Mathematical Methods 2

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Complex Numbers
Lecture - 02
The triangle inequality

So in this lecture we look at the triangle inequality from the point of view of omplex numbers. So, the triangle inequality is basically the statement that given any 2 points now that the line joining the 2 points will be the shortest possible distance. If you take a third point and then you take a journey including the third point, then necessarily that distance covered will be greater than if you just took a straight path.

But this is a statement that we will derive using complex numbers in this lecture.

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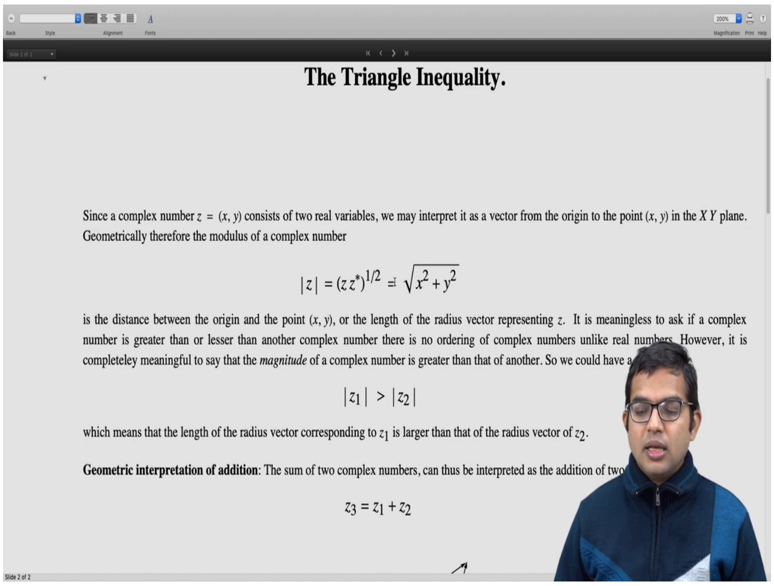

So, let us first of all look at how you know complex numbers have this geometrical interpretation. So, we know that a complex number consists of 2 real variables. So, that is a real part and an imaginary part, each of them is a real number and so we can in fact, interpret a complex number as a vector from the origin to the point x,y in the x y plane.

And so there is this complex plane and so any complex number can be thought of as a point in this x y plane. And in fact it is helpful to imagine this complex number as a vector which is formed by joining the origin to the point x,y.

So, the modulus of a complex number therefore is nothing but the length of the vector, which is formed by joining the origin to the point x,y right. So, we have seen that the modulus of z is just the square root of x squared plus y squared. So, it is the distance between the origin and the point x,y in the 2D plane which is the length of the radius vector.

However there is no ordering in the complex field. So, if you have any 2 complex numbers it is meaningless to ask if one is greater than the other. So, there is no ordering, which is naturally present for complex numbers unlike in the case of real numbers. So, we know that all real numbers sit on the real number line.

So therefore, it is always possible to ask if a certain real number lies to the left of another real number or does it lie to the right of it or of course to the possibility of the 2 of them being the same is always there. There is definitely a notion of equality of 2 complex numbers, but inequality of complex numbers is only as far as saying that they are not equal.

You cannot say one of them is greater or lesser than the other, but on the other hand it is certainly possible to ask for the relation between the magnitudes of 2 complex numbers. So, you can definitely say that mod of z 1 is greater than mod of z 2 or mod of z 1 is less than mod of z 2 and because mod of z 1 and mod of z 2 are both real numbers.

So, this is well defined and so it is just simply a reference to the lengths of the radius vectors of each of these complex numbers right. So, there is a geometric interpretation possible for the addition of 2 complex numbers as well. We have seen that when you add 2 complex numbers their real parts add and the imaginary parts add.

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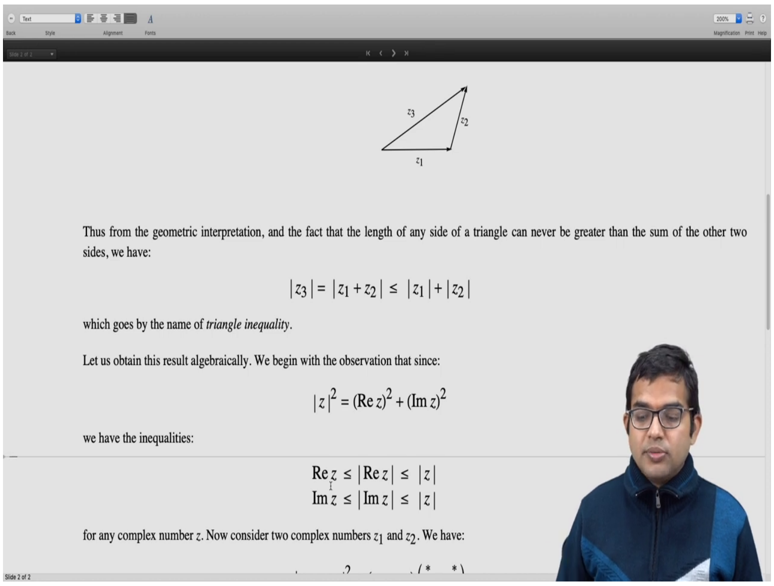

So, we can think of the addition of 2 complex numbers as the addition of two vectors. I have you know drawn this vectors for z 1 and z 2. So in fact, if a, you know z 1 starts at the origin and goes all the way up to. We place the tip of one vector at the bottom of the second vector and obtain the sum of two vectors as another vector. We are free to move vectors in parallel.

So and therefore the sum of these 2 vectors will turn out to be just this vector formed by joining you know the bottom of z 1 to the tip of z 2 after the bottom of z 2 has been placed at the tip of z 1, being careful to move it in a parallel fashion. So, you cannot mess with the directionality of z 2, but you have the freedom to move it in a parallel manner.

So, this is a geometric interpretation. So, this is what we would call the parallelogram law of addition of vectors. So, there is this geometric interpretation and soon we should also be able to understand the triangle inequality from a complex number point of view right. So, a natural you know idea which comes up next would be that of the triangle inequality.

What is the triangle inequality? So, the length of any side of a triangle is less than or equal to the sum of the lengths of the other 2 sides right, that is the statement of the triangle inequality. The less than rule holds if it is a real triangle; if it is a degenerate triangle you also have the less than or equal to.

So, this is just like a commonsense idea that if given any 2 points the shortest distance between those 2 points is simply along the line joining those 2 points. And if you happen to take a detour via a third point meaning you have 2 different line segments, if you add them the sum of these two lines necessarily must be greater than or equal to just going in one straight direction.

Equality will hold if it so happens that the third point also lies on the first line segment itself and therefore then it is a degenerate triangle it is not quite a triangle at all ok. So, the way to say this in terms of complex numbers is mod of z 3 must be less than or equal to mod of the sum of z1 and z2. So, mod of z 3 is the same as mod of z 1 plus z 2 right because this is the geometric interpretation of you know addition of complex numbers, mod of z 1 plus z 2 is less than or equal to mod of z 1 plus mod of z 2 right. So, that is the triangle inequality.

So, let us prove this using a completely algebraic approach. So, we begin with the observation that since mod of z square is the same as the real part of z 2 square plus imaginary part of z 2 square. So, this is really nothing but stating the fact that mod of z is equal to square root of x square plus y square that for mod of z square is x square plus y square, but x is real part of z and y is imaginary part of z.

So, immediately we have these 2 inequalities. So, real part of z must be less than or equal to mod of real part of z right. I mean real part of z can be positive or negative but mod of real part of z is a positive number so this inequality or trivially. And so in fact mod of real part of z must be less than or equal to mod z right.

So, you are adding 2 positive numbers to get a third positive number. So, to be more precise you are adding 2 non negative numbers to get a third non-negative number. So, this inequality certainly holds real part of mod of real part of z must be less than or equal to mod of z and likewise mod of imaginary part of z is less than or equal to mod z right.

So, saying that imaginary part of z is less than or equal to mod z is a weaker inequality. So, in general in fact you can say mod of this itself is less than or equal to mod of z right for any complex number z. So, we will exploit this in our argument ahead. So, let us consider 2 complex numbers z 1 and z 2.

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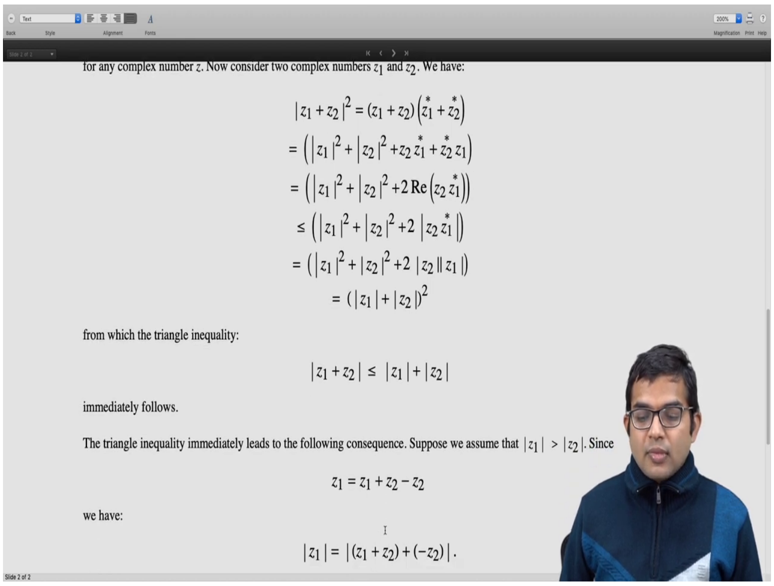

So, we have mod of z 1 plus z 2 the whole square is nothing but this complex number z 1 plus z 2 times the complex conjugate of this complex number z 1 plus 2. But the complex conjugate of the sum of 2 complex numbers is the same as the sum of the complex conjugates of these two complex numbers. So, we have z 1 star plus z 2 star here.

So, z 1 plus z 2 the whole thing times z 1 star plus z 2 star and so then we just expand it out. So, we have so when you write down z 1 times z 1 star that can be written as mod of z 1 square and likewise z 2 times z 2 star will become mod of z 2 square and then we have these 2 cross terms z 1 times z 2 star plus z 2 times z 1 star right. So, you so this is just simply a consequence of expanding all these and identifying some of these terms appropriately.

So, then we observe that in fact you know these last 2 terms are actually complex conjugates of each other z 2 times z 1 star, the whole star is nothing but z 2 star times z 1 star the whole star but z 1 star star is z 1. So, indeed these are complex conjugates of each other. So in fact if you are adding a complex number to it is complex conjugate that is going to give you just 2 times real part of that complex number right.

So, this is something which follows from the definition itself. So, all I am saying is that I have a complex number z and so which can be written as x plus I times y, so z star is x minus i times y. So, if I add z z plus z star I am going to get 2 x, so this y part is going to just cancel.

So in fact therefore since I have identified these 2 to be complex conjugates of each other and I am adding these 2 it is the same as 2 times real part of either of these it does not matter real part of z 2 z 1 star is same as real part of z 2 star z 1. Now, we exploit this inequality, which we wrote down right at the beginning. So, the real part of a complex number is going to be less than or equal to the modulus of that complex number.

So in fact this is less than or equal to mod of z 1 square plus mod of z 2 square plus 2 times mod of the complex number itself. As far as the inequality is concerned in place of 2 times real of this we just replace it by 2 times mod of z 2 z 1 star.

And then we rewrite the same thing like here, because mod of the product of 2 complex numbers is simply the product of the mods. So, and also the mod of the conjugate complex conjugate of a complex number is the same as the modulus of the complex number itself. So, mod of z 1 star is the same as mod of z 1, so we have 2 times mod of z 2 times mod of z 1.

So, this is an inequality here from here to here and then we immediately observe that this is nothing but it is a perfect square. In fact, we have mod z 1 is a real number square plus mod z 2 another real number square plus 2 times product of 2 the same 2 real numbers. So, which is nothing but mod of z 1 plus mod of z 2 the whole square.

So, what we have managed to show is that the mod of z 1 plus z 2 the whole square is equal to mod z 1 plus mod z 2 of this whole square right. If you take the square roots on both sides we have the triangle inequality which is the statement that the mod of the sum of 2 complex numbers necessarily is less than or equal to the sum of the magnitudes of the 2 complex numbers right. So, this is the triangle inequality.

Now there is an immediate consequence of the triangle inequality and so suppose without loss of generality we assume that mod of z 1 is greater than mod z right. Now since z 1 can be written as z 1 plus z 2 minus z 2 all I have done is added z 2 and then subtracted z 2 from a. So, z 1 is equal to z 1 plus z 2 minus z 2. And then if I take the modulus of this number z 1 it is the same as the modulus of a complex number plus another complex number.

So, minus z 2 is also a complex number if z 2 is a complex number minus z 2 is another complex number. So, I am adding 2 complex numbers one of them is z 1 plus z 2 and the other one is minus z 2 and I am taking the the modulus of the sum of 2 complex numbers.

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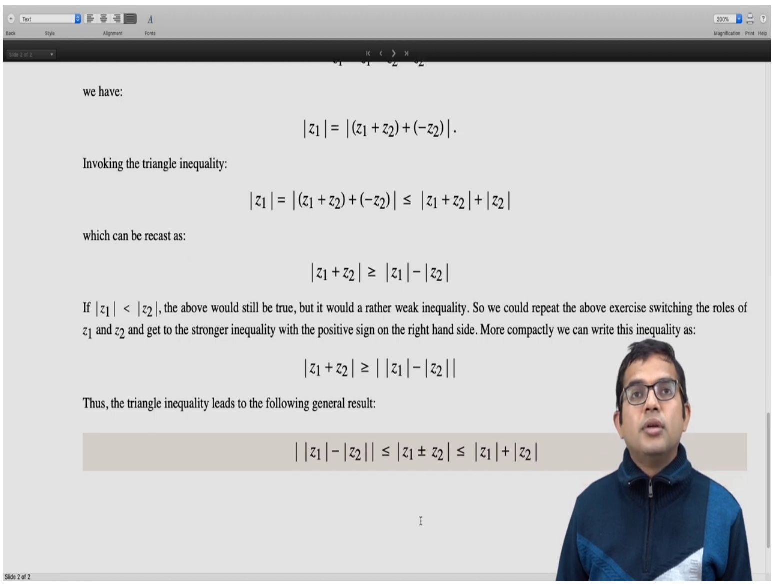

But what have we just shown we have shown that the modulus of the sum of 2 complex numbers must be less than or equal to the sum of the individual moduli; if I invoke this triangle inequality so then I have mod of z 1 is equal to mod of this whole stuff must be less than or equal to mod of z 1 plus z 2 plus mod of minus z 2.

But mod of minus z 2 is the same as mod of z 2 which I directly write down. So, I have mod of z 1 is less than or equal to mod of z 1 plus z 2 plus mod of z 2, but I can bring this mod z 2 to the other side and then rewrite this inequality as mod of z 1 plus z 2 is greater than or equal to mod z 1 minus mod z 2 right.

So, I mean I could have of course done the whole exercise even if mod z 1 were less than mod z 2. All of these are arguments which still hold. But then this would be a very weak inequality right because all it is if I have on the right hand side I would have mod z 2 minus mod z 1 right. But if well I mean if so I would get a negative number on the right hand side if I had if mod z 1 were to be were to be less than mod z 2 right.

Everything would just go through as it is and then if you have got a negative number on the right hand side if I am claiming that the modulus of a complex number is greater than or equal to a negative number which is a you know a statement which has no content in it. Because of course, the modulus of any complex number is going to be greater than 0, so it is a it is not even a weak inequality but it is a non-statement in some sense.

So, only if I have the stronger inequality has some content. So, that is why I had to make this assumption. So, in general we could actually write this inequality as a modulus of this overall thing. So in fact, mod of z 1 plus z 2 will be greater than or equal to modulus of mod z 1 minus mod z 2 right would hold that is the strong form of writing this inequality right.

So, more compactly so you see that this has given different bounds. So, we have an inequality in which mod of z 1 plus z 2 it should to be greater than or equal to something, we already showed what mod of z 1 plus z 2 less than or equal to something all of that can be compared together and you know we can write this general desire right.

So, first of all we write mod of z 1 plus z 2 is greater than or equal to modulus of mod z 1 minus mod z 2 like I just said and then we have this much more compact is it. So, I write down I bring this stuff to the left most and so mod z 1 minus mod z 2 is less than or equal to mod of z 1 plus z 2 and which is less than or equal to mod z 1 plus mod z 2. So, and if you observe I have suddenly brought a plus or minus sign here.

So in fact, you can argue that it does not matter whether there is a plus sign or a minus sign right, because you know if this holds for any 2 arbitrary complex numbers z 1 and z 2 it would hold also for the 2 complex numbers z 1 and minus z 2 alright. So, you know in certain applications it is more useful to use the minus sign.

So, we might as well make it explicit and write down mod of z 1 plus or minus z 2 here in the center and so this is you know necessarily less than or equal to the sum of the moduli of z 1 and moduli of modulus of z 2 and it is greater than or equal to mod of z 1 minus mod of z 2 the magnitude of this quantity right.

So, the magnitude is there so that this inequality is not redundant it has some content for sure. But the key point from this whole exercise is to realize that really it is just 1 inequality we manage to show the triangle inequality and then we applied it in a clever way to itself and to get a an inequality, where we have managed to show what mod of z 1 plus z 2 is greater than or equal to right using the triangle inequality itself ok. So, that is all for this lecture.

Thank you.