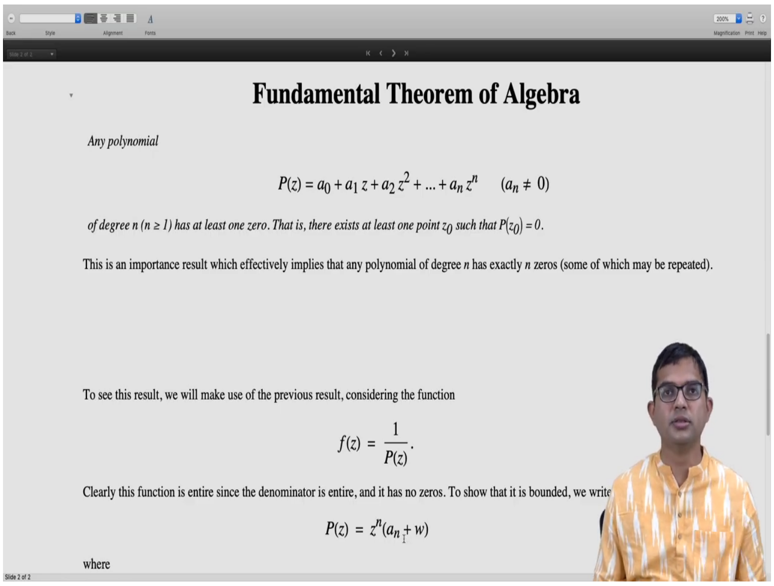

And from Liouville's theorem follows an extremely important result which is called the fundamental theorem of algebra. So, it says that any polynomial P of z, which is can be written as a naught plus a 1 z plus a 2 z squared so on all the way up to a n z to the n; a n not equal to 0 because it is an n-th degree polynomial has at least one zero, right.

So, that is you will be able to find the point z naught a complex number z naught such that P of z naught is equal to 0. So, this is known as the fundamental theorem of algebra. And so, in fact we have made use of this when we looked at some properties of linear vector spaces and you know finding the eigenvalues of a matrix and so on. So, there are very important consequences of this theorem.

And so effectively what it means is that any polynomial can be factorized into you know exactly n factors. Some of those may be repeated so we know that you can take a polynomial and write it as z minus z 1 times z minus z 2 times z minus z 3 all the way up to z minus z n, right. Some of these may be repeated. So you will actually get you know z minus z 1 the whole power 2 for example then. But, the sum of these you know powers of these factors will all add up to exactly n.

So, there are going to be exactly n roots right, which is a consequence of this because the way you argue is. If it is true that a polynomial of degree n has at least one zero then you take this find this 0 and divide this polynomial by z minus z naught then you will get a polynomial of degree n minus 1.

And if a polynomial of degree n minus 1 also must have at least one zero, right. And then you keep on reducing it until you reach you know just a constant, right. So, basically you know at every level you can argue that there is a factor, so there is a naught. And therefore in fact it is always possible to take any polynomial and factorize it into its factors which are connected to the zeros.

So, a very important theorem that has lots of important consequences, but let us see how it can be beautifully argued directly from Liouville's theorem. So, the way to do this is to consider this function f of z equal to 1 over P of z.

So, provided, I mean suppose we make the hypothesis that P of z has no zeros, right. So, in fact it is like a contradiction of this theorem. Suppose it is true that f of P of z has no zeros and then we will argue that this function f of z is equal to 1 over P of z must be bounded, right. So, the way to do that is to write P of z as z to the n times a n plus w.

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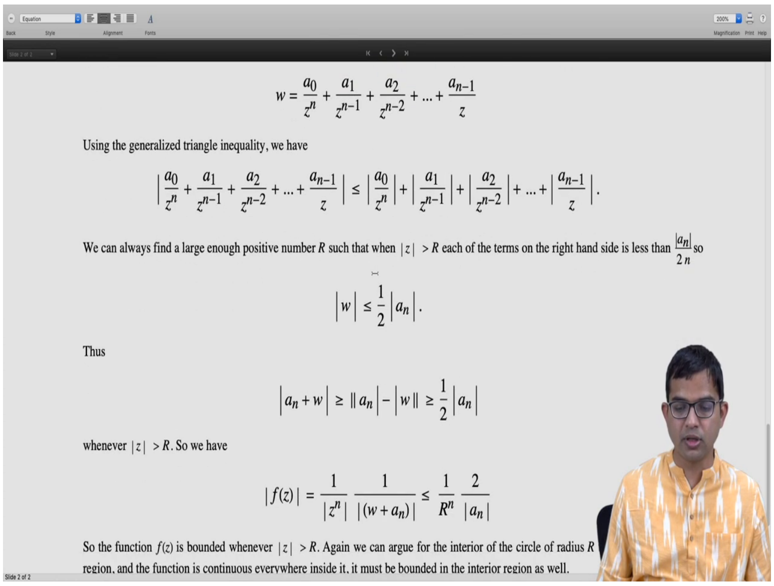

So, you pull out this factor z to the n and so you have, you know write the w is basically a naught divided by z to the n plus a 1 divided by z to the n minus 1. So, on all the way up to a n minus 1 divided by z, and then also you have a n, right. I have separated out a n and then I have written it as this whole stuff is just w.

So, then we argue I mean it is the generalized triangle inequality. So, mod of w is mod of this sum is less than or equal to sum of the moduli, right. So now, we can always find a sufficiently large R, such that whenever mod z is greater than R you know after all you have you know z z to the n z to the n minus 1; all of these guys sitting in the denominator, right.

So, you can always find a sufficiently large R such that each of these quantities is sufficiently small basically, right. It is always possible you can choose an R such that this guy, this quantity becomes small, this quantity becomes small because after all you have the power to make z as large as you want.

So, if you can always find an R such that each of these quantities is less than mod a n by 2 n. So, now you will see in a moment why we want to make it less than; so, after all this this is a constant. So, given any constant you will be able to find an R such that or each of this is smaller than that constant.

So, it turns out for our purpose it is enough to choose this constant to be mod a n divided by 2 n. So, what it means is mod w is less than or equal to you know n times mod a n divided by 2 n which is nothing but mod a n by 2, right. So we managed to show that there is always an arc such that mod z, when mod z is greater than R mod w will be less than or equal to half mod n.

And then, we can argue that this mod of a n plus w; that appears here right, so this guy is greater than or equal to mod of you know mod of a n minus mod of w. So, this is basically the triangular inequality applied in the other direction, right. So, we want to get a greater than or equal to symbol here so, we have to do mod of a n minus mod w.

But, mod of minus w is greater than or equal to half mod a n. So, using this result we just obtained, we managed to show that mod of a n plus w is greater than or equal to half mod a n. And so this is what we want to show, whenever mod z is greater than half.

So, if we combine this so immediately we have this result; that mod of f of z which is nothing but mod of 1 over P of z which is nothing but 1 over mod of z to the n times 1 over mod of w plus a n. This is going to be less than or equal to 1 over R to the n times 2 divided by mod a n.

So, basically what we have managed to show is that you know in the region with mod z greater than R this function mod of because this function f of z is bounded right. So, it is bounded it can never exceed a certain value right, you can always find an R such that this condition works, right.

So, basically mod f of z is bounded. And so since f of z has so, we have assumed that P of z has no zeros. So, if P of z has no zeros; so it is an analytic function P of z. Therefore, 1 of P of z is also very nice continuous function there is no difficulty because there is no 0.

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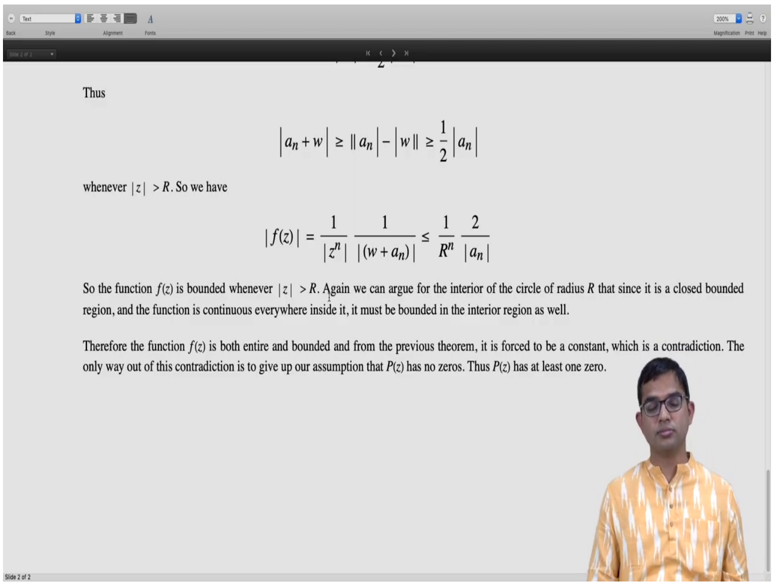

And so, this function is both entire and it is also bounded, right. So, we can also argue that in the interior of this circle of radius R. I am not going to go into the details of this argument. Basically the point is that inside a finite closed bounded region; if your function is continuous, since there is no there is no 0 for the denominator this function f of z is continuous everywhere inside, inside this region.

And any function which is continuous in a closed bounded region will have a maximum. It can, it cannot blow up at any point right. So, this is you know one can argue based on just the definition of continuity here, right. So, therefore it is bounded outside of this region and bounded inside this region.

Therefore, we have managed to show that this function f of z is analytic everywhere and it is also bounded and provided f of z P of z has no zeros, right. But then that the only way this function f of z can be bounded and also entire is if f of z is a constant, right. But f of z is not just a constant because P of z is a genuine polynomial of degree n, so there is a contradiction. Therefore the only way this can happen is if P of z has at least one zero, right. So, this proves the result.

Ok, so we have gone over this argument, you know somewhat in a fairly detailed way, but basically the point is that it is a direct consequence of Liouville's theorem which says that a function which is entire and bounded is a trivial constant. And we have argued that if a polynomial P of z does not have any zeros.

Then this will force a function like f of z is equal to 1 over P of z to become just a constant which is a contradiction. And therefore, any polynomial of degree n must have at least one zero. Which we have also said immediately implies that a polynomial of degree n, we will have n factors although some of these may be repeated, ok.

So that, so we have seen such a beautiful and powerful result which has applications in all fields of mathematics come out of some simple arguments involving the Cauchy's integral formula which comes about in the field of complex analysis, but this result it is self has you know wide applications in many fields of mathematics. That is all for this lecture.

Thank you.