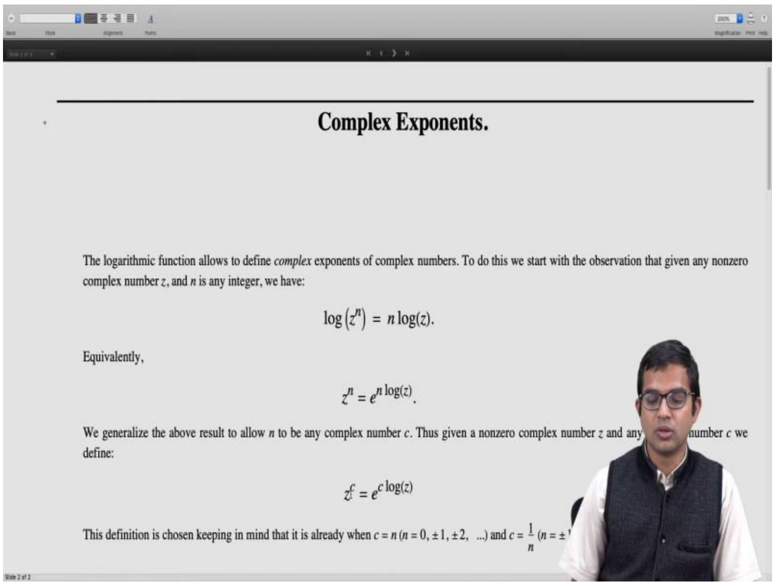
Mathematical Methods 2
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Module - 02
Complex Variables
Lecture - 17
Complex Exponents

We have seen how the exponential function can be generalized to include complex variables. We also saw how the logarithmic function appears naturally when we ttry to invert the exponential function. Using the log function, it turns out that we can also give a meaning to a complex exponent, so that is to say you take some complex number to the power of some other complex number. That is what we will discuss in this lecture.

(Refer Slide Time: 00:51)

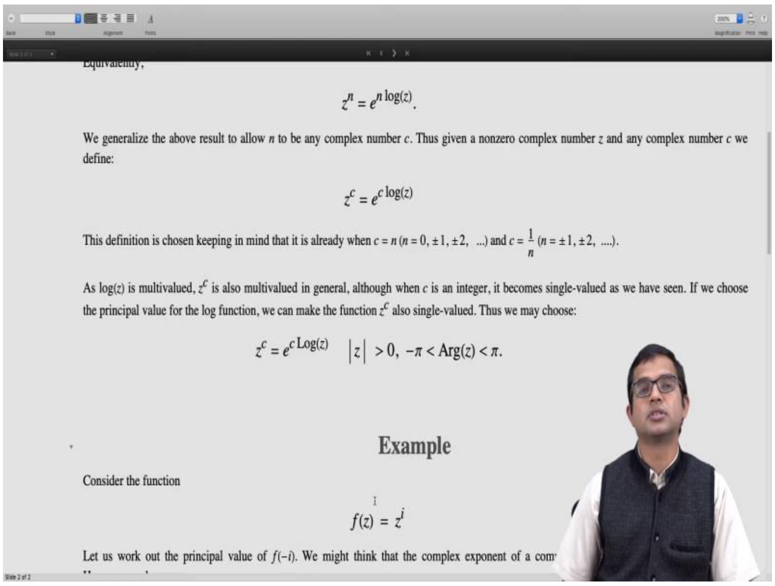


The starting point is the observation that if you take a complex number and take its power to some integer, then we have this property log of z to the n is equal to n times log z, or equivalently z to the n is equal to e to the n times log z. So, this is a property which we have already looked at when we introduced the log function.

This is a result that we will generalize now to allow for n to be any complex number c. So, n is no longer restricted to be an integer right, but for any complex number c we can define z to the c to be this complex number which comes when you compute e to the c times log z.

This is chosen keeping in mind the general principle that whenever you generalize a function it should return for you the same value at least when you put values for which it was already defined.

(Refer Slide Time: 01:58)

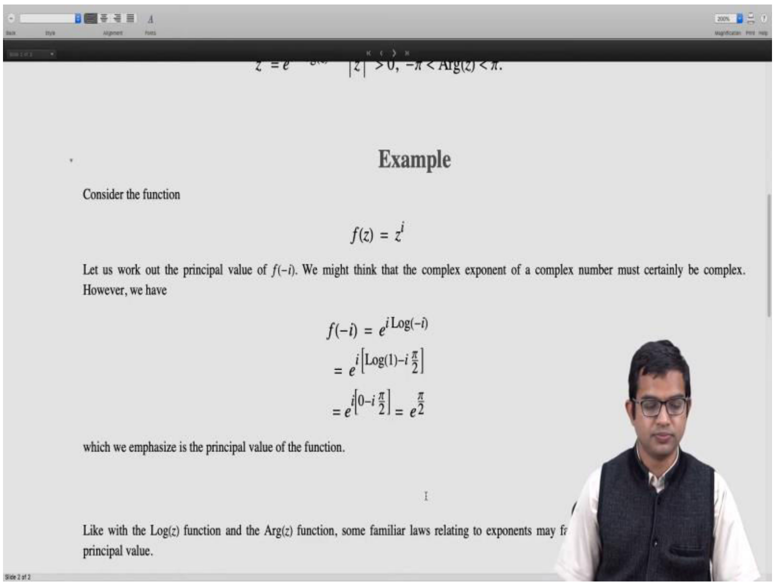


So, in this case, if you put c equal to n or c equal to 1 over n, it will return the same complex number which we had earlier for z to the n. So, of course, because log z is multivalued, so is z to the c in general right. So, when c is an integer we have seen that it becomes single valued.

But in general for an arbitrary complex number c, this function z to the c is a multivalued function. But like we did with the argument function and with the log function, it is possible to restrict it to what is called a branch. We may choose to work in a special branch called the principal branch where the function takes the principal value. There are other values for the function since it is multi-valued. At some point, we will discuss multivalued functions in some more detail, but for our purpose here we just want to observe that since log z is a multivalued function so is z to the c. And we can make it single valued if we restrict log of z to be capital Log of z which is the principal value.

So, we can choose z to the c to be e to the c times capital Log of z or principal value of Log of z mod z greater than 0 and this angle which is represented by the capital Arg of z function is restricted to lie between minus pi and plus pi. And it is possible to choose any range which extends over a region of 2 pi right, but the convention is to choose minus pi to plus pi.

(Refer Slide Time: 03:51)



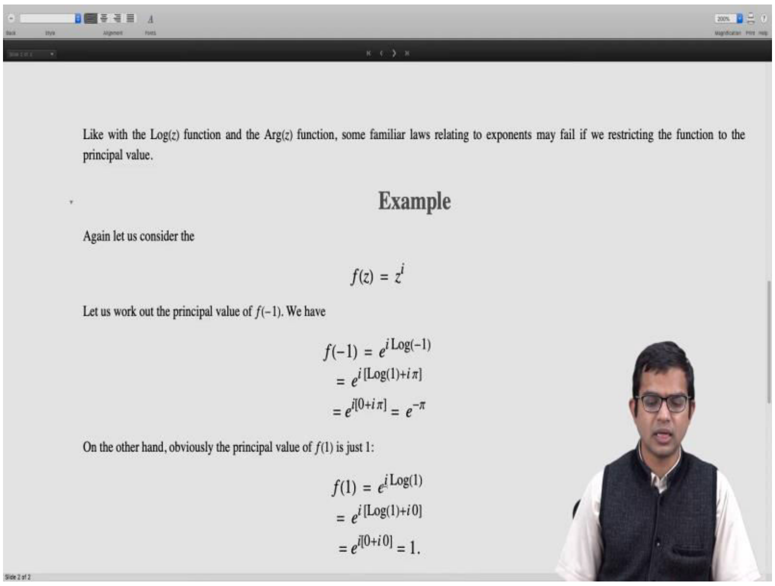
If we do this, then it becomes single valued. Let us look at an example. Suppose, we think of the function z to the i. Suppose you wish to work out the value of minus i to the power i. So, you might think that this is a very complicated looking operation. So, yeah, minus i to the i must be a complex number. However it turns out that the principal value of this quantity minus i to the power i is in fact simply obtained.

So, let us look at what happens. If I want to compute f of minus i, I have to do e to the i times Log of minus i. And Log of minus i to compute it, I have to first put the modulus of this which is Log of 1 minus i times pi by 2. So, if it is minus i, then we can ascribe an angle of minus pi by 2 for this quantity.

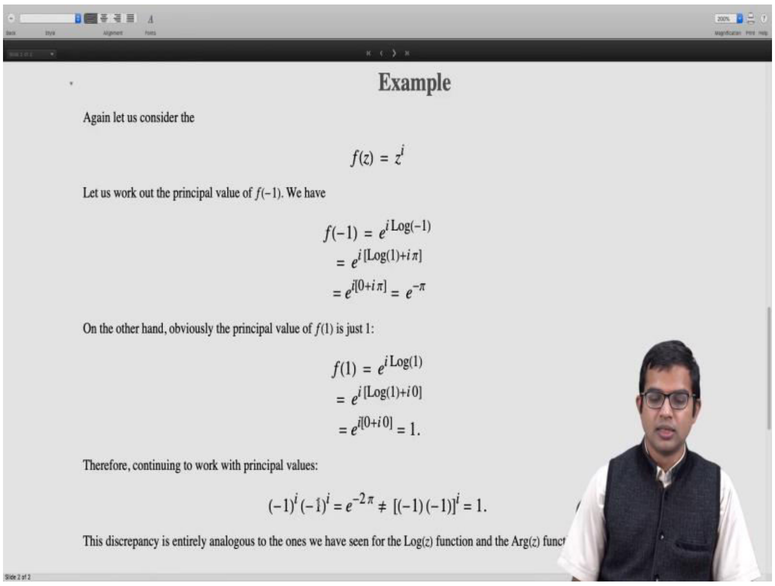
And so then we see that Log of 1 is a real positive number. So, Log of 1 is a 0. And so we will be just left with i squared minus i squared will be plus 1. So, it is just e to the pi by 2. So, in fact, it turns out that minus i to the power i the principal value of this is a real number e to the pi by 2.

So, like the Log function and the Arg function some familiar laws relating to exponents may fail if we restrict this function to the principal value. So, let us look at an example of how this can happen. So, again if you consider the same function f of z is equal to z to the i, and if we work out the principal value of f of minus 1.

(Refer Slide Time: 05:32)



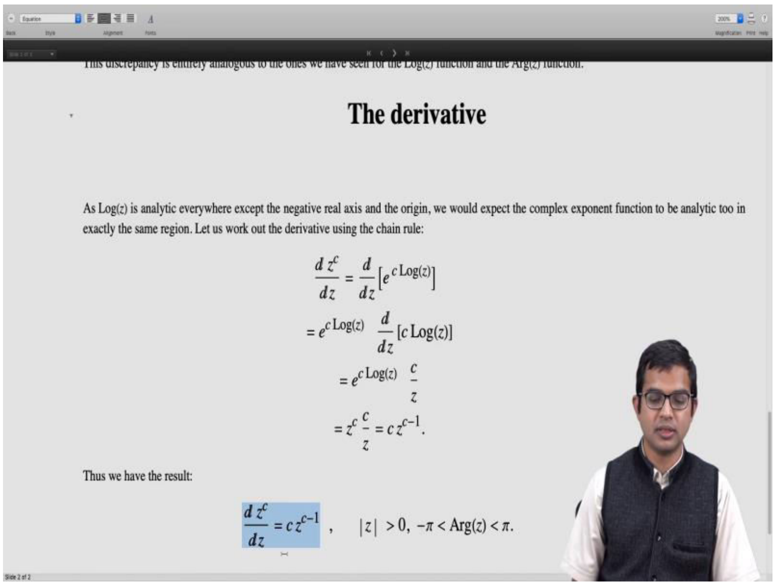
So, my f of minus 1 is e to the i Log of minus 1 by a procedure which is very similar to the previous one, but now we have instead of minus i we have minus 1. So, e to the i times Log of 1 plus i pi and which is e to the minus pi in this case i squared is minus 1. So, e to the minus pi right. So, on the other hand, f of 1, it is just 1 because it is 1 to the i right. (Refer Slide Time: 06:24)



So, the key point is that if I take minus 1 to the power i times minus 1 to the power i and if I take the principal values, I would get e to the minus pi times e to the minus pi which is e to the minus 2 pi. But this is not going to be equal to the product of these two functions, these two numbers to the power i. We are used to working with identities like z 1 to the power c times z 2 to the power c must be equal z 1 times z 2 the whole power c .

So, this kind of an identity may not hold if you restrict to the principal value. So, here for example, you see minus 1 times 1 is just 1. So, 1 to the power i is going to be 1. So, clearly 1 is not equal to e to the minus 2 pi right. So, this is similar to the kind of difficulties we encountered when we worked with capital Log of z or capital A Arg of z function right. So, some familiar identities involving the Log may not quite hold if you are working with the principal value.

(Refer Slide Time: 07:32)



So, this function is analytic wherever Log z is analytic. And we have seen that Log z is analytic everywhere except along the negative real axis. And so we can use the chain rule to explicitly get the value of the derivative when this function is analytic. So, d by d z z to the c is the same as d by d z of e to the c Log z. And so you are going to get e to the c Log z times c divided by z which in turn is immediately seen to be just z to the c times c divided by z or just c times z to the c minus 1 right.

So, this is the sort of familiar expression we have for some real variable to some exponent, it is going to give you z exponent times the variable to the power exponent minus 1, so that is what we are seeing except now we have both c and z are complex numbers.

So, this function is analytic wherever the Log function is analytic and we have seen how if you choose this branch of the Logarithmic function, indeed the Log function is going to be analytic. And therefore, in this region, mod z is greater than 0. And when the argument of z lies between minus pi and plus pi, you have this result. So, z to the c is not only analytic and its derivative can also be explicitly written like here. Ok, so that is all for this lecture.

Thank you.