Computational Mathematics with SageMath

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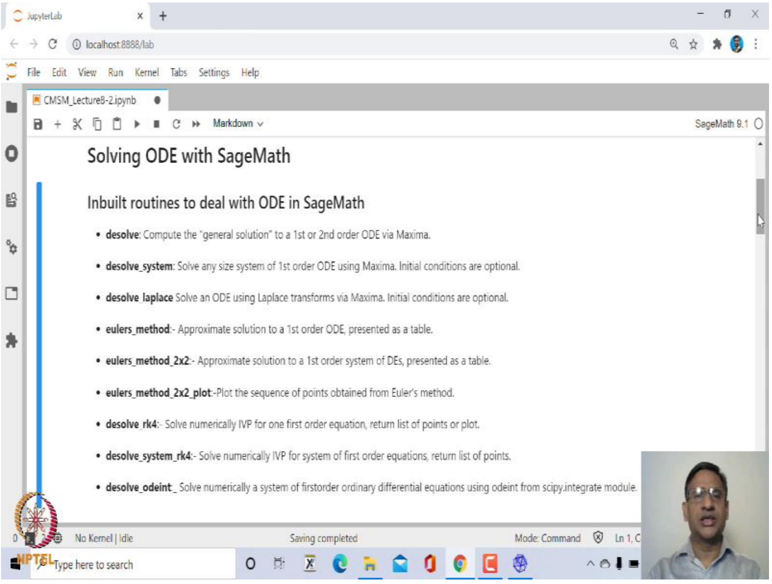
Lecture - 50

Solving 1st and 2nd order ODE with SageMath



Welcome to the 50th lecture on Computational Mathematics with SageMath. In this lecture, we will explore Solving Ordinary Differential Equations of order 1 and 2 using SageMath. First let us look at what are the facilities or inbuilt methods which are available in SageMath in order to solve differential equations.

There are several functions available in SageMath - one is desolve to find a general solution of 1st and 2nd order ordinary differential equations and it uses maxima in the background.

It also has desolve underscore system, we already made use of this when we looked at application of eigenvalues and eigenvectors to solve system of ordinary linear differential equations.

Then you have desolve_laplace it solves initial value problems using Laplace transforms and then it also has numerical methods like Euler's method. You also have Euler methods 2 cross 2 that is 1 st order system of ordinary differential equations.

Then you can also plot graph of the solution of system of ordinary differential equations.

Then you also have rk4 method including rk4 method for system of linear system of 1st order differential equations.

You can also use desolve_odeint, this actually solves the numerically a system of 1st order ordinary differential equations, and it uses scipy integrate module.

So, these are the some of the default facilities available in SageMath in order to solve differential equations.

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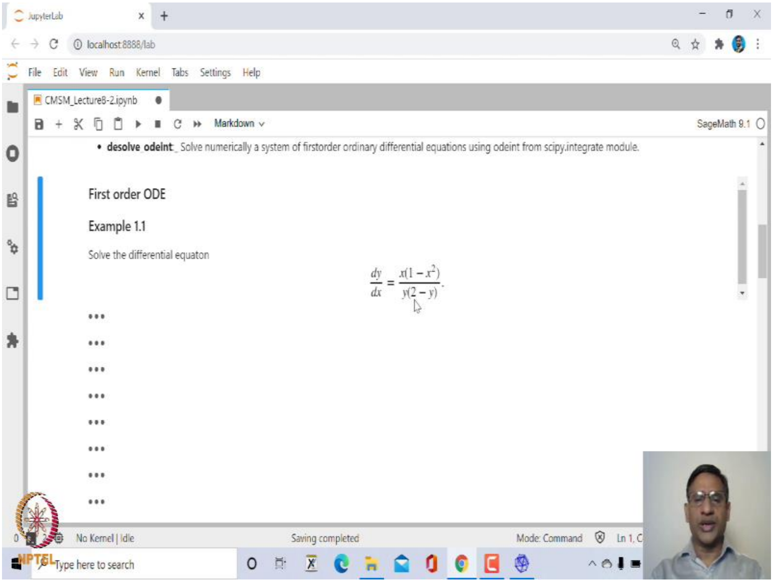

Now, let us start with an example. Suppose we want to solve this differential equation, 1st order differential equation

dy by dx is equal to x into 1 minus x square upon y into 2 minus y.

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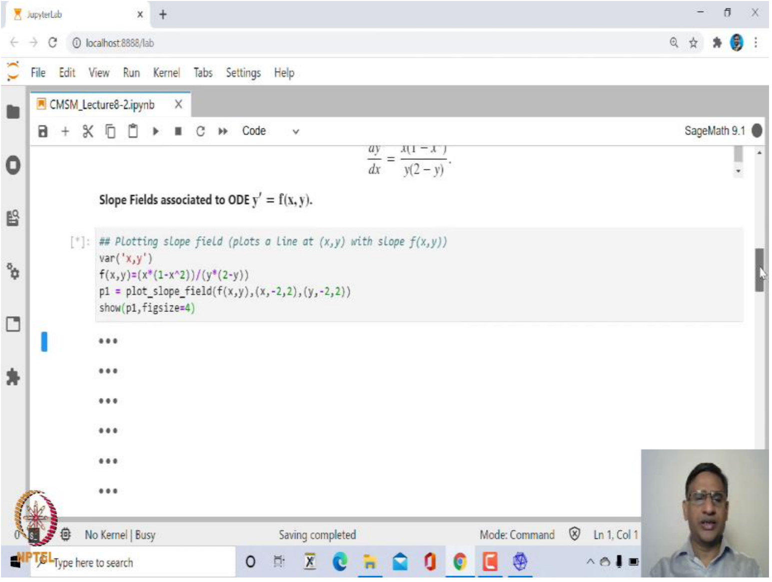

So, first thing we will look at how to plot slope fields associated to this differential equation. When we say slope field, what does it mean? dy by dx is nothing, but slope of the curve y equal to f(x). So, this is your y equal to f(x)

So, at every point you have the slope which is given by this value. At each point the slope field will give you a tangent to this curve at that particular point (x, y). Let us see how we can plot. Sage has an inbuilt function known as plot_slope_field in order to plot slope field of a given function.

Here we define f(x, y) is equal to the right hand side of this ordinary differential equation and then we use plot_slope_field and then let us reduce the figure size is equal to 4 and then you will see, right.

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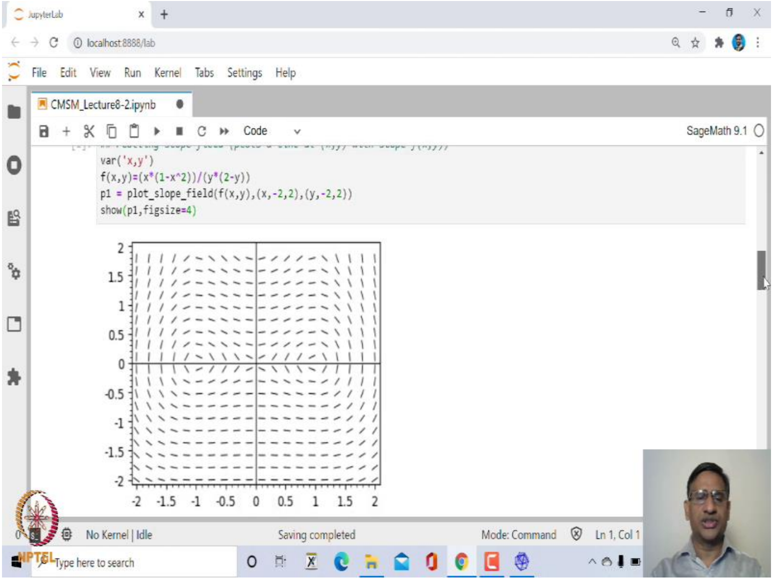

This is the slope field. For example, at this point, if I look at this tangent is along this line and so on. So, at each point it plots a small line which you can think of as if a line chopped off from the tangent line and in the direction of the slope of the function.

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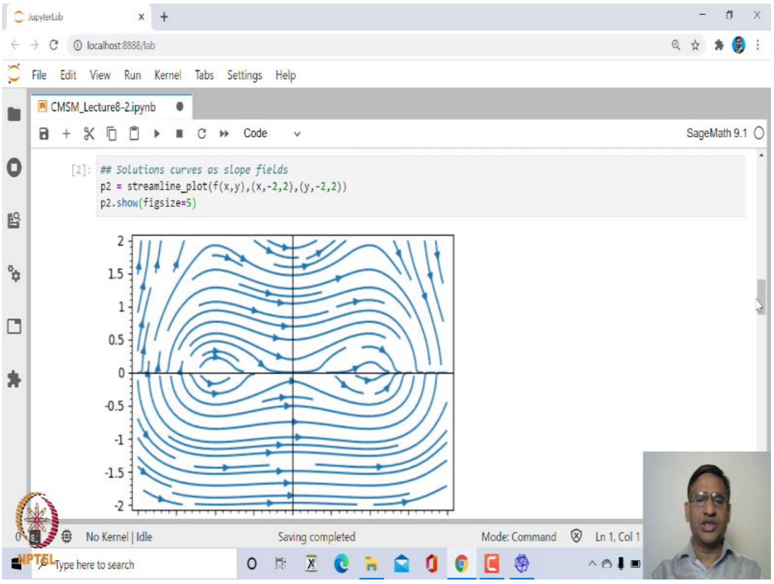

You can also plot streamline plot, you can use streamline_plot. This will plot the curve, not just the tangent, but it will plot the curve. Let us look at. Again we are plotting x between minus 2 and 2, y value also between minus 2 and 2. So this is the streamline plot.

If we think of let us say this point is 0 comma 1. So, this is the curve passing through 0 comma 1. It actually will be a solution of this differential equation with initial value x(0)=1. So, this is how the solution curve will look like. We also call the solution curve as integral curves.

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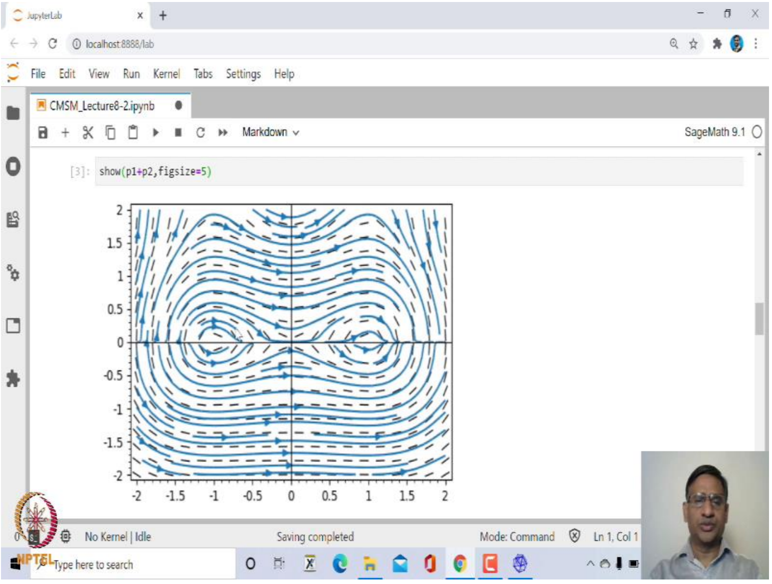

We can try to plot both these things together, the solution curve, that is, integral curves along with the slope fields. You can see here for example, if I look at this curve this is how the slope field is moving. Of course, it is quite easy to see that no two slope curves, no two integral curve will intersect.

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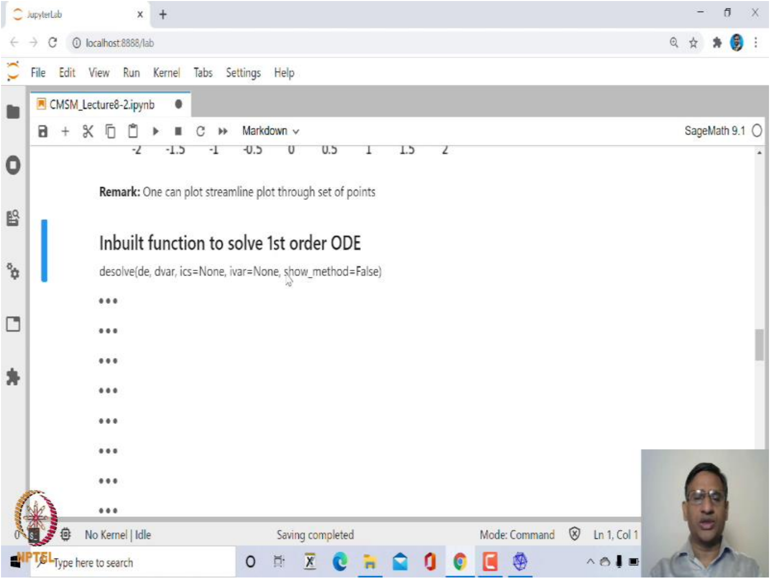

Now, let us look at, you can also plot this streamline plot through a set of points. If we wanted to plot only one solution curve let us say, through (0,-1), for example, let us say this curve. We can simply mention an option called starting start points. Then you will see the option, you can look at help on streamline_plot it will tell you how to use set of points from which you want to plot the solution curve or this integral curve.

Now, as I said Sage has inbuilt function namely desolve to solve ordinary differential equation of order 1 and 2. In this case, let us see how it works. The function is desolve you need to give the differential equation, you need to declare the variable and initial condition, if any otherwise it will give you general solution. You can also mention what is the method it is using. It has some certain methods which you can display.

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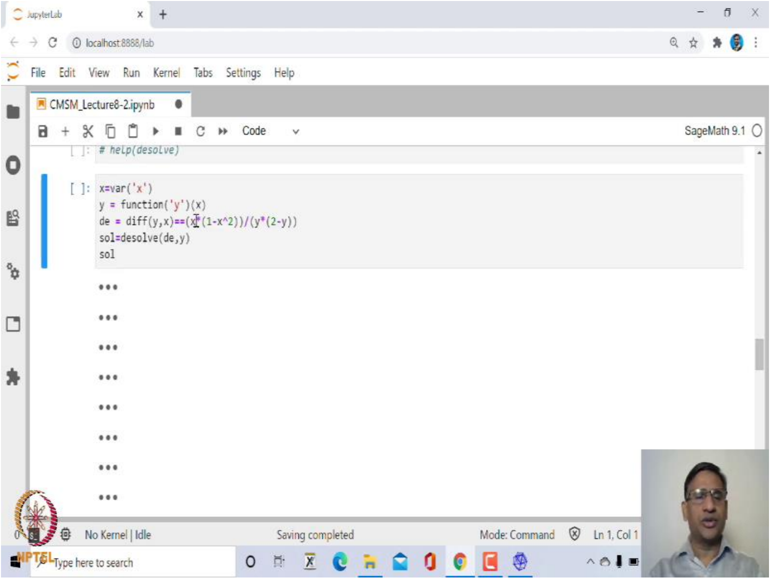

You can of course, take help on desolve and go through this help document in order to explore more about this function.

Let us make use of this. First we need to declare x as a variable and then y as a function of x. So, function inside this say double quote or single quote y and then it is function of x and then define the differential equation. The differential equation is diff of y with respect to x, the derivative of y with respect to x, that is, dy by dx is equal to the value of the function f(x, y).

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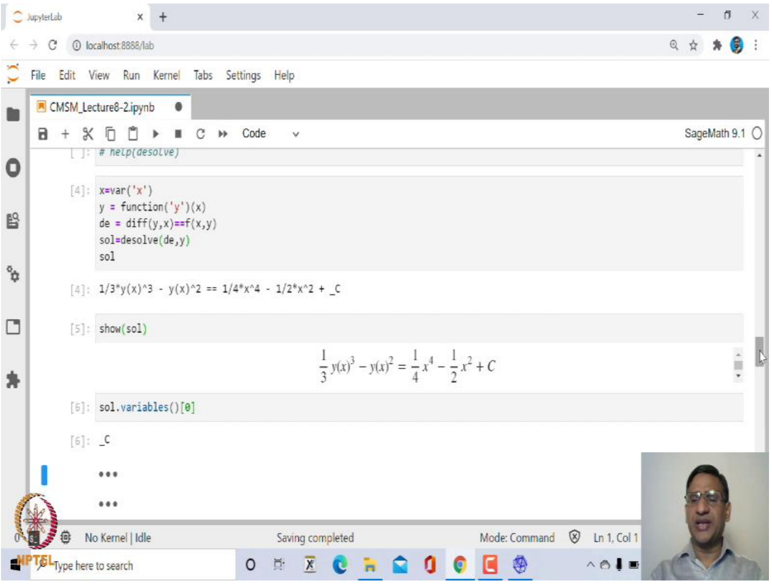

I could have simply said here f(x, y), since we have already defined f(x,y). And, then simply say desolve(de), differential equation with respect to variable y, other things are default ones. Let us run this and see what is the solution? It takes a bit of time. So, this is the solution. It is the solution which is implicitly defined in x and y, and here this underscore c is actually the constant. Since it 1st order differential equation it will have one constant and you can even ask it to show this solution. So, this looks in pretty good form, and this variable, underscore c, you can obtain by this solution dot variables. It can have more than one variables. For example, if I look at a 2nd order differential equation it may have two variables and then you can obtain this. So, it says that this variable is underscore c.

You can replace this by some other constant and try to plot graph of this this function or you can plot set of solution curves.

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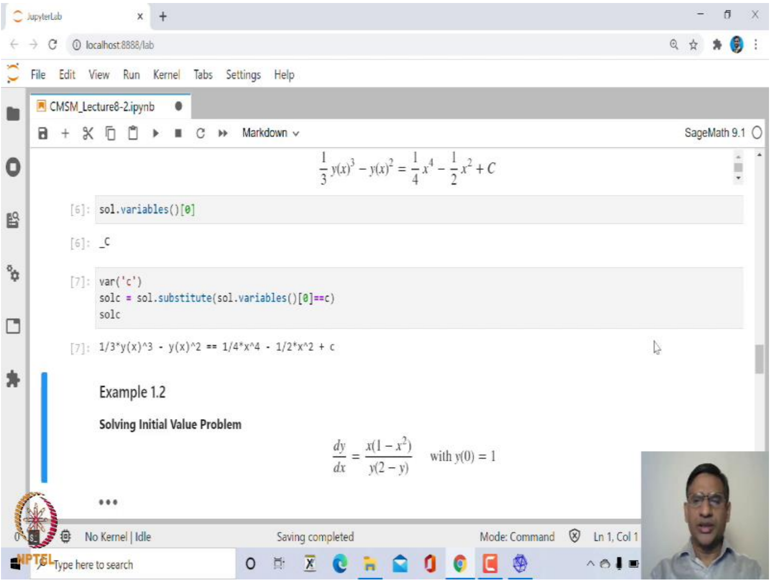

If you want to, let us say, substitute this instead of capital C if you want to make it small c simply say solution.substitute and then variable variables 0 substitute it as c, that can be also done. Now, you can see that this is replaced by small c okay.

Now, let us see how we can solve this initial value problem. So, same problem

dy by dx is equal to x into 1 minus x square upon y into 2 minus y and this initial guess here, y(0) is equal to 0. This is what is known as initial value problem.

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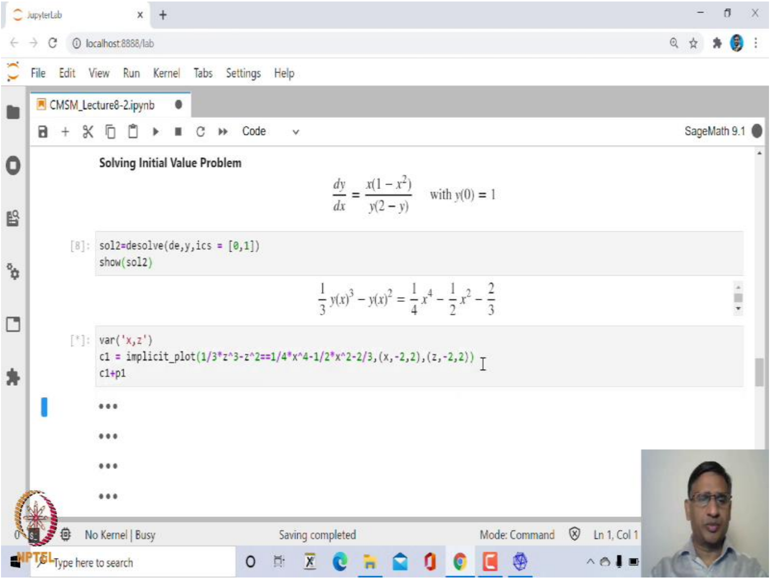

So, if you have to solve this, all you need to do is, again you call desolve and you need to mention the initial conditions. You can even say ics is equal to initial condition is equal to 0 comma 1. So, this is x0, y0, that is, how the initial condition is given. So, let us ask it to solve. This is what you get, this is the solution.

Now, let us try to plot graph of this solution curve and then see how it looks like. If you want to plot graph of this function, this is the output, in terms of y, x. We will just replace y(x) by a variable z and x will be as it is. We have to use implicit_plot. So, I have just replaced, wherever there is y(x) by z and then plot this between x between minus 2 and 2, z between minus 2 and 2.

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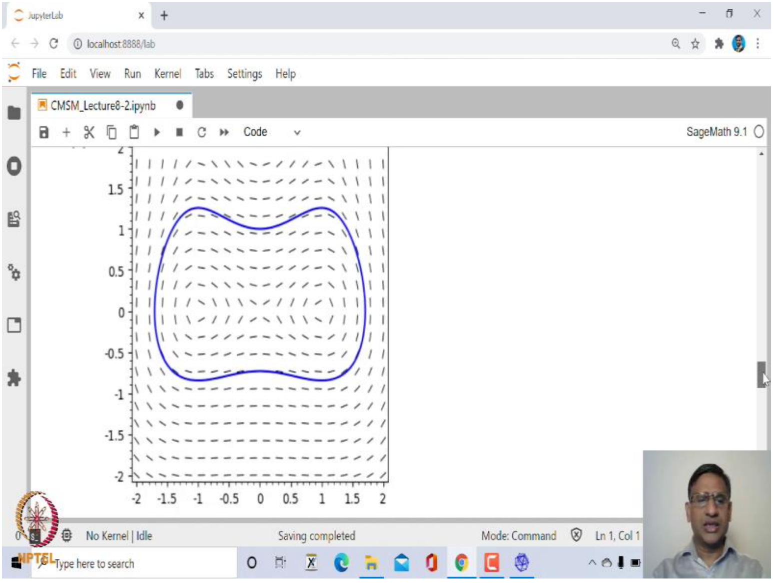

Let us plot along with the slope field. You can see here, this is how the solution curve looks like and this is the slope field.

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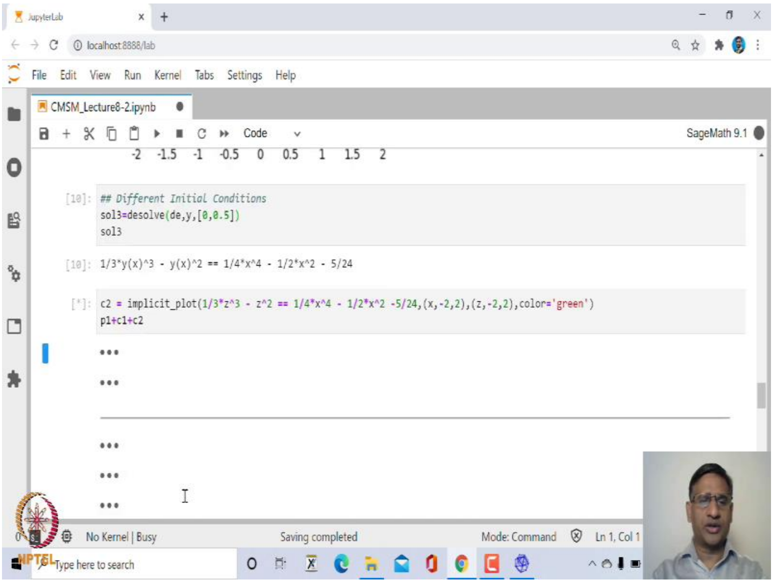

If I give you another initial condition, let us say here, first one was y(0) is equal to 1. Now, I am saying y(0) is equal to 0.5 and then if you try to solve, this is how the solution looks like.

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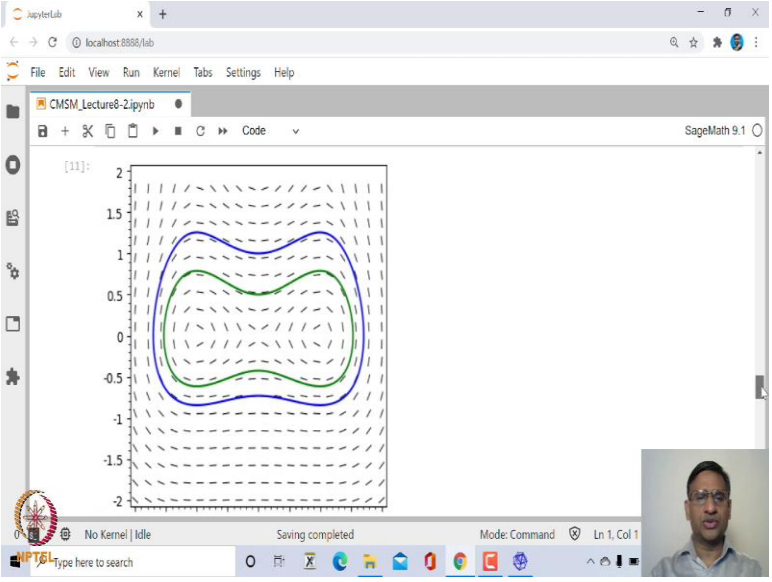

And, again you can plot graph of the first solution, the first initial value problem, that is, (y) is equal to 1 and this is y (0) is equal to 0.5. So, several curves you can plot together, and you can see that this these plots are exactly same as what you got using streamline plot.

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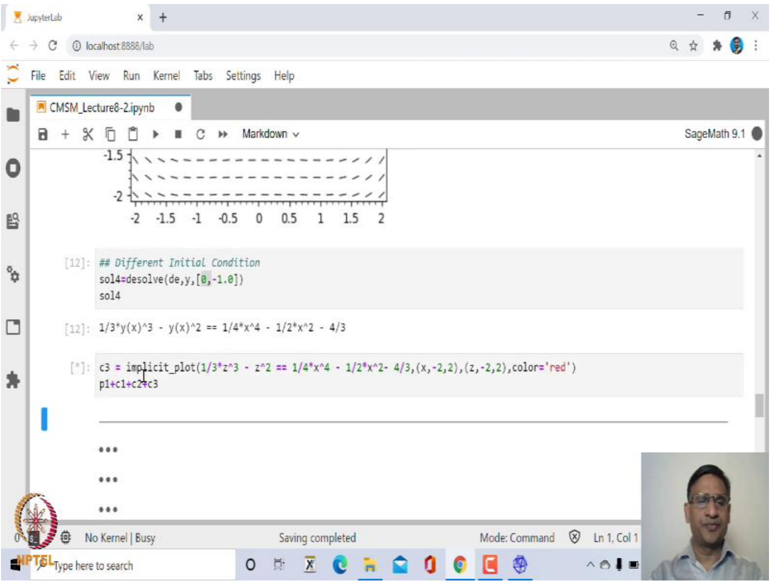

Let us also add one more solution, let us say when y(0) is equal to minus 1, then how the solution looks like? The solution is this form.

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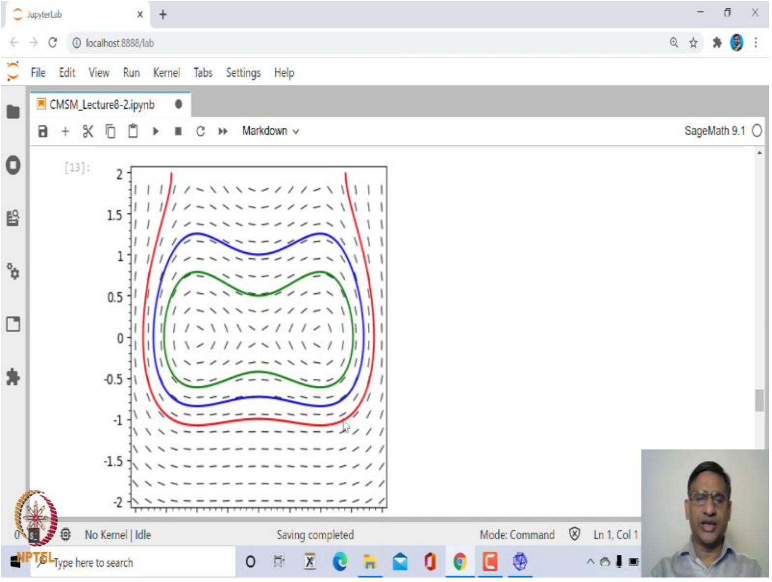

Then if you try to plot graph of all these solution curves together this is how it looks like.

This red colour one is the integral curve of this initial value problem, where y(0) is minus 1.

This is how, not only solve the ordinary differential equation, but you can also visualize solution curve using SageMath. This helps you to see, how the solution looks like.

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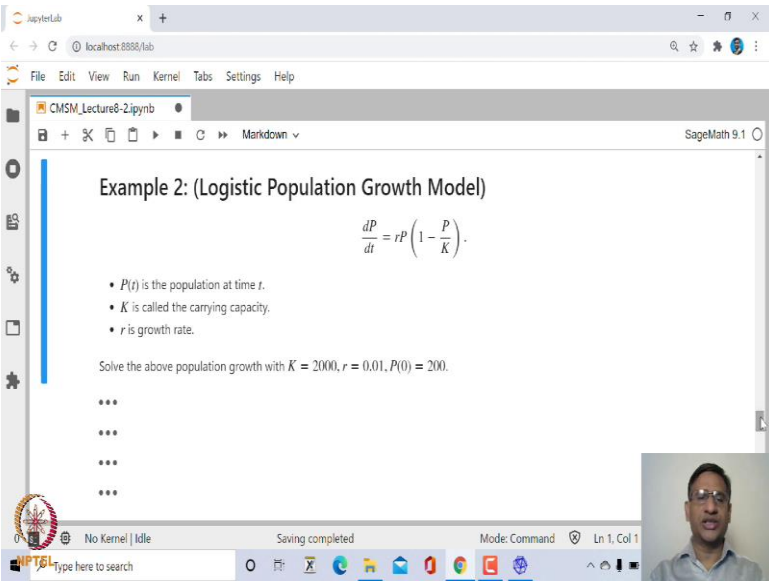

Let us look at one more example of 1st order ordinary differential equation.

This is a population growth model, also known as logistic population growth model. Here P is the population at a time t of certain species or it could be population of human beings, and this r is the growth rate and K is what is known as the carrying capacity. If the population grows more than this carrying capacity there is a chance of catastrophe.

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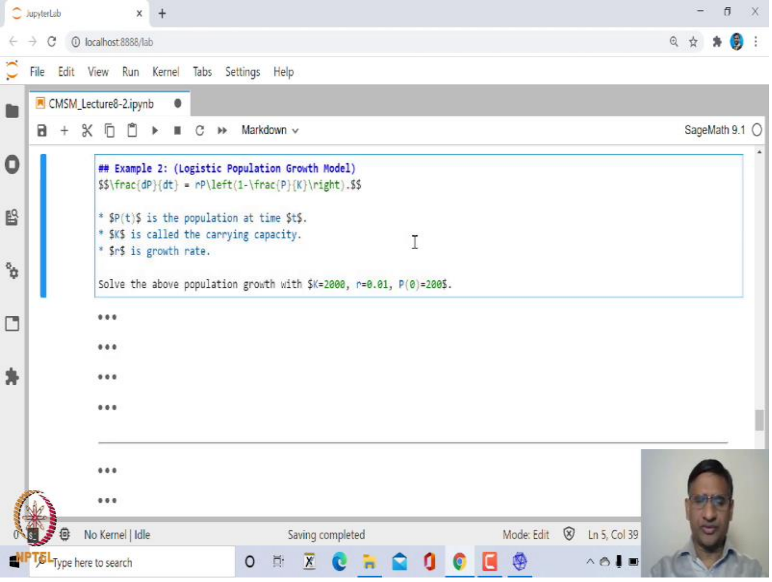

We want to solve this differential equation, where K is let us say 2000, that is the current carrying capacity, r is 0.01 which is 1 percent 1 percent growth and initial population is 200.

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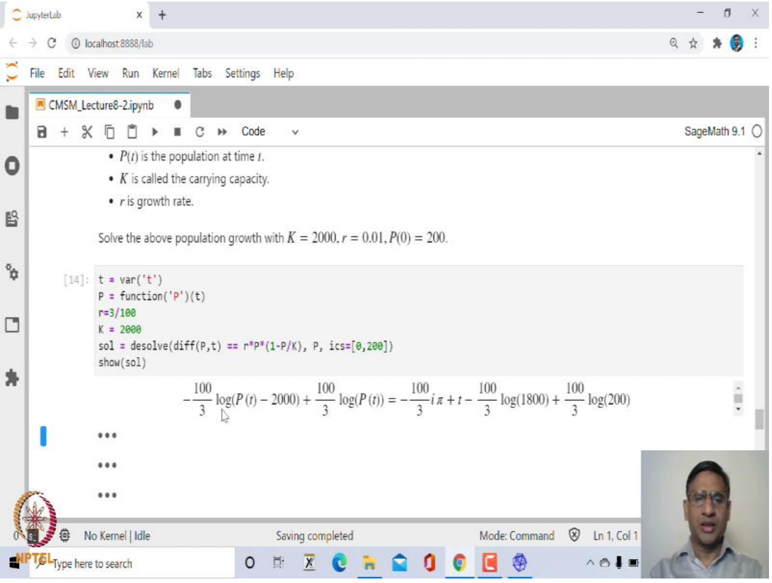

So, how do we do that? Again let us just define this differential equation.

This t is a variable and P is the population function of t, r is 3 by 100, K is 2000 and then solve this solve. Solution is equal to desolve differentiate P with respect to t and r into P into 1 minus P by K and initial guess is 0, 200. This is how the solution looks like. You can see here that throughout there is a constant minus 100 by 10. So, you can replace that or you can multiply throughout by minus 3 by 100. Then what you left in this case, you can solve for log. There is an inbuilt function to log solve in terms of log. It will find the value of P. Let us use that.

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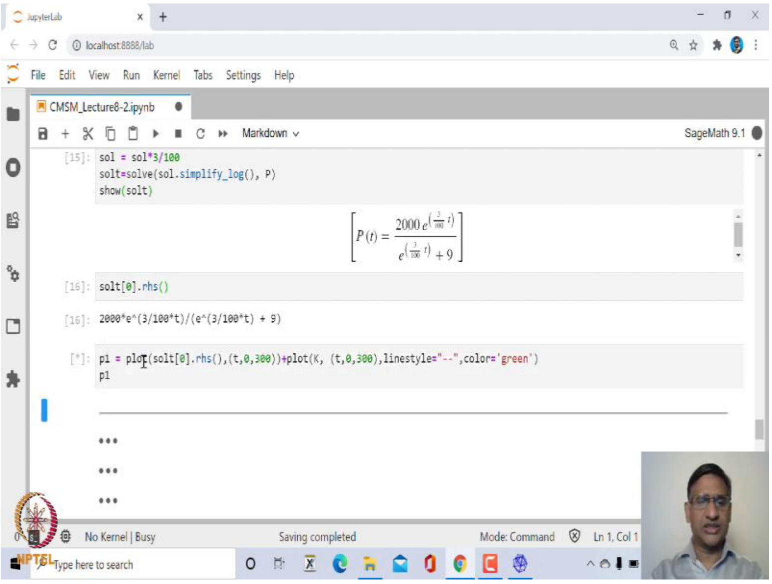

So, first we are multiplying everything by -3 by 100 and then solve this simplified version of this solution which is in in the form of equation for P. This is what you get. This is the solution or this is the population at time t. This is exponential function of 3 by 100 t.

Now, let us look at what is the right hand side of this. This is the right hand side. You can even plot graph of this population.

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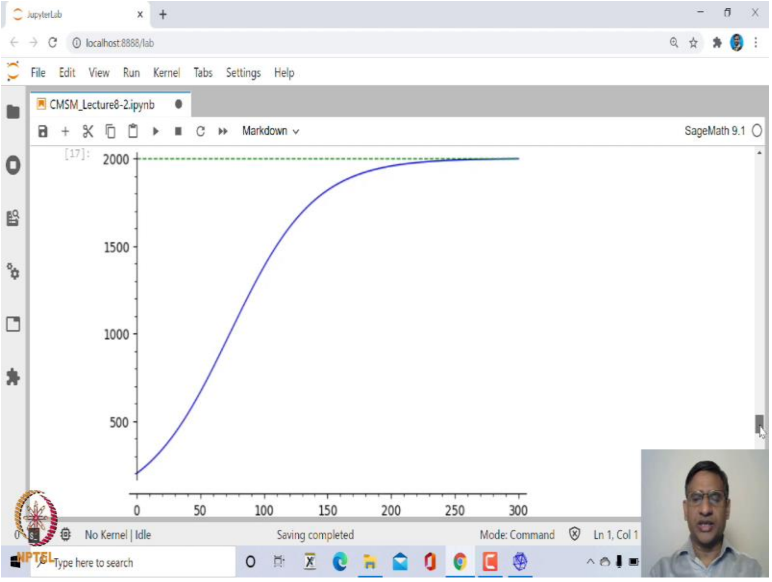
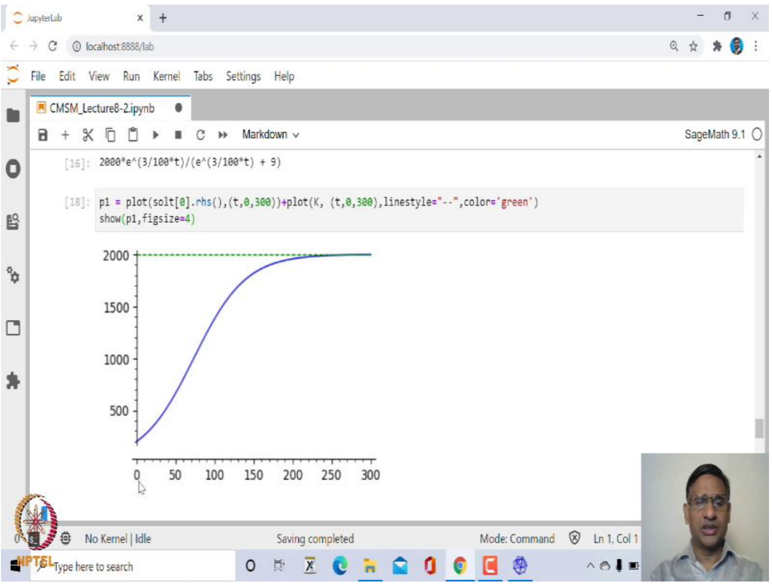


When you plot graph of this population model, this is how it looks like. It is a kind of S shape curve and this is the carrying capacity.

You can see here that over that period of time this population will be very close to carrying capacity. That is a kind of equilibrium in this case, you have. You can try to change the carrying capacity, rate of growth rate etcetera, and then see how this population changes.

So, this kind of thing will help you to visualize this kind of model and look at various parameters and change these various parameters and see how the population changes. (Refer Slide Time: 15:03)



Let me figsize is equal to 4. That is how the solution curves looks like. The period is 0 to 300, if instead of 300 make it 500, then you will see that it will be also be very close to this carrying capacity. Now, what happens in case you have initial population more than carrying capacity?

You can see what happens? Initial population, instead of 200, suppose you have 5000, then then what happens, you can try to do that.

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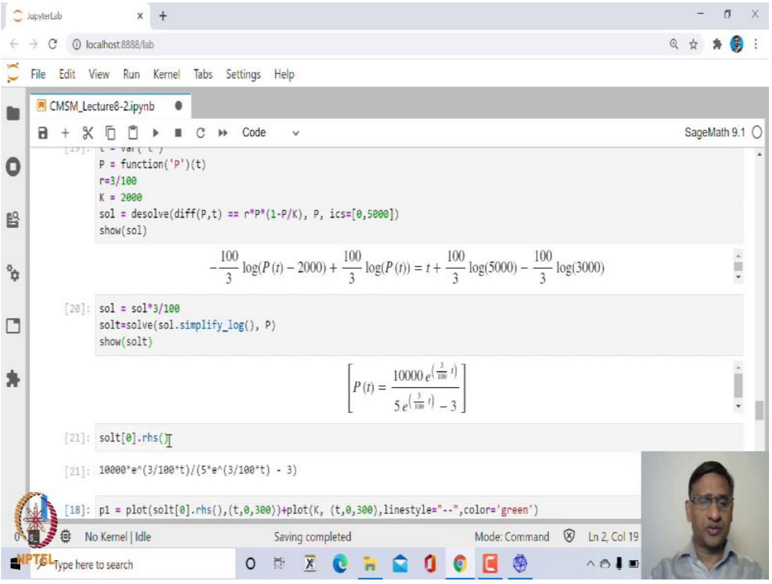

So, suppose if I say 5000, this is what you get. Let us solve this and let us plot the graph of this.

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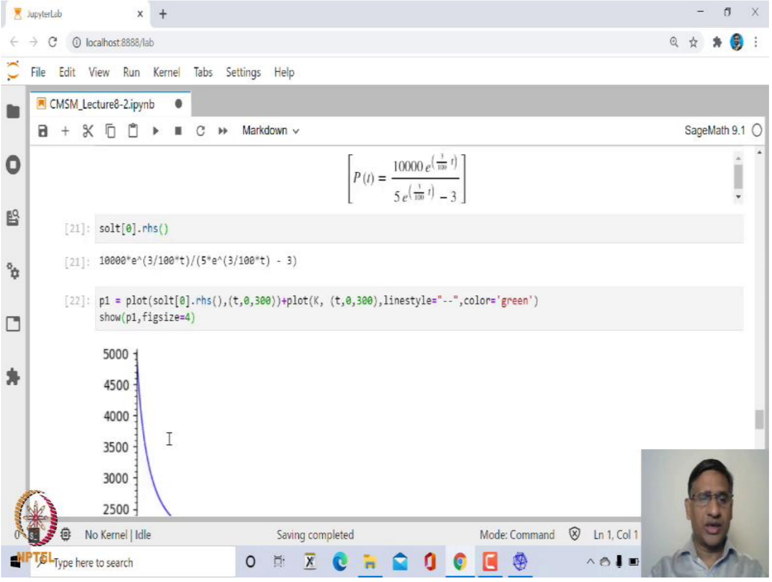

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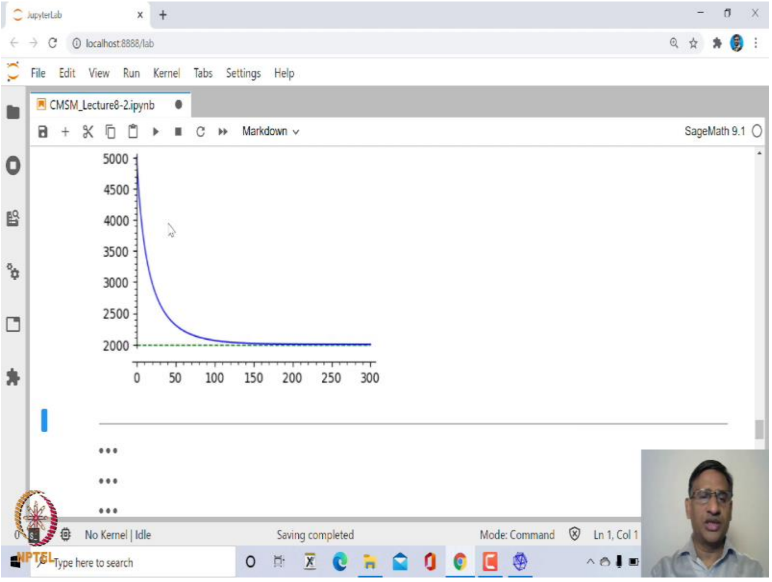

So, you can see here now, it is going down and then down and then after a certain period it becomes very close to the carrying capacity. So, these are the things you can explore using SageMath.

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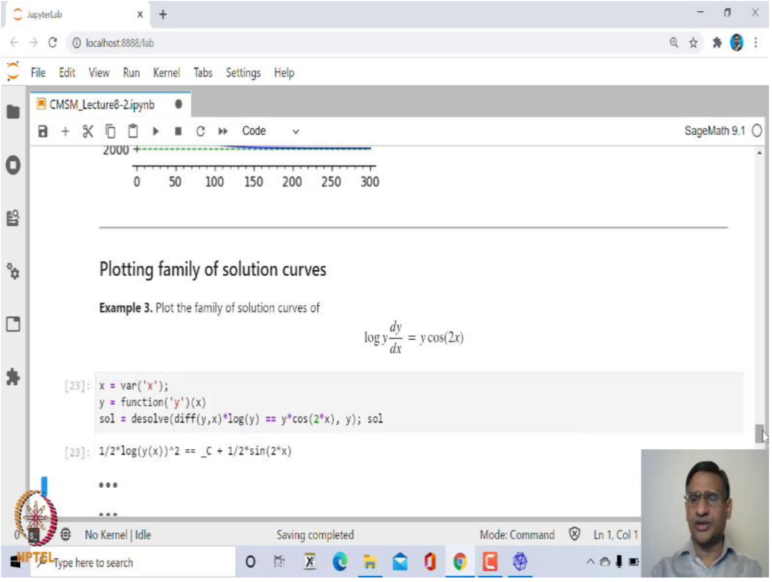

Let us look at. We already saw that plotting this streamline plot gives a family of solution curves. This is one way of plotting, but if you want you can you can plot that manually. Let me just quickly tell you how to how to do this.

Suppose you have a differential equation

log(y) into dy by dx is equal to y into cos 2x

and try to solve this differential equation. The solution is in terms of underscore c.

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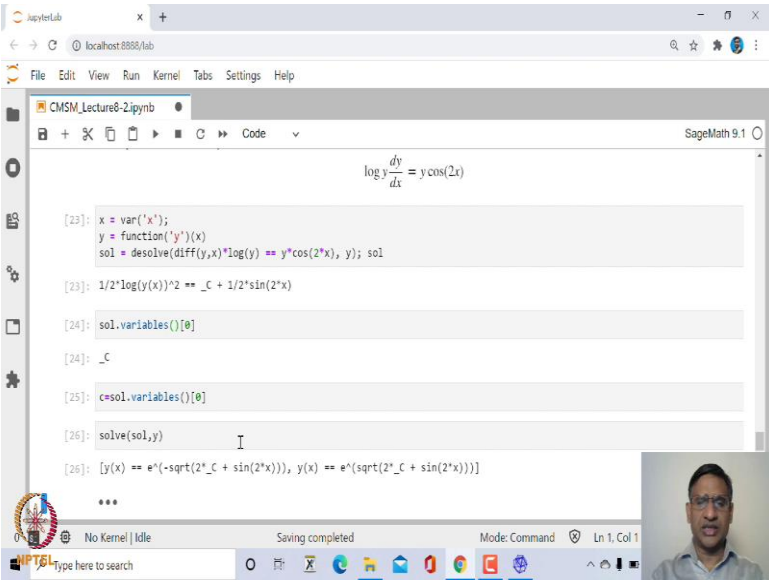

That constant we can see as variables 0 and then we can replace that by a small c and then you see the solution. This is the same thing.

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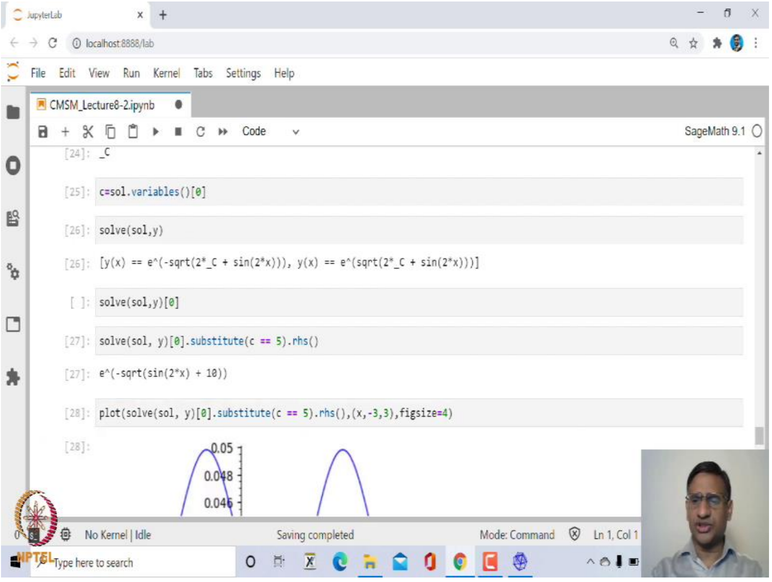

Next let us substitute this capital C by small c =5, then you see how the solution looks like. Now, let us plot graph of this function.

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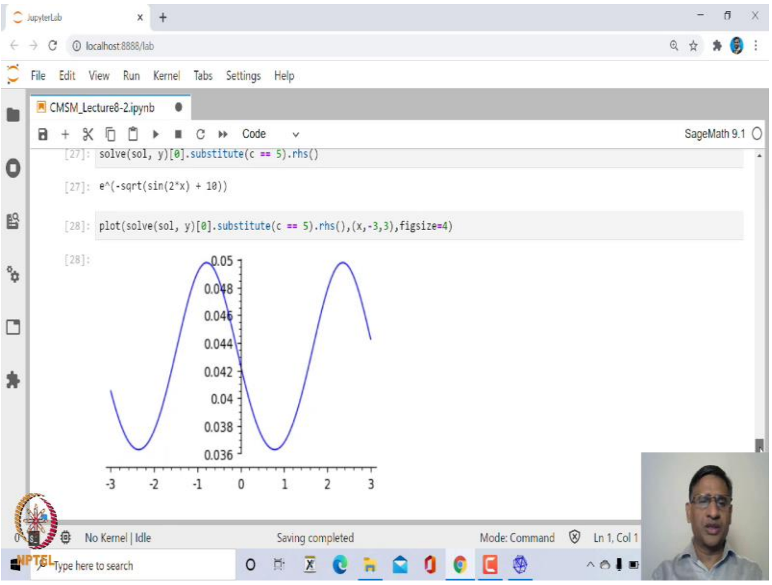

So this is a single solution curve with c value, the constant of integration equals to 5.

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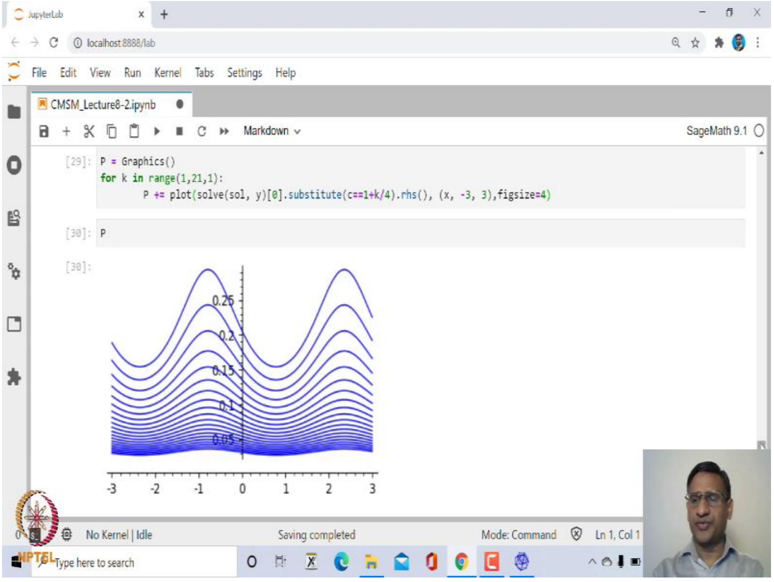

But, instead of just one c equal to 5, suppose you want to do it for multiple c values. Let us say you take c value between 1 and 20 with a step length equal to 1. So, this is what we are doing we start with initially empty graphics and in that you just keep on adding the solution curve. So, this is what you will see. Let us plot graph of this function. This is the family of solution curves. You can you do the streamline plot and then you will see that the streamline plots will also look like exactly similar to this, ok.

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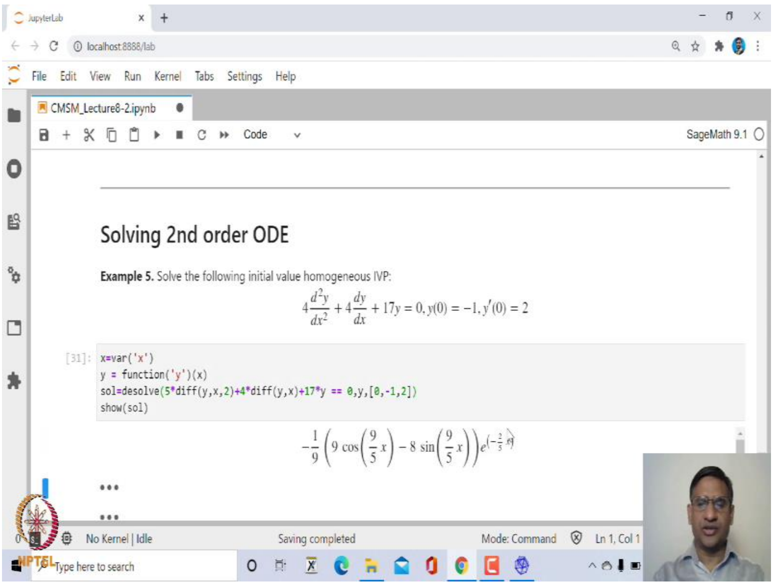

Next let us look at how to solve a 2nd order ordinary differential equation. This is a 2nd order ordinary differential equation initial value problem. This is actually a homogeneous 2nd order ordinary differential equation, these are initial conditions. We want to solve this.

We define this differential equation by taking y as a function of x. This is the differential equation, 5 into second derivative of y with respect to x plus 4 into derivative of y with respect to x plus 17 y is equal to 0 and the initial condition is 0, minus 1, 2. Here this is x, this is y0 and this is y'(0). This is the default value and then this is what the solution looks like.

Here it gives you y(x) equal to this. So, this is the solution curve. Of course, you can find derivative of this, second derivative, and try to see whether it satisfies the differential equation, that should be an easy task. I will leave that as an exercise.

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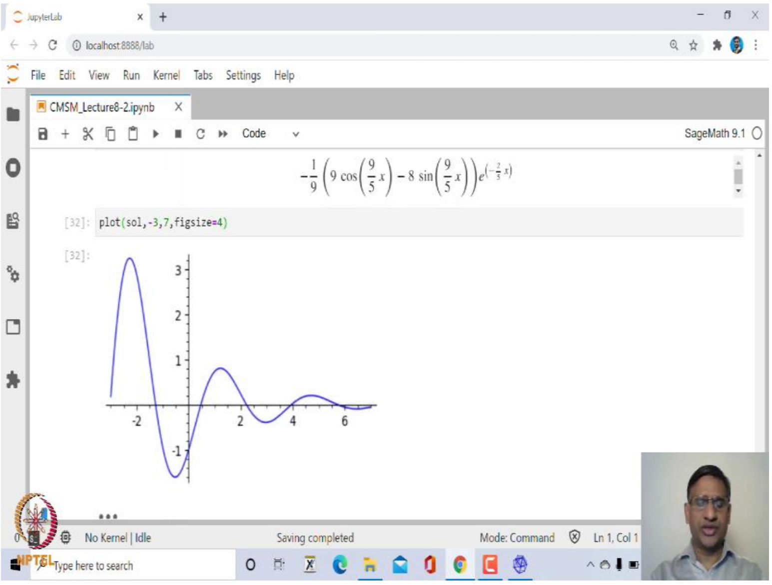

Now, you can plot the graph the integral curve, this is how it looks like. You can see here, as a x increases, this will become very close to x axis. So, this is a kind of some wave and as you see when the time increases the wave will die down.

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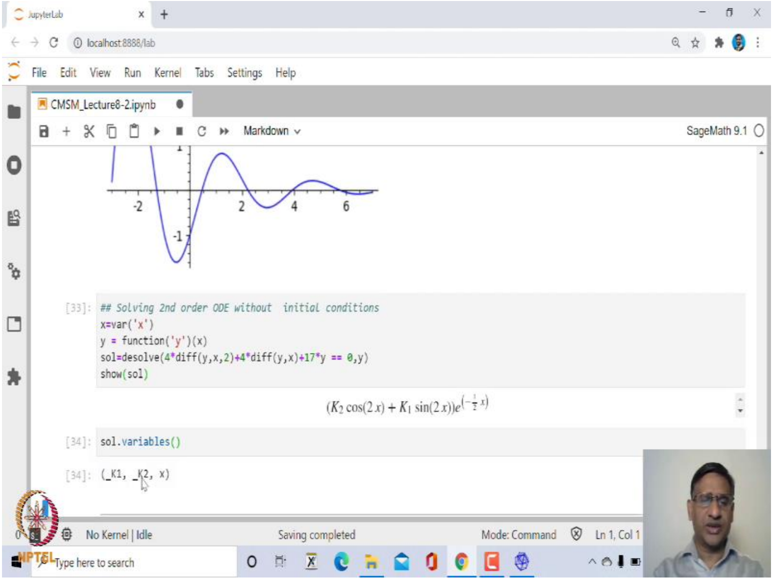

You can also solve the same problem without initial condition. If you do not mention the initial conditions, this will give you solution in terms of two constants here K1 and K2. K1 and K2 are two constants. You can see here, what are the variables involved. It says that K1 and K2 and x are the variable. These are the two constants.

Again, you can replace these two constants by some fixed value and try to plot graph this function.

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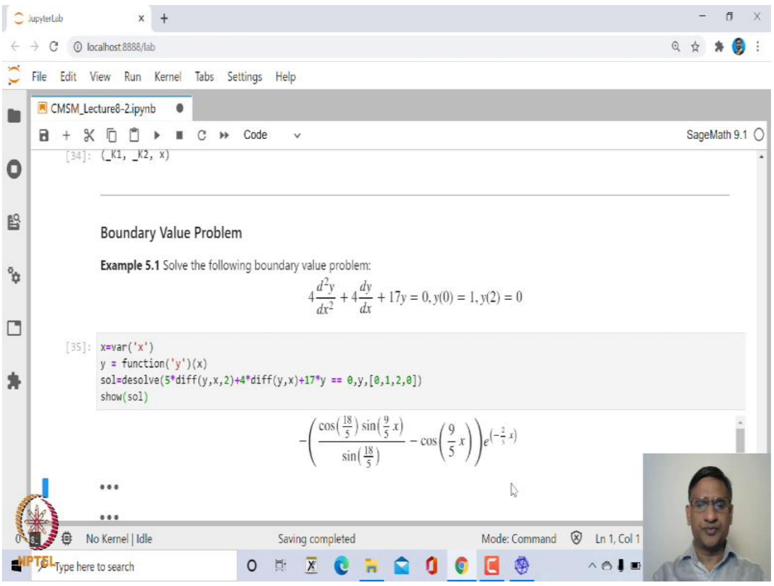

You can also solve boundary value problem. In the same differential equation suppose the boundary conditions are something like, y at 0 is 1 and y at 2 is equal to 0. Let us solve this differential equation.

In stead of initial condition now you have to give as a list of four elements, this is x0, y0, this is x at 2 and y at 2. So, this is x0, y0, and this one is x at a 2 and this is value of y at 2. If you solve this this is how it looks like.

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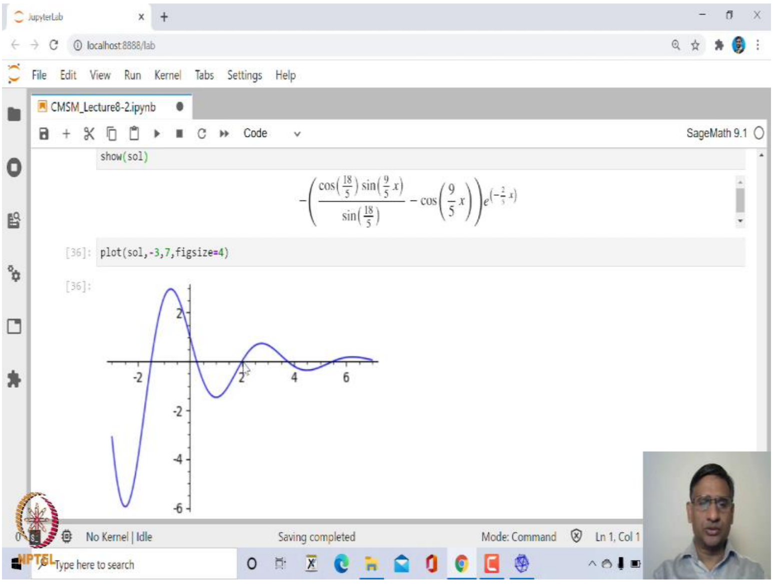

If you try to plot its solution curve, just to make sure that it satisfy the boundary conditions. You can see here, at x equal to 0, y value is 1 and x equal to 2, y value is 0. So, this is the solution curve.

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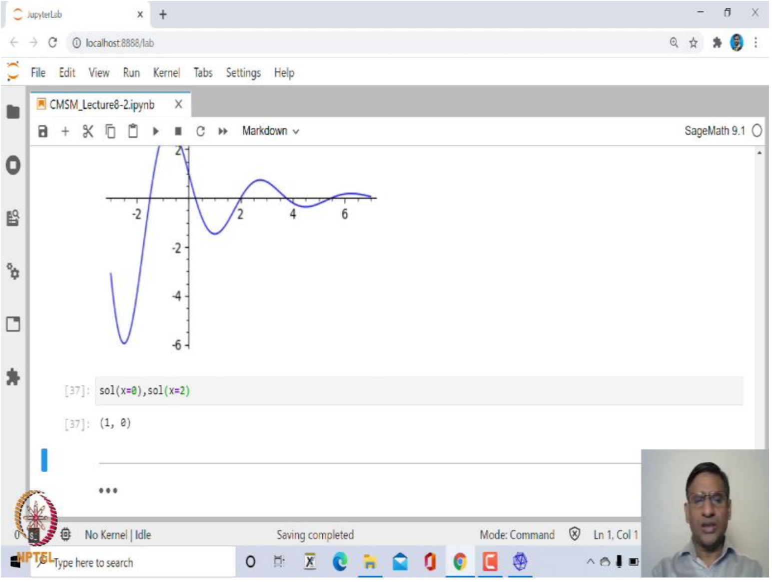

You can also check whether the solution at x equal to 0 is 1 and solution at x equal to 2 is 0. So, that is what we get.

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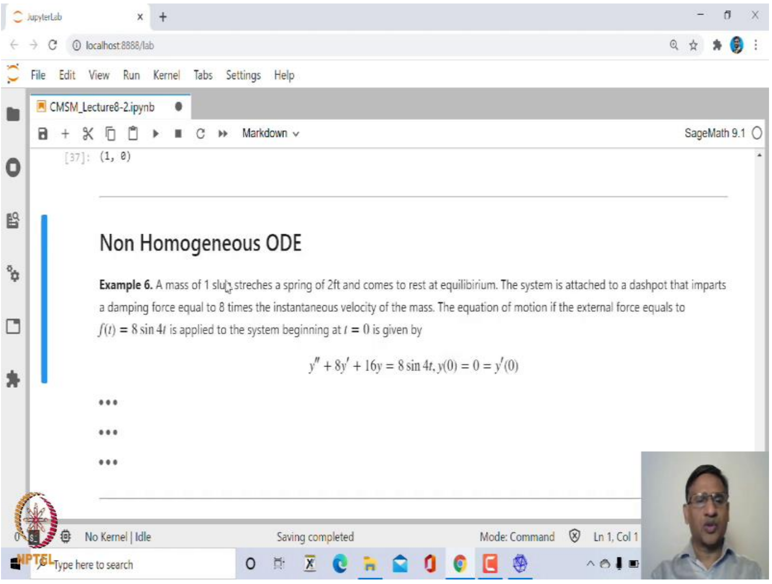

Let me look at one more problem which is non-homogeneous ordinary differential equation. This is a 2nd order differential equation, this is non-homogeneous because right hand side is nonzero and this is initial value problem because y at 0 is 0 and y dash at 0 is also 0.

Every differential equation will have some physical meaning and this differential equation also has a physical meaning. A mass of 1 slug stretches a spring of 2 feet and comes to rest at equilibrium. The system is attached to a dashboard and that imparts a damping force equal to 8 times the instantaneous velocity of the mass that is where you see this 8 times y dash. The equation of the motion in this is the external if the external force equal to 8 times 4 sin t is applied. That is the force on the right hand side is applied to the system starting at t equal to 0. Then this particular problem is governed by this particular differential equation.

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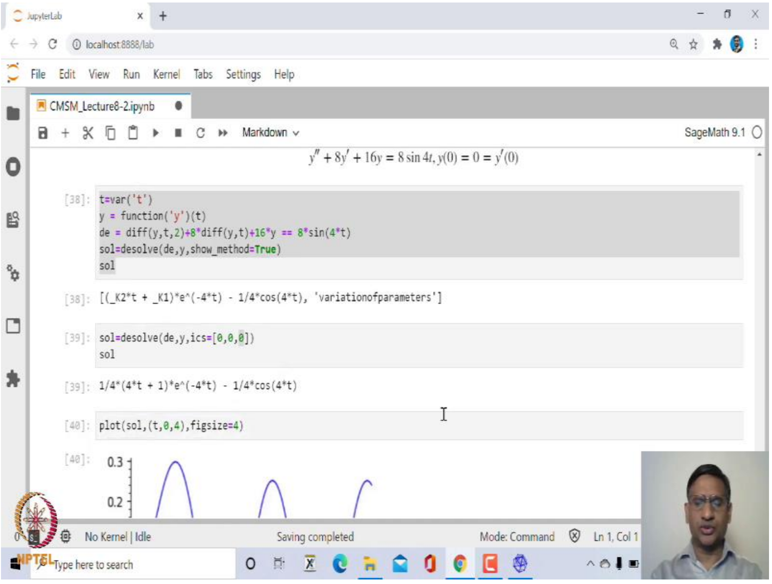

If you try to solve this this, so, again the solution is exactly similar. You use desolve, but the right hand side here is mentioned as equal to 8 into sin(4t) and then you can also give option called solution underscore method is equal to true. So, it says that it uses a method called variation of parameters.

You can look at what are the other inbuilt methods. So, depending upon the differential equation, the nature of the differential equation, Sage will find appropriate method and it will solve using that method. Of course, you could create your own user defined function in order to solve such differential equations; that should not be a very difficult.

And, this is a general solution of this ordinary differential equation of course. You can find a particular solution of this and all these things using variation of parameters.

Let us solve this initial value problem. In this case the initial value is 0, 0, 0, that is, x at 0, that is, y at 0 is 0 and y dash at 0 is also 0. This is how the solution looks like and you can try to plot graph of this function.

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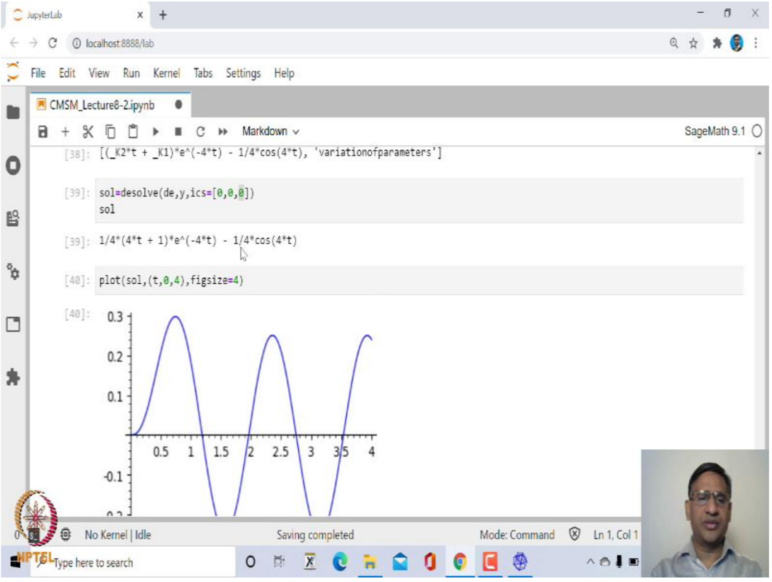

You can see here that is again a kind of wave.

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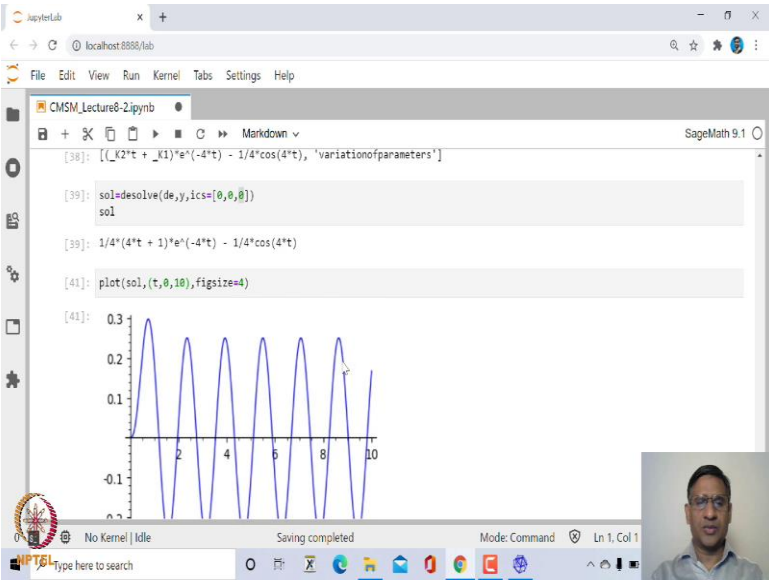

Instead of 4 let us make it 10, this is how it looks like this is a wave function. So, that is how you can solve ordinary differential equation of order 1 and 2 using inbuilt Sage function desolve. You can not only solve this analytically but you can also plot the solution curves.

So, that is the advantage of using this kind of a software. You can use it for visualizing the nature of the solution.

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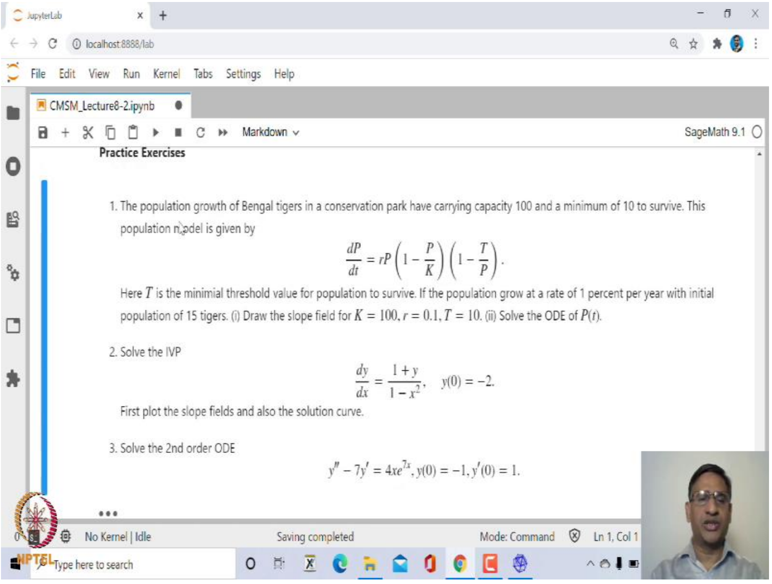

Let me leave you with few simple exercises. These are the 3 exercises.

One is again a population growth model and it is actually about Bengal tigers in the conservation park having carrying capacity 100 and minimum of 10 to survive. This is the minimum survival value of the surviving population. This is governed by this population growth model. So, here T is the minimal threshold value of the population to survive.

If the population grows at one percent per year with initial population of 15 tigers, then you can draw the slope field with the following values; K is equal to 100.

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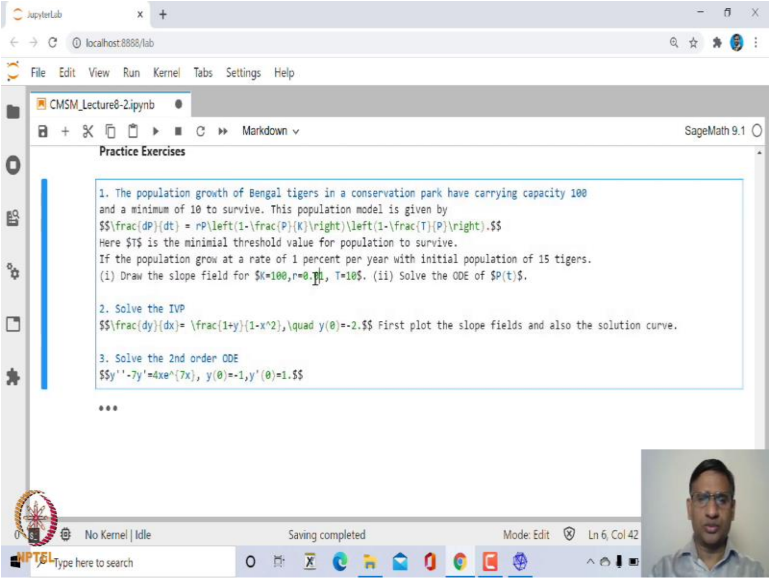

r should be actually be 0.01 not 1. T is equal to 10. Solve this for a population at time t.

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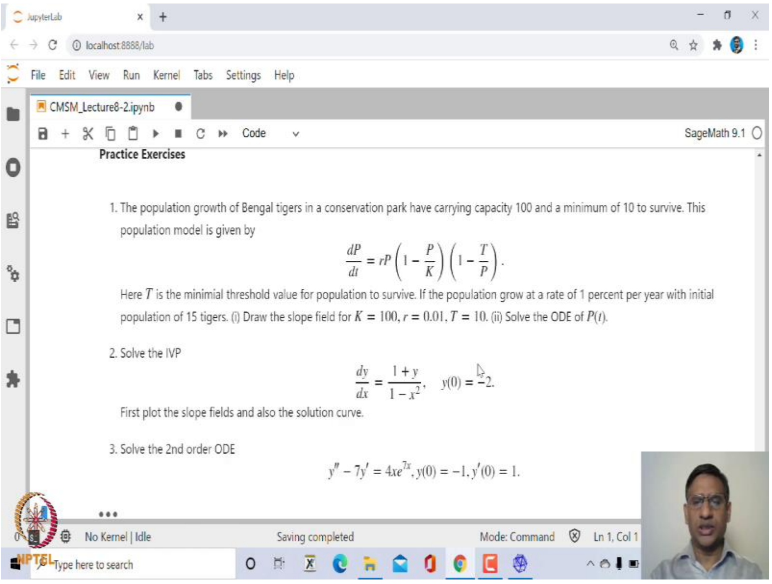

The next problem is solve this initial value problem which is the simple one. Also plot the slope fields along with the solution curves.

And, then solve a 2nd order differential equation, which is initial value problem, you can also include boundary value problem in this case.

So, these are the straightforward problems, I have left as practice exercise.

Thank you very much and I will see you in the next lecture, in which we will look at how to solve ordinary differential equation numerically. We will try to explore the some of the methods like Euler method and other things. We can create our own function as well as we will use in-built Sage functions.

Thank you very much.