Mathematical Methods 2

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Complex Numbers
Lecture - 03
The de Moivre formula

So in this lecture, we look at what is called the de Moivre formula. This is a formula which is very elegant and which actually follows from Euler's result that we have seen in the earlier lecture and it has some very interesting consequences. So, we look at an example of how this can be used.

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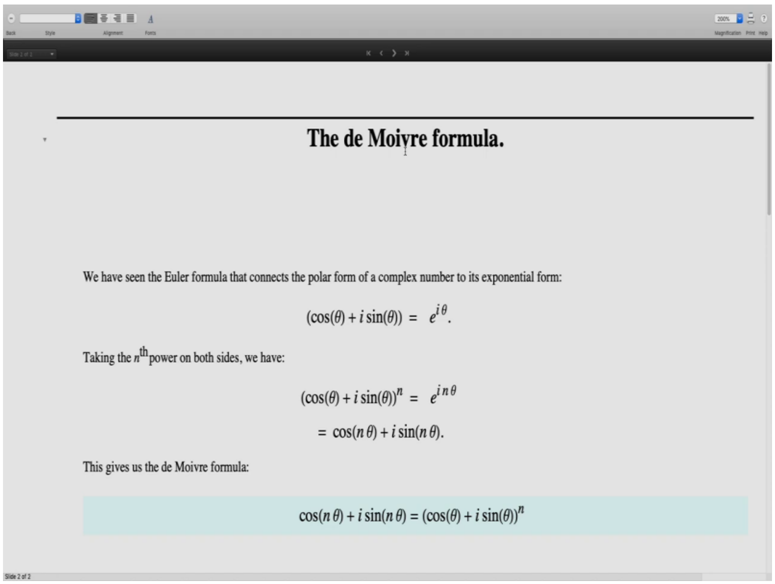

We have seen the Euler formula which says that the real part of e to the i theta is just cos theta and the imaginary part of e to the i theta is sin theta. So, e to the i theta is cos theta plus i sin theta. So, if you take the nth power on both sides. So, we get cos theta plus i sin theta the whole power n. Taking the nth power on the right hand side is much easier because it is exponential. It is just e to the i n theta.

So, we see that we are taking the nth power of a complex number whose modulus is unity and so, when you take the nth power, the modulus remains unchanged and it is only the phase which keeps increasing in value. And then it rotates around the circle in fact. So, what you have is e to the i n theta.

So, this is equal to cos n theta plus i times sin of n theta. (Refer Time: 01:58) where we have once again applied the you know the same Euler formula. So, in fact, this leads us to the de Moivre formula, which is the statement that cos n theta plus i sin n theta which is equal to cos theta plus i times sin theta the whole power n. So, this result allows us to extract some useful trigonometric identities.

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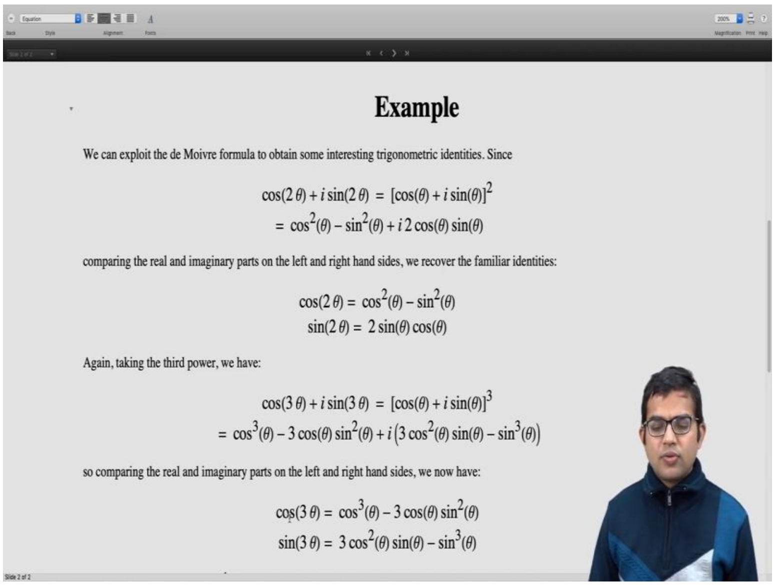

So, we will look at an example here. If I put n is equal to 2 here. I get cos 2 theta plus i times sin 2 theta is equal to cos theta plus i sin theta the whole square right. So, if I expand, then I have cos squared theta and then, i square sin square theta is same as minus sin squared theta; then plus i times cos theta sin theta plus i times sin theta cos theta which is same as i times 2 cos theta sin theta.

And then, when two complex numbers are equal, it necessarily means that the real part of each complex number you know is the same and also, the imaginary parts separately of these two complex numbers must be the same. So, the real part here on the left hand side is cos 2 theta and the real part on the right hand side is cos squared theta minus sin squared theta.

So, we have cos 2 theta is equal to cos squared theta minus sin squared theta and likewise, the imaginary part of the left hand side is sin of 2 theta, imaginary part on the right hand side is 2 cos theta sin theta. So, we have the result that sin 2 theta is equal to 2 sin theta cos theta. So, both of these are familiar trigonometric identities and so, we have just managed to obtain these results using de Moivre's formula.

So, if we had taken the third power instead of the second power right, we would have got this result; cos 3 theta plus i sin 3 theta is equal to cos theta plus i sin theta the whole to the power 3. So, carefully collecting terms, we have cos cube theta, then we have plus you know 3 i sin theta the whole squared cos theta.

So, that will give us the minus 3 sin squared theta cos theta plus if you take this square with the cos sin term and then, i is just i. So, then you have 3 cos squared theta sin theta with i and then, when you take i sin theta the whole cube. So, i squared will give you a minus 1 and then, you have 1 i extra. So, you get minus i sin cube theta right.

So, now once again, we can collect the real part and the imaginary part separately and then, compare the left hand side and right hand side which leads us to yet another useful trigonometric identity. So, cos 3 theta is the same as cos cube theta minus 3 cos theta sin squared theta and likewise, sin 3 theta is equal to 3 cos squared theta sin theta minus sin cube theta. So, there is nothing to prevent us from going to higher powers.

So, we have seen that when we take n equal to 2, it gives us a result which we are all completely familiar with and when we take n equal to the third power, we get a result which perhaps we have seen; but maybe some of us have not seen, but it is a way of writing cosine of some number times theta in terms of this cosines and sines of theta.

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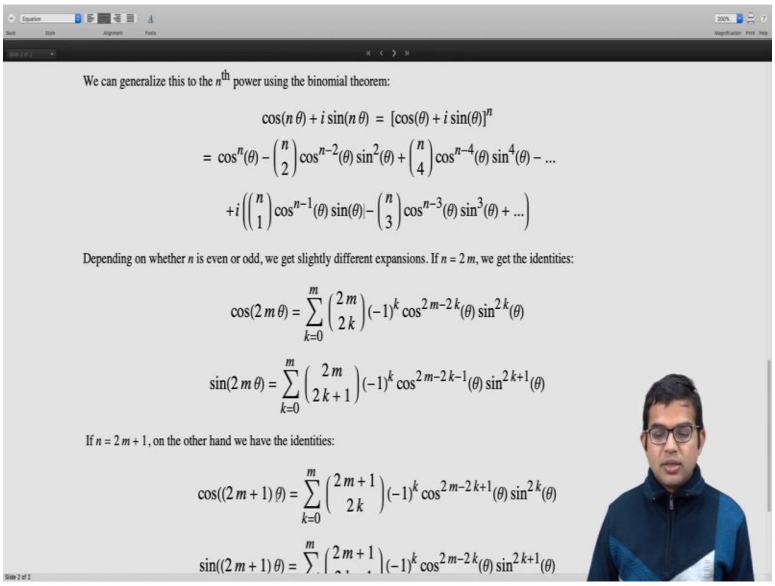

So, that is why it has some use. In fact, you can go ahead and generalize it to the nth power using the binomial theorem. So, cos of n theta plus i times sin of n theta is equal to cos theta plus i sin theta to the whole power n and so, this is the nth power of a sum of two terms and when we expand using the binomial theorem, so and then, we collect terms also carefully.

So, when I took cosine n of theta is a first term, then I if I take cosines cosine of n to the minus 2 theta, then it is going to have a n over n, n choose 2. But there is also going to be i squared right, i squared will give me a minus sign and there is a sin square of theta and then, likewise, have first term, second. This is the third term.

The fourth term we have i^3 which is minus i. The fifth term again, so where you are going to get i to the power 4 which will be just 1. So, all of these contribute to the real part. And likewise, the imaginary part is going to be the even terms in this case.

Actually, this is the first term and the second term, the fourth term and so on will all come with i's right. You have to be careful with the signs. So, because i cube is going to be minus i right. So, this is something that if we carefully collect terms, we have i times n choose 1 cos to the n minus 1 theta times sin theta minus n choose 3 cos to the n minus 3 of theta times sin cube of theta and so on right.

So, now some thought reveals that of course, each of these series are not going to keep on going forever right. We have seen examples, so they are going to end. So, in fact, you are going to have cosines and sines in each of these you know the sum of which will always be n.

So, if you have the n you know if you have the kth power of sin theta, then you are going to get n minus kth power of cosine of theta in each of these and you know you have to be careful about exactly how you know these powers are apportioned between cosines and sines and so, you see that here you know different some of these terms are part of the imaginary part and the other terms are in the real part right.

So, and this also depends on whether n is even or odd, we will see how at what point, it is going to terminate each of these series. So, we can you know carefully collect all these terms, if some little bit of thought you can show you know you can convince yourself of this following result.

If n equal to m, we get these identities cosine of 2m theta. I am directly writing in place of n 2m cosine of 2 m theta is equal to summation over k and so you, so this is something that you can check; k will go from 0 to m, 2 m choose 2 k. So, we see that only you know even powers appear.

So, we see that n is equal to 2 m. So, of course, n itself will appear and then n minus 2, n minus 4, even powers in cos theta appear and then, even powers in sin theta also appear right; cosine of 2 m of theta. And likewise, when we do sin of 2 m theta, this is going to contain odd powers in cos theta and odd powers in the sin theta as well so that the sum also always necessarily has to add up to this n which in this case is 2 m.

So, we have 2 m choose 2 k plus 1 minus 1 to the k cosine of 2 m minus 2 k minus 1 of theta sin to the power 2 k plus 1 of theta. So, this sine also has odd powers and cosine also has odd powers. The sum of these two necessarily always is m which is 2 m in this case right. And so, if n is an odd number on the other hand, we have the following identity.

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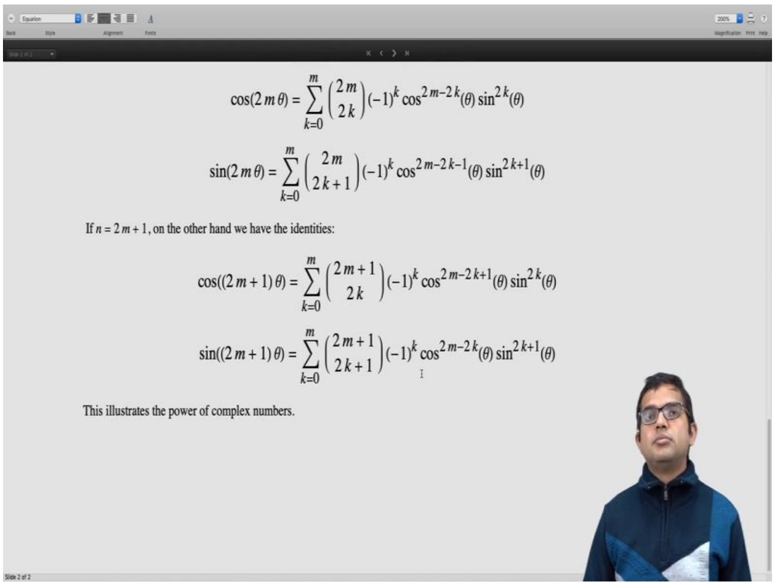

So, now you get cosine of 2 m plus 1 theta is you know summation over k going from 0 to m. So, again you get powers of this, this is the sine powers are all even here and the cosine powers are all odd in this case right. So, you have to; this two have to add up to 2 m plus 1. So, one of them has to be odd, the other one has to be even and on the other hand, the sine expansion is going to contain terms in which the cosine powers are all even and the sine powers are all odd right.

So, this is a set of identities which all follow from the joint use of the de Moivre formula along with the Binomial theorem. So, it is for you to check all these coefficients correctly and so, there are also some other interesting series which one can derive with the use of the de Moivre formula. So, you can look up some text books, maybe there will be some homework along these lines.

But, so, the whole point of this lecture is to illustrate that complex numbers will give some extra power right and particularly, when used in conjunction with you know other rules like the binomial theorem, you can derive these beautiful trigonometric identities. That is all for this lecture.

Thank you.