Computational Mathematics with SageMath
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Lecture - 43
Solving System of linear ODE using Eigenvalues and Eigenvectors

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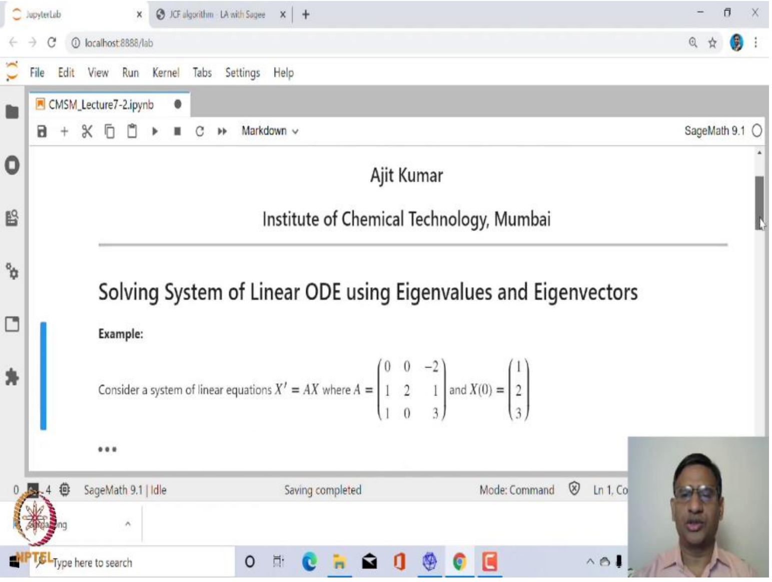

Welcome to the 43rd lecture on Computational Mathematics with SageMath. In this lecture, we will look at Solving System of linear Ordinary Differential Equations using Eigenvalues and Eigenvectors. So, let us start with an example. So suppose you have a system of ordinary differential equations as X dash t is equal to AX. So, X here, X is X 1, X 2, X 3. So, X dash will be X 1 dash t, X 2 dash t, X 3 dash t, and AX is this, where A is this matrix X. So, what will you get?

X 1 dash is going to be minus 2 times X 3, and X 2 dash is going to be X 1 dash t plus 2 times X 2 dash t plus 3, plus X 3 dash t, and X 3 dash t will be X 1 dash t plus X 3 dash t and this is ordinary, this is initial value problem. So, X 0 which is 1, 2 and 3 column; that means, X 1 at 0 is 1, X 2 at 0 is 2, and X 3 at 0 is 3. That is the system of linear ordinary differential equation we are looking at, and we want to solve this.

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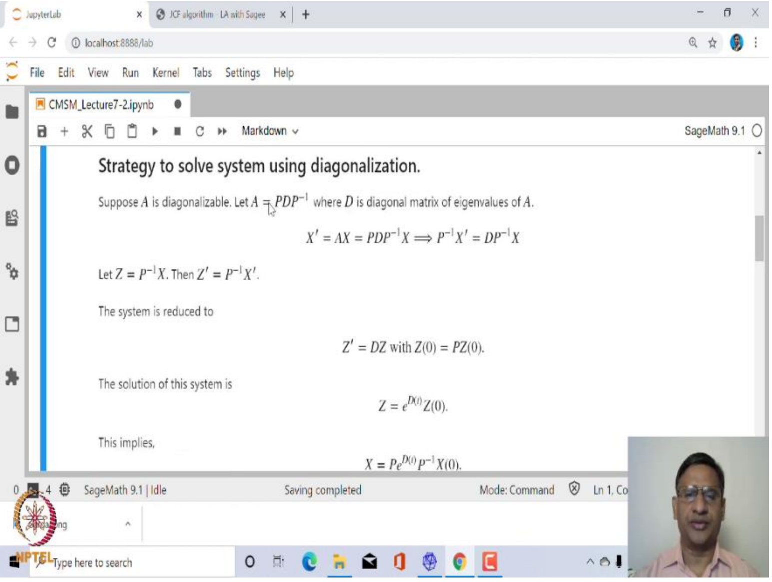

Now, how do we solve this system? So, so, first let us assume that this matrix is diagonalizable. Suppose, this matrix is diagonalizable, then we can write A as P into D into P inverse, where D is the diagonal matrix of eigenvalues, P is the matrix which diagonalizes this E, right? So, in this case, now what is the X dash? X dash has become X dash, which is AX, which has become P into D into P inverse X.

And this implies that if you multiply both sides by P inverse, then what you will get? P inverse X dash is equal to D into P inverse X, right? So, in case we substitute this P inverse X as Z. So, if I take Z is equal to P inverse X then Z dash will be nothing but P inverse X dash. So, this differential equation now has become, Z dash is equal to DZ, where Z at 0 is P times X at 0.

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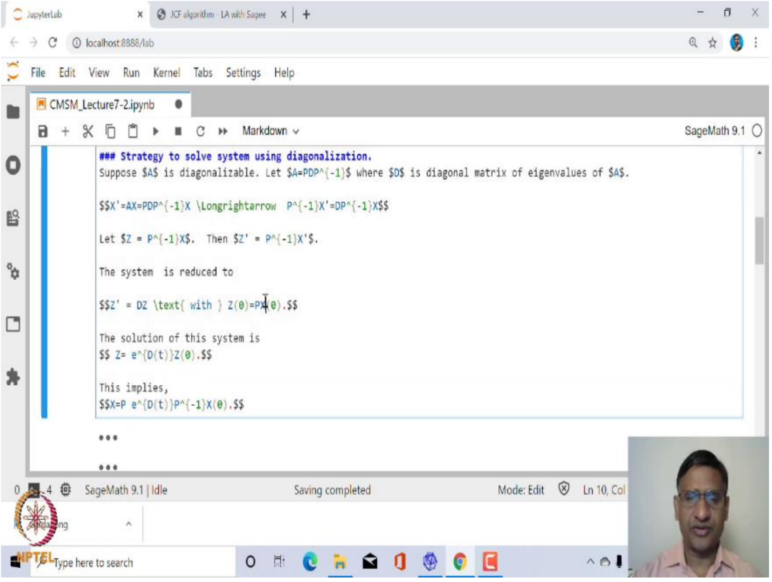

This would be X at 0. So, let me change that, this is X at 0, right?

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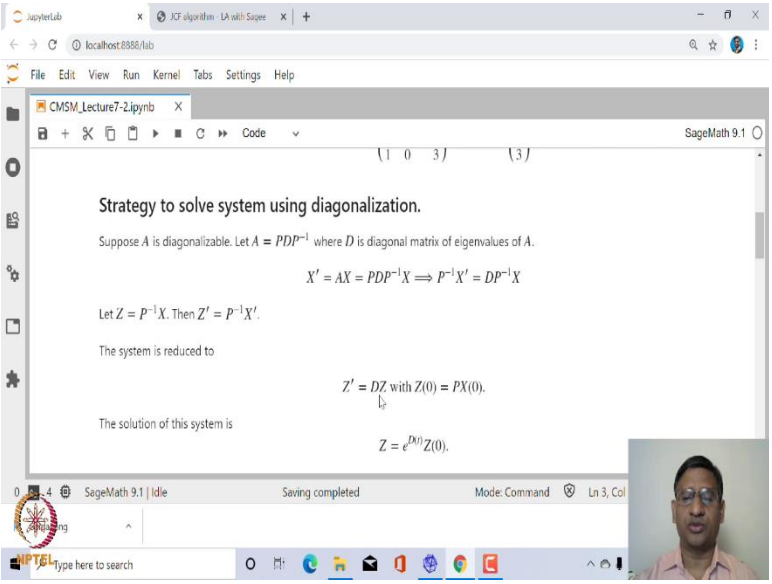

So, that is the, the differential equation. Now, Z dash is equal to DZ. And where Z at 0 is P into X at 0. Now what is D? D is diagonal matrix. So, what you are looking at? You are looking at differential equation Z 1 dash t is equal to some lambda 1 times Z 1, and Z 2 dash t is equal to some lambda 2 times Z 2, Z 3 dash t is equal to some lambda 3 times Z 3.

So, those are very easy to solve. This differential equation D, dX by dt is equal to some constant times X is easy to solve. I am sure, you must have already solved this. The solution is in terms of exponential.

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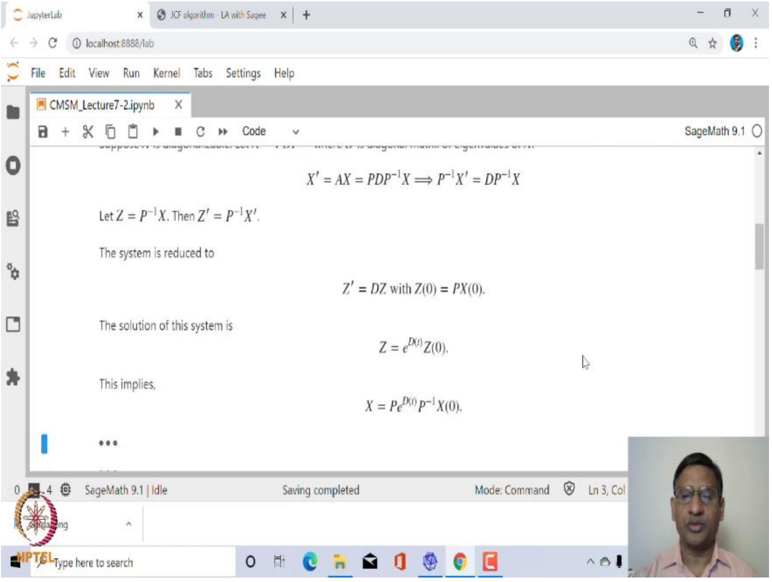

So, what will be the solution of this system? This will be, Z is equal to e to the power D t, D times, this is D into t here, into Z at 0. So, e to the power t is going to be the exponential of the, let us say lambda 1t differential diagonal matrix, exponential of lambda 1 t, exponential of lambda 2 t, exponential of lambda 3 t, diagonal entries and multiplied by Z 0, so that is the solution.

So, in case the matrix is diagonalizable, the solution can be obtained by taking the exponential of diagonal matrix and the diagonal matrix, the diagonal entries are nothing but eigenvalues. So, this, this, and now if you substitute in this, X is going to be P inverse, X we have substituted as P inverse, Z we have substituted as P inverse X therefore, X is going to be P times Z.

So, you just write P into this, that is the solution. So, that is how we can solve this system of ordinary differential equations using eigenvalues, eigenvectors. In particular, when the matrix is diagonalizable.

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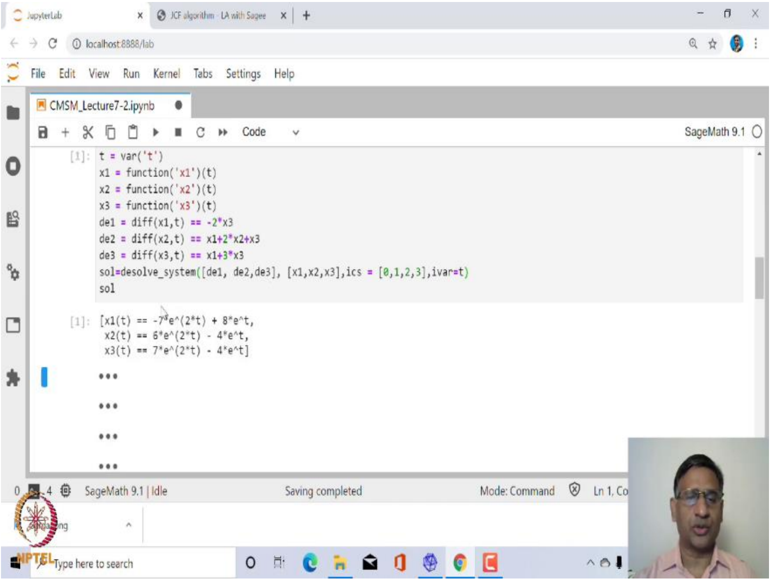

Now, let us see how we can use this in SageMath. However, Sage has an inbuilt function to solve a system of ordinary differential equation. This is known as de underscore solve, desolve underscore system. And what you need to do is first declare the variable with respect to which you want to solve, and x1 is a function of x1 t, x2 is function of x2 t, x3 is a function of x3 t. Then define the first differential equation, which where derivative of x1 is, is equal to minus 2 times x3, that is what you have this matrix, right?

And then x2 dash, the 2nd differential equation in this system is x2 dash t is equal to x1 plus 2 x2 plus x3, and 3rd equation in this system is x3 dash t is equal to x 1, x1 t plus 3 times x3. Now you just call desolve underscore system. Then give the list of differential equations and mention the variables with respect to which you want to solve. And mention the initial conditions. So, here initial condition is 0, 1, 2, 3 that simply means that x1 at 0 is 1, x2 at 0 is 2, x3 at 0 is 3, and mention the initial variable this is t here. So, if I ask it to solve and then it will give you the solution. It may take a few seconds, but it will give you the solution, right?

So, this is the solution. x1 dash t is minus 7 into e to the power 2 t plus 8 into e to the power t, 2nd x 2 t is 6 e to the power 2 t minus 4 e to the power t and x3 is 7 e to the power 2 t minus 4 into e to the power t, that’s the solution.

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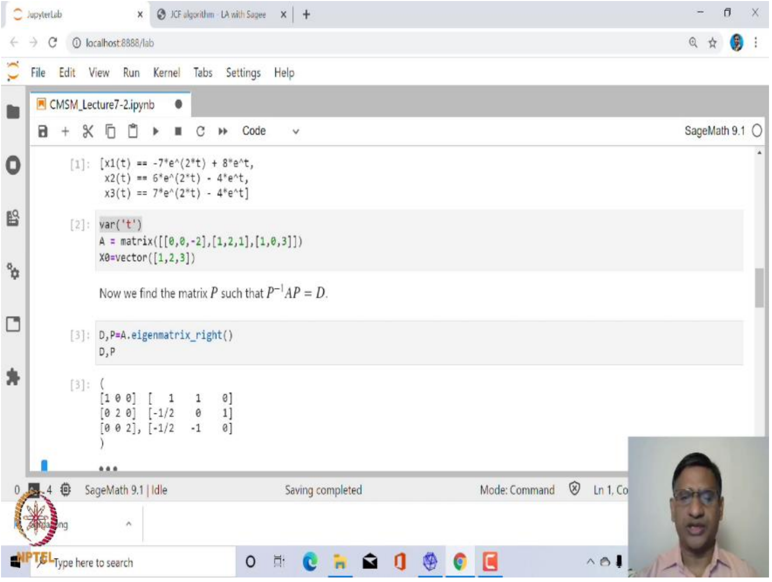

Now, we want to look at how we can solve this using diagonalizability method, right? So first let us define this coefficient matrix A and the initial vectors in X 0. So, these are the matrices and declare t as a variable. And next what we need to do? We need to find the matrix P, such that P inverse AP is D. That is, we need to diagonalize this matrix and that we know how to do that.

So, we can simply say A dot eigen matrix underscore right, this will give me the diagonal matrix of eigenvalues and this matrix P. So, let us, let us run this. So, these are, this is the diagonal matrix, so it has 2 eigenvalues 1 and 2; 2 has multiplicity 2, whereas 1 has multiplicity 1, and the eigenvector with respect to eigenvalue 1 is 1, minus half, minus half, with respect to 2 there are 2 eigenvalues 1, 0, minus 1, and 0, 1, 0, ok? So, these are the, the D and P.

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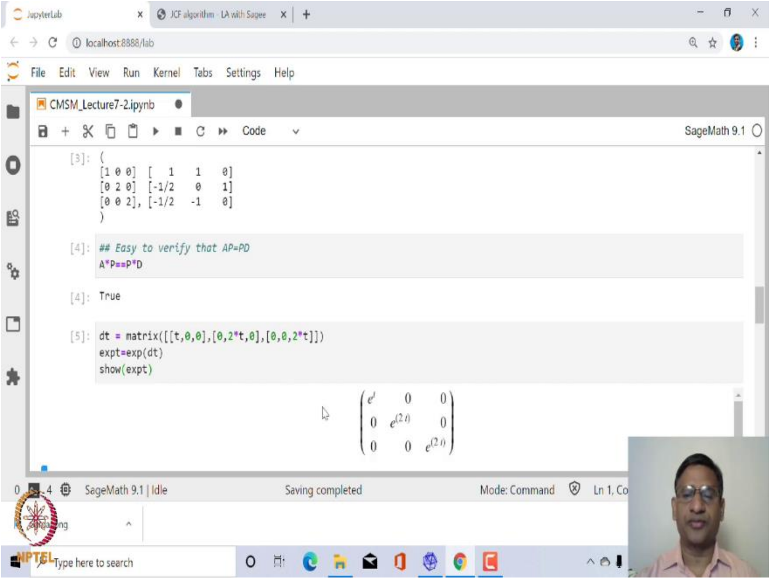

Now, what do we do? Next, we will, we will, you can also check that whether A into P is P into D, that is correct, we have, we have this. Now what we need to do? Let us find the exponential of this diagonal matrix. I am calling this dt, and what is the exponential in this case? First, we find out the matrix d into t, t times d which is the diagonal matrix t, here it is 2 t and here it is 2 t, right? This is lambda 1 times t, this is lambda 2 times t and this is also lambda 3 times t. Lambda 3 and lambda, lambda 2 and lambda 3 are 2, right? And then Sage has inbuilt function to find exponential of a matrix as well. So, if I say exponential of dt, it will give you the exponential. So, let us look at what is exponential of this matrix. So, this exponential is the, exponential of a diagonal matrix is nothing but the diagonal matrix of the exponential of the diagonal entries so, e to the power t, e to the power 2 t, e to the power 3 t, right?

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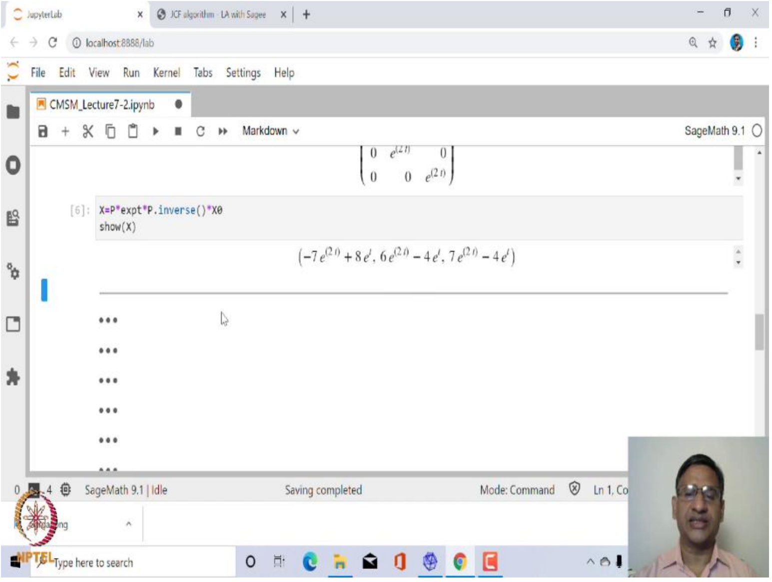

Now, what is the solution? The solution is X is equal to P into exponential of dt into P inverse times X 0, right? And so when you do that, this is what you get. So, the first solution is minus 7 times e to the power 2 t plus 8 into e to the power t and the 2nd component is 6 e to the power 2 t minus 4 e to the power t, 3rd component is 7 into e to the power 2 t minus 4 into e to the power t. And if you compare this with the solution which we got, obtained using inbuilt function, they are the same, right? So, this is how you can solve a system of linear ordinary differential equation using, using this diagonalizability. So, you, once the matrix is diagonalizable, then solving this system is very easy, right? Of course, the price you pay in the diagonalizing, this the matrix, right?

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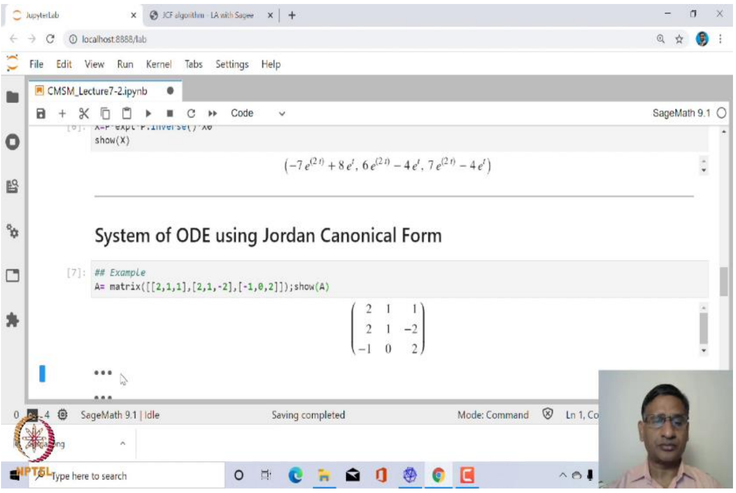

Now, what if that matrix is not diagonalizable? So, how do we solve that? So, in order to solve such matrices, such a system of ordinary differential equations, one has to use a notion of Jordan canonical form.

So, in case the matrix is not diagonalizable, you, what you can, you can ask for what's the best you can do, right? As we looked at, what is the best possible curve that you can fit to given system of, given set of points.

So, similarly here we can ask, in case the matrix is not diagonalizable, what is the best possible way, or what is kind of matrix which is best, or very close to the diagonal matrix?

So, this is what is known as diagonal Jordan canonical form. So, let us look at an example. Sage has inbuilt function to find Jordan canonical form, let us look at 2 examples. From there I will explain, but this is very useful concept in linear algebra.

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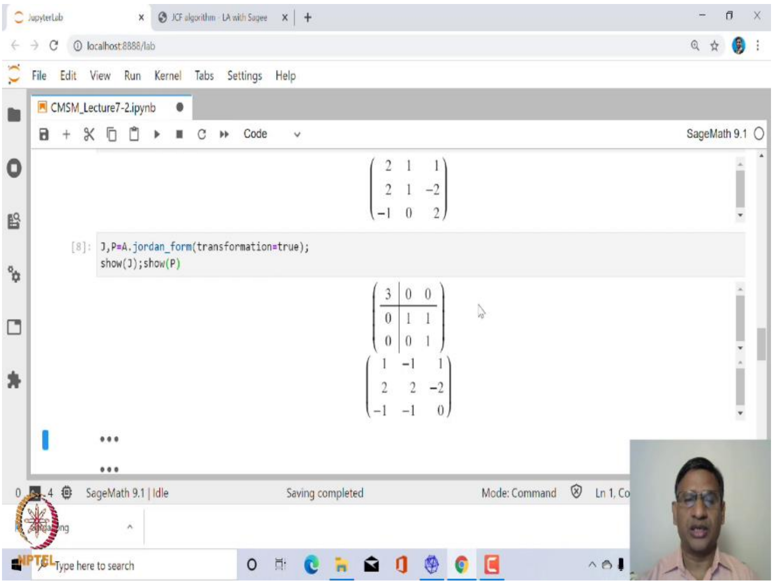

So, suppose you have a matrix A, which is this matrix, and you can find Jordan canonical form of this matrix.So, there is inbuilt method called A dot jordan underscore form, and you need to mention what kind of, here the transformation is true, that option. So, it will give you 2 matrices; one is J, which is Jordan canonical form, Jordan matrix, and P is the matrix which converts A into this Jordan form.

This will be, this will be a matrix of, a kind of Jordan basis, ok? So, let us ask it to show what we have got. So, this J is, it has 2 blocks; each block is called a Jordan block, so here first block is 3. So, this means that A, A has an eigenvalue 3 with multiplicity 1, whereas A has another eigenvalue namely 2 with multiplicity again 1, that is why this matrix is not diagonalizable. It is easy to check that this matrix is not diagonalizable.

So, this is what is called Jordan block. So, Jordan block is basically going to be a matrix of this form, where the diagonal entries will be a particular eigenvalue and just above the diagonal will be all ones, right? So, this is, and this is the, the matrix P, the first column, this is going to be eigenvector with respect to eigenvalue 3.

And these 2 vectors is, what is called actually, generalized eigenvector, with vectors with respect to the eigenvalue 1, ok? So, I am not going to get into too many, I mean much details on finding Jordan canonical forms, ok?

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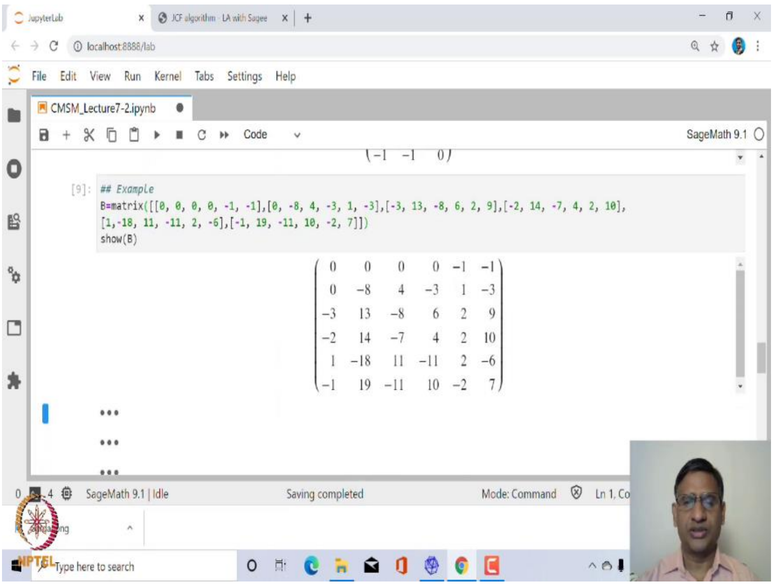

So, now let us see another example, slightly bigger example. So, this is a matrix, this, this is a 6 cross 6 matrix.

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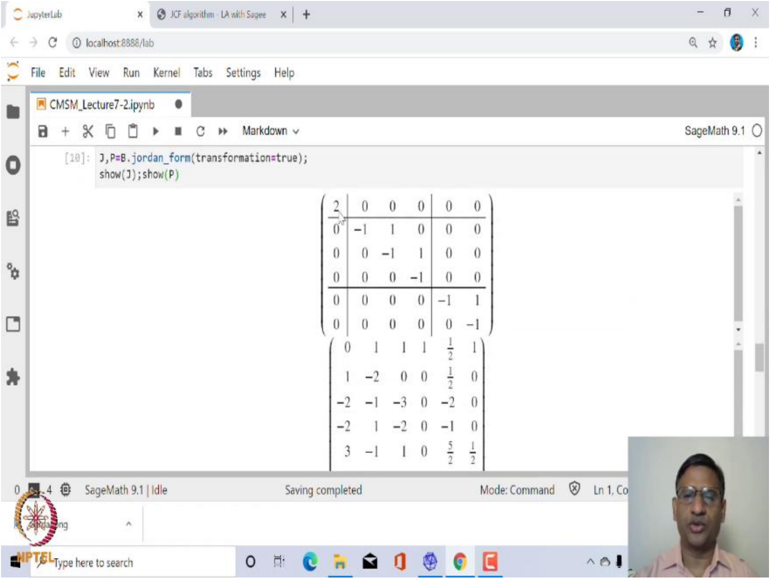

And let us find out Jordan canonical form of this matrix. So, when you find Jordan canonical form of this matrix, this, this is what you get. So, you have 3 Jordan blocks here, 1 block is with respect to eigenvalue 2, other 2 blocks are again with respect to eigenvalue 1, sorry minus 1.

So, this is a 3 by 3 block of eigenvalue minus 1, and this is 2 by 2 block, Jordan block, with respect to eigenvalue minus 1. So, you can have more blocks, or several blocks with respect to a given eigenvalue, ok? And you can, you can check that in this case, the, the characteristic polynomial of this matrix is going to be x minus 2 into x minus, minus 1, that is x plus 1 to the power 5.

So, this is easy to check that this is the characteristic polynomial, which is x minus 2 into x minus x plus 1 to the power 5. So, it has only 2 eigenvalues, namely 0, sorry 2 and minus 1, minus 1 has algebraic multiplicity 5, right? And when you, so this is the matrix P. So, this is a, a matrix consisting of Jordan basis, this is the eigenvector with respect to eigenvalue 1, these are all eigen, not eigenvectors, these are what is called generalized eigenvectors of A with respect to eigenvalue minus 1, right?

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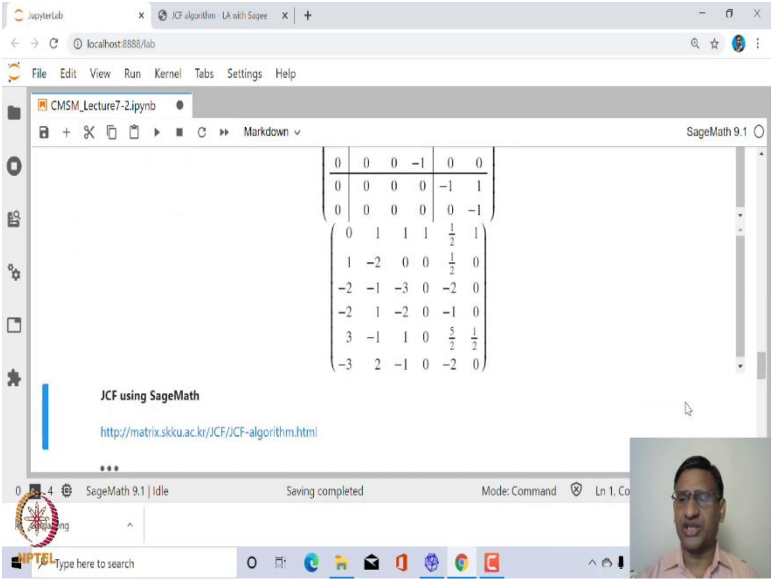

So, this is how you can find Jordan canonical form of any matrix, and if you want to look at step by step procedure or some kind of algorithmic approach of finding Jordan canonical form of a matrix, you can visit this site which is created by myself and Professor Sang-Gu Lee.

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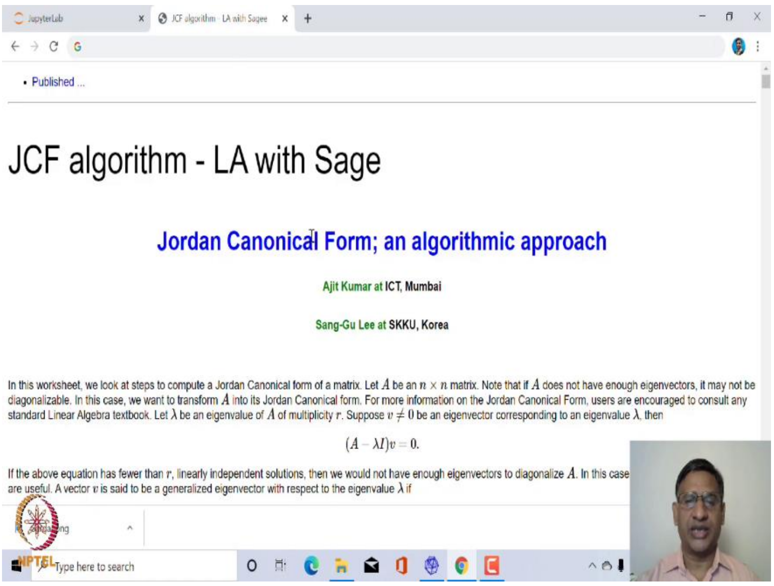

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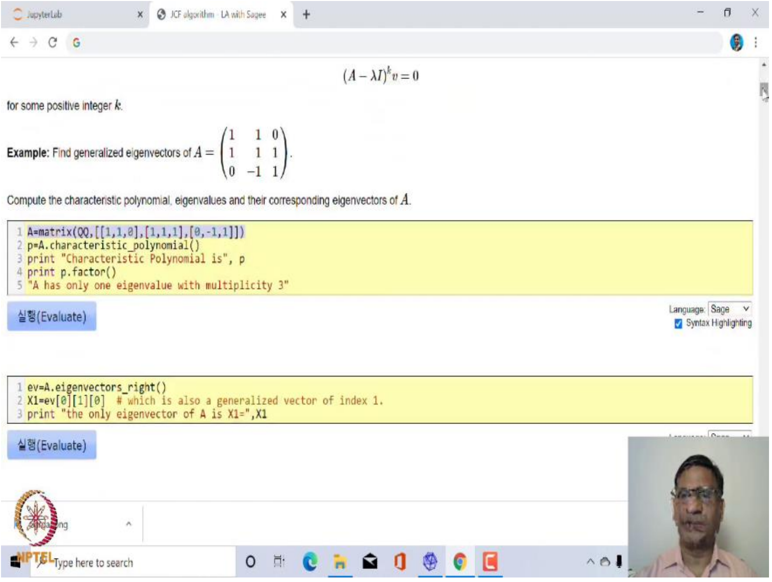

So, that is the website it gives you several examples along with the, the, some kind of algorithms.

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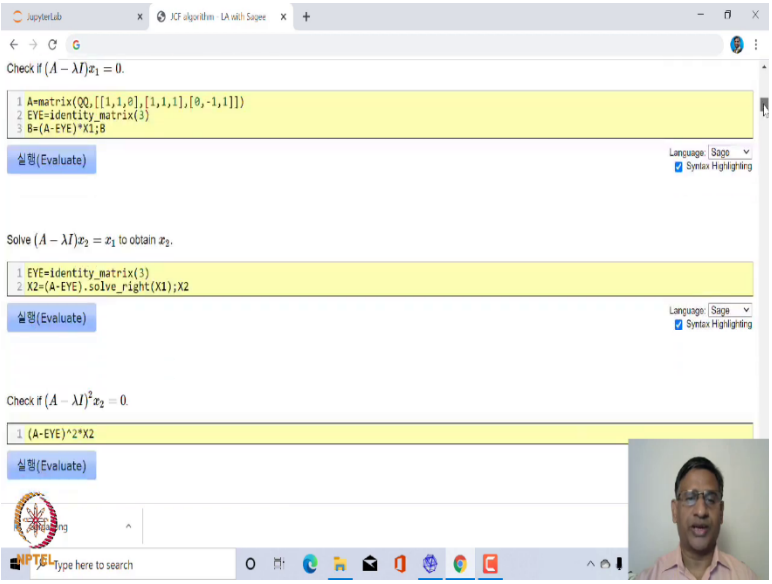

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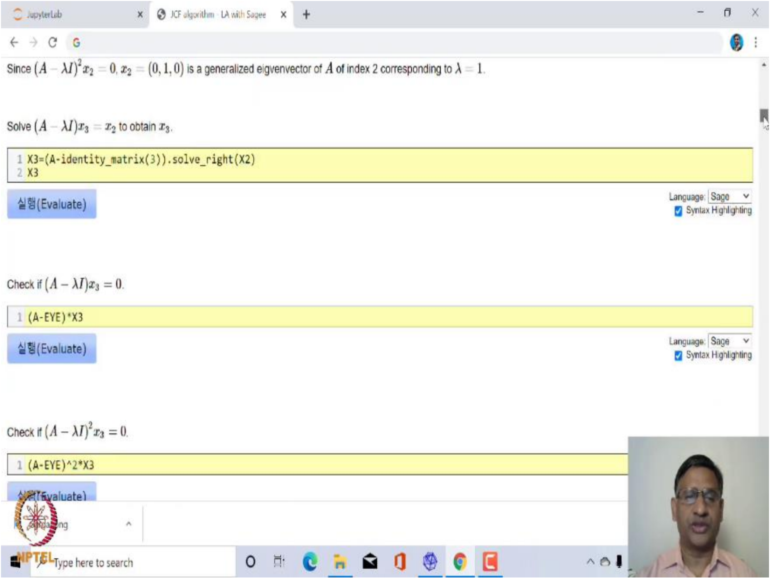

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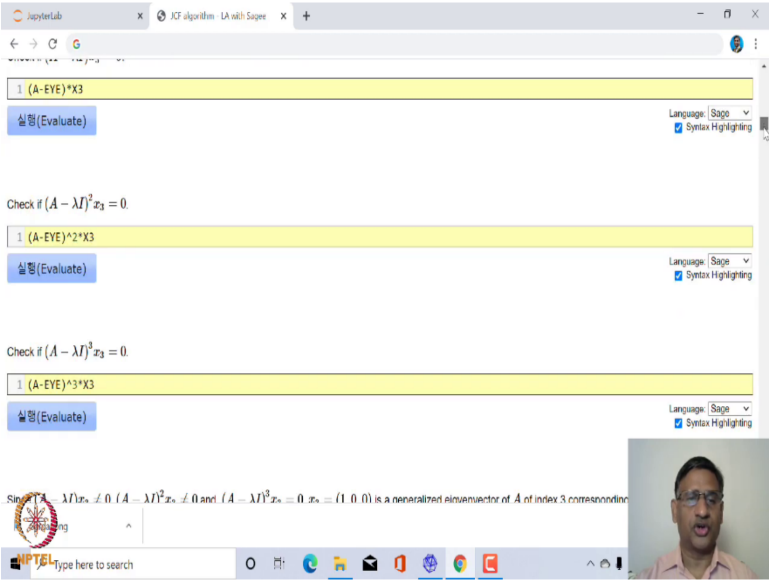

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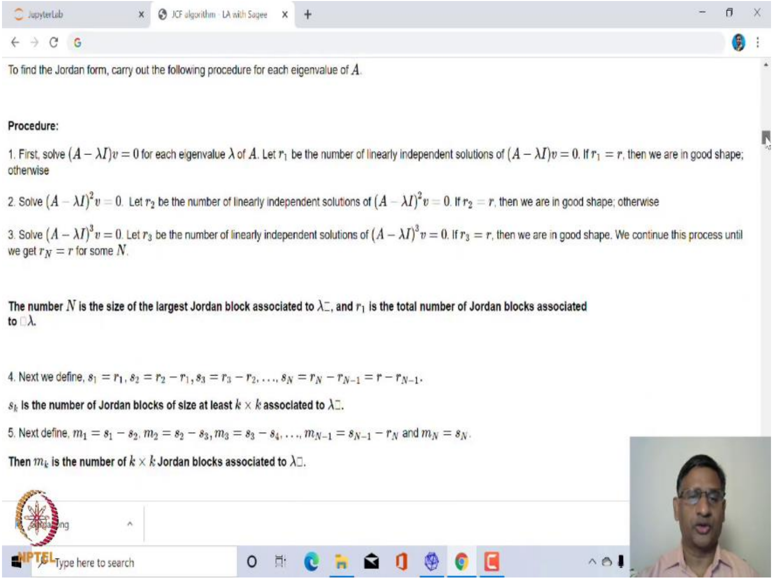

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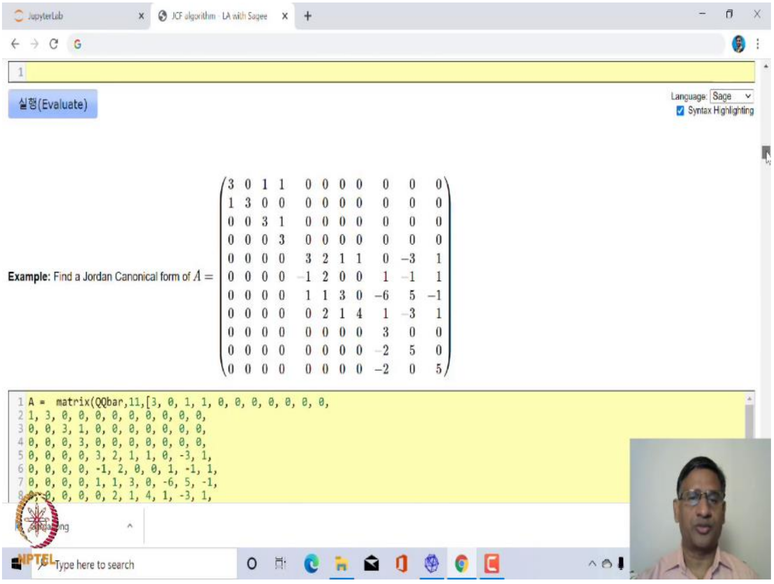

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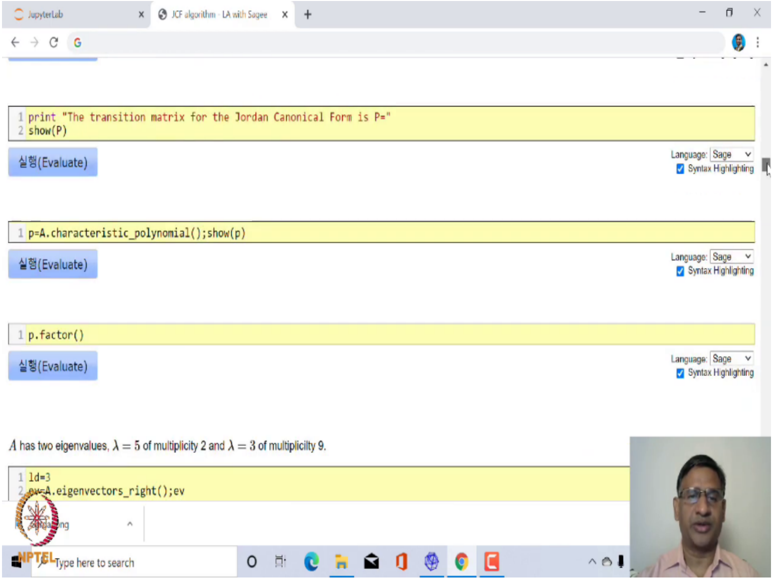

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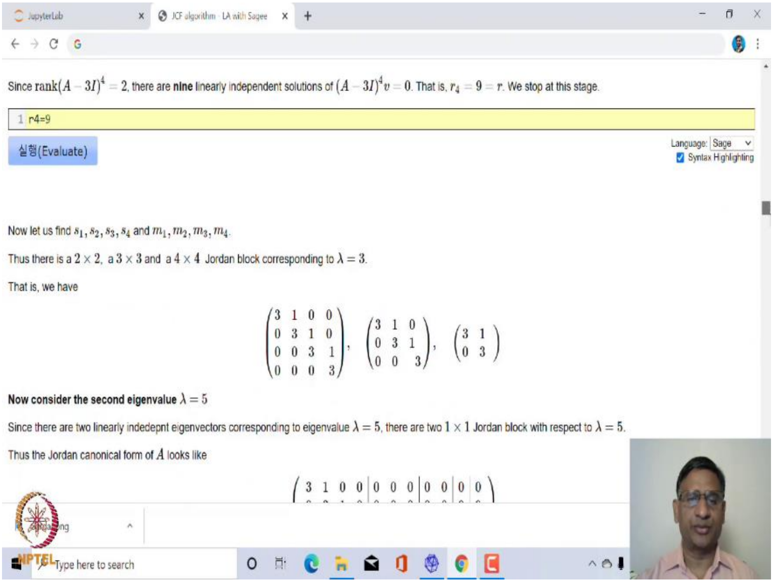

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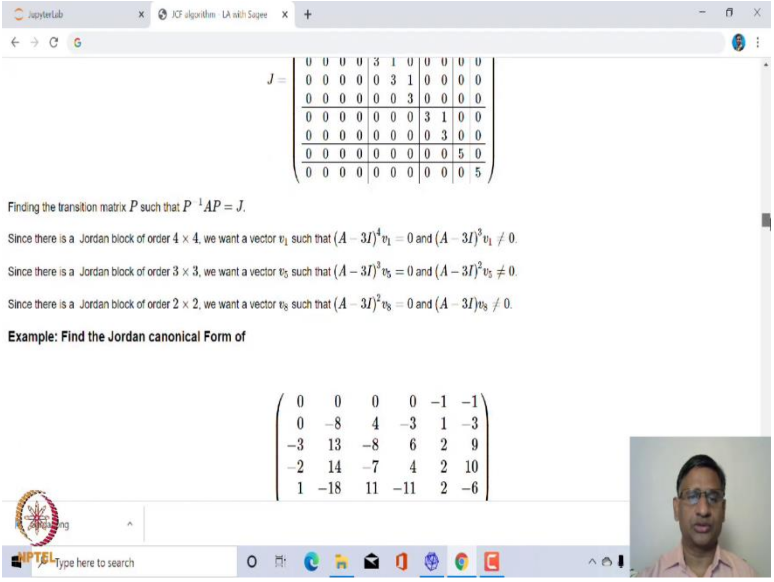

How to find Jordan canonical form of a given matrix, so there are several examples also.

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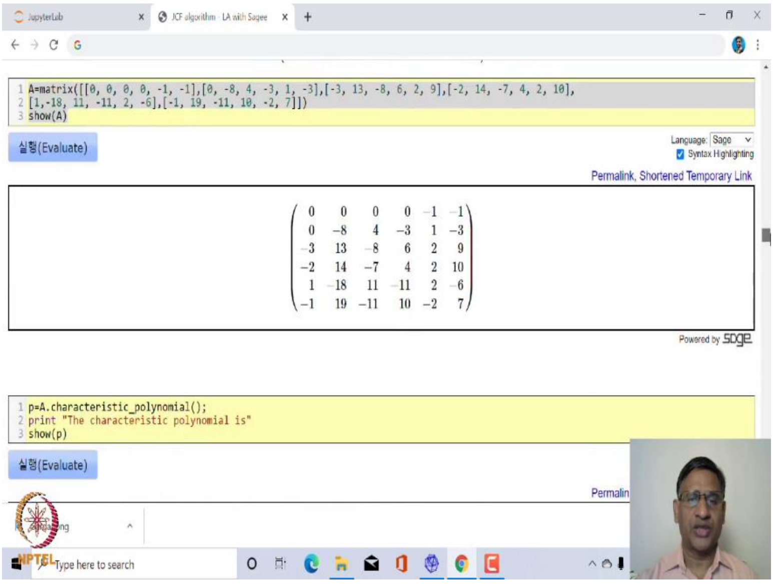

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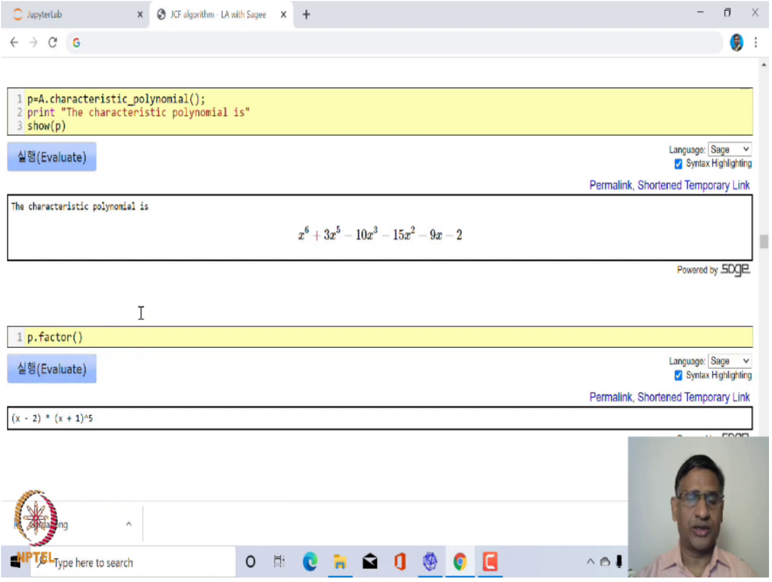

And it also has an inbuilt Sage cell. So, you can, you can run each of this, ok? So, you can look at this website.

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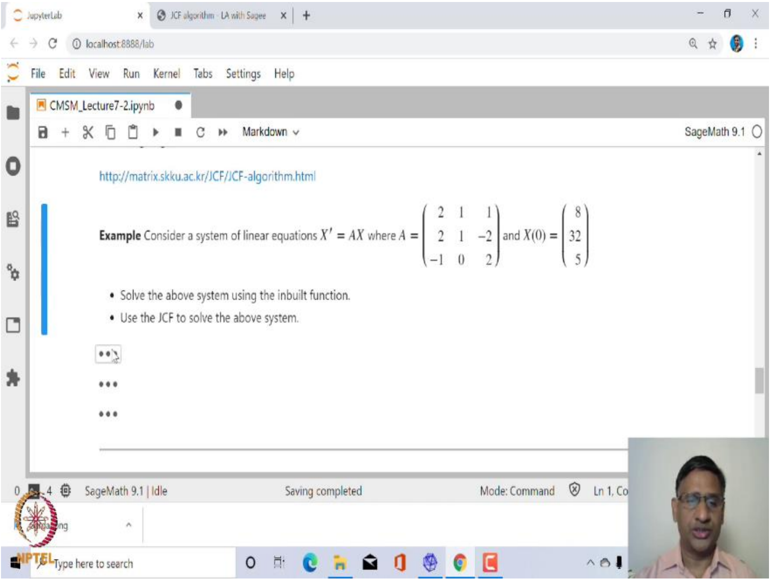

Now, let us look at how we can solve a system of ordinary differential equation using Jordan canonical form.

So, the procedure is exactly similar, except there, A was split into, into P inverse DA, in case A is diagonalizable. In this case, now it will be split as P inverse JP, and so Jordan canonical, that exponential of a, Jordan of a matrix A will be actually written in terms of exponential of Jordan blocks. An exponential of Jordan blocks is also very easy to, to find and this is what it is, it uses.

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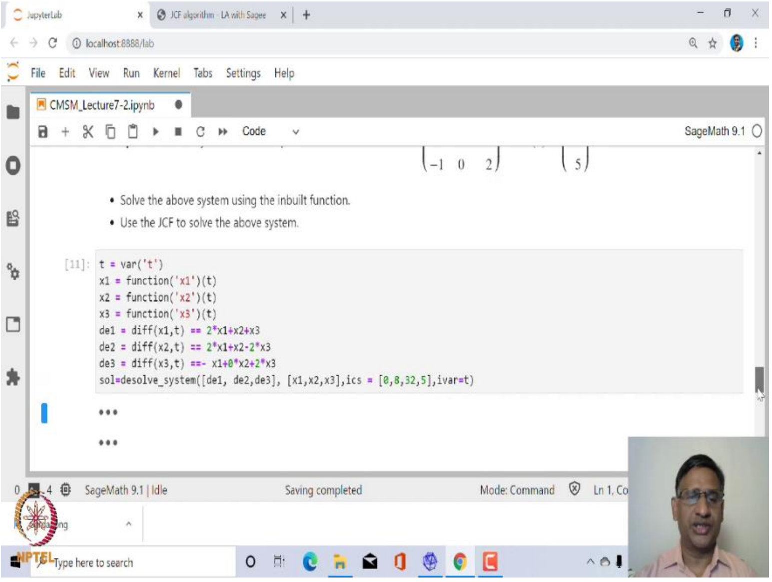

So, let us look at this example, and you can first, again solve this system using inbuilt function, inbuilt functions, desolve system, these 3 differential equations.

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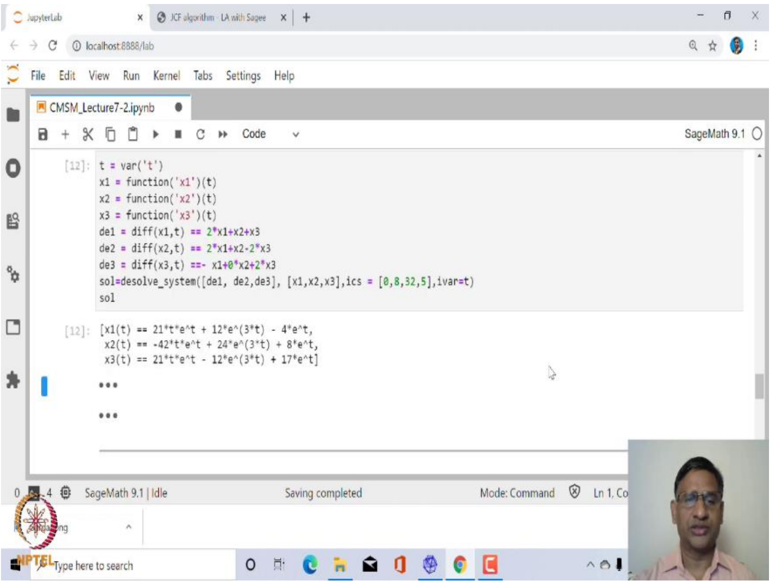

So, let us, let us ask it to show what is the solution we have obtained.

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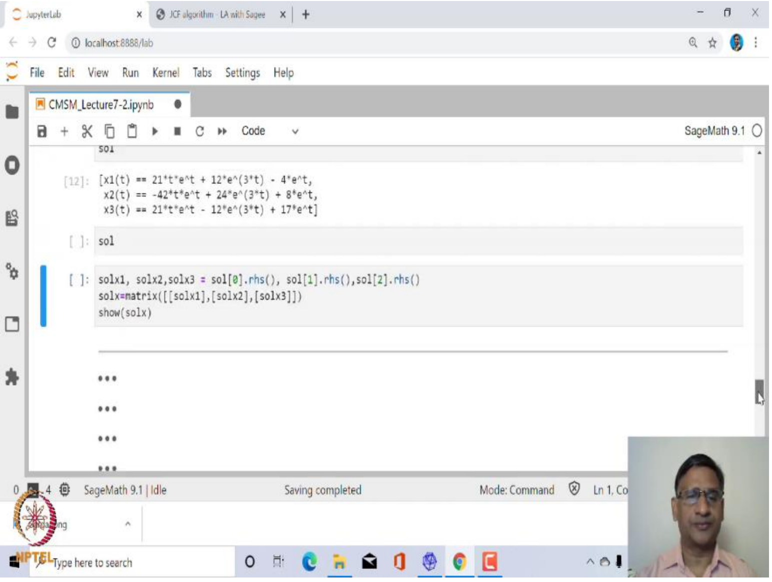

So, that is the solution, now let us look at how we can solve this using a Jordan canonical form.

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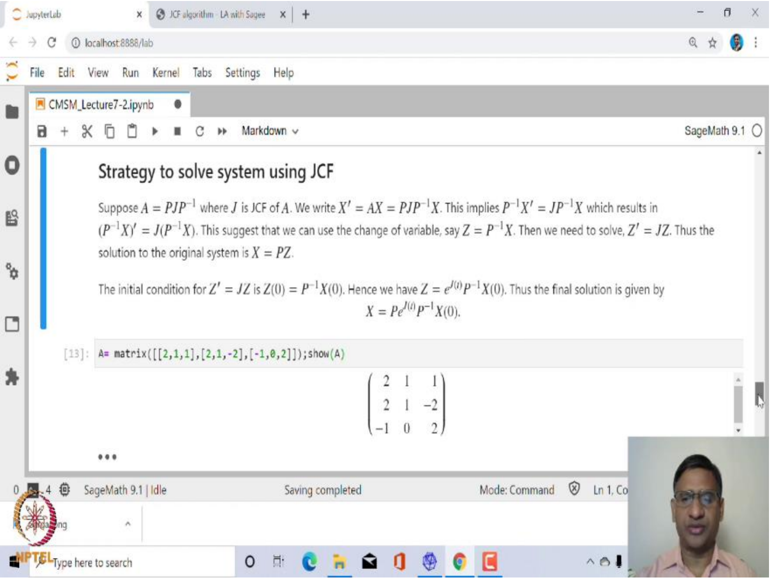

So, what is the strategy we are going to adopt? So, here A is P into J into P inverse, J where J is Jordan matrix of A, consisting of Jordan blocks, right? Now, so in this case, and we have the equation X dash t is equal to AX. Therefore, this is going to be P into J into P inverse. So, multiply both sides by P inverse, you have P inverse X dash is equal to J into P inverse X.

Now, if you substitute P inverse X is equal to Z, then the system may become Z dash is equal to J into Z, and with initial condition X, X 0 is equal to some, some matrix. So, therefore Z0 will be P inverse X 0, right? So, what will be the solution in this case? Solution of Z dash t equal to Z, this is going to be e to the power J of t times P inverse X 0, right, e to the power Jt.

Now, J is Jordan canonical form of A, and finding Jordan canonical form of this, exponential of Jordan canonical form is an easy task, you can always find this, so that is the solution. Now, let us look at, so this is exactly same as diagonalizable matrix, except that you need to find exponential of Jordan canonical matrix, right? So, let us take, define A as this matrix, which is the, the coefficient matrix of this system X dash t is equal to AX.

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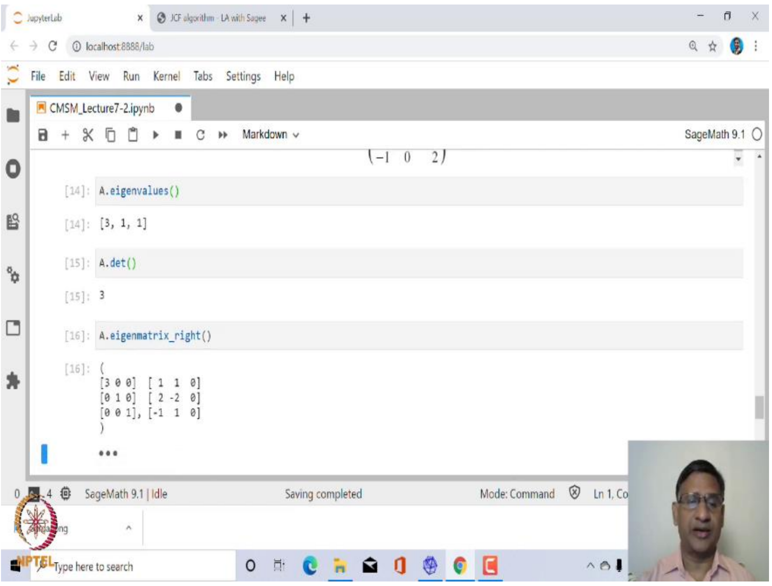

And let us find eigenvalues of A. So, eigenvalues of 3 and 1 and 1, but if you look at the determinant of this matrix is 3. So, determinant is 3 right, so this, and let us look at what is the eigenmatrix of A? So, eigenmatrix of this, you see that one eigenvalue is 0, one, this is a 0 matrix. So, it cannot be diagonalizable, this matrix is not diagonalizable, that is what it means. It does not have enough eigenvectors to make a basis, right?

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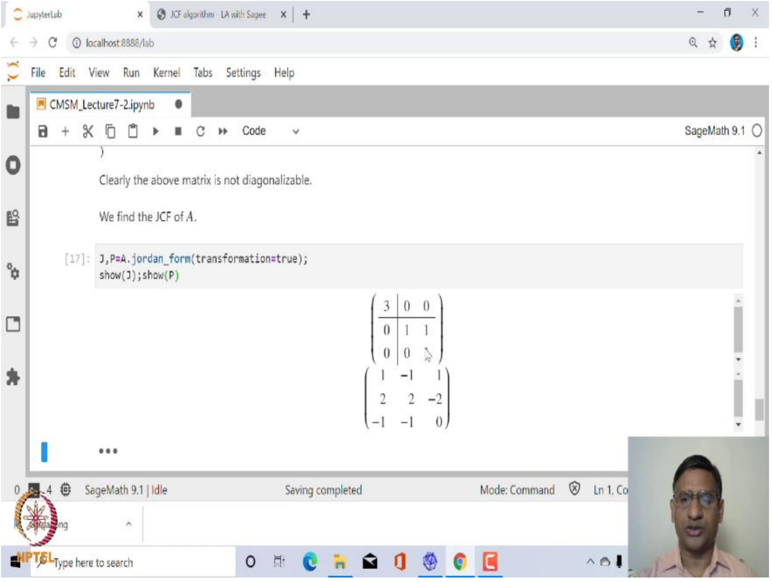

So, in that case, we will find Jordan canonical form of A. So, let us find Jordan canonical form of A, and this is the Jordan canonical form. So, it has 2 blocks, one is 3, other one is Jordan block with respect to eigenvalue 1, which is 1, 1, 0, 1. And this is the diagonalizing, this is a matrix P whose first column is eigenvectors with respect to eigenvalue 1, and these 2 generalized eigenvectors corresponding to eigenvalue 1, right?

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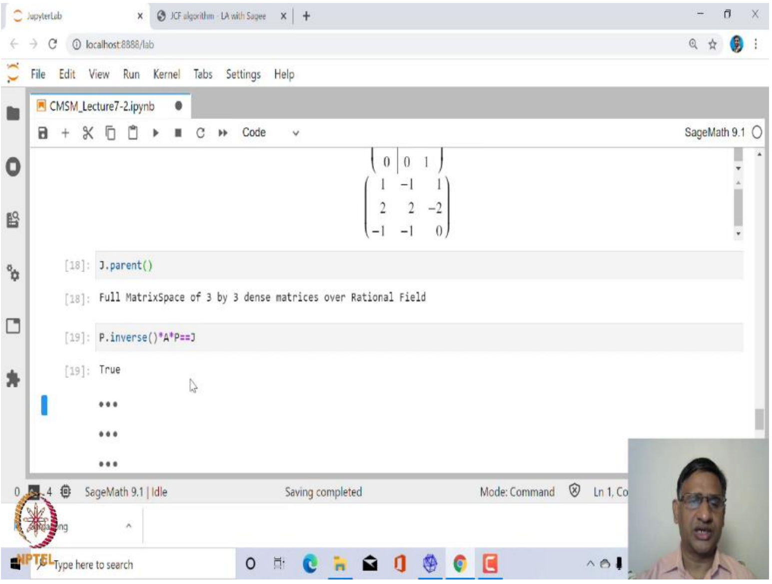

So, now what we will do? Let us, you can ask what is the, the parents of J? You can ask whether P inverse AP is J, that is true, this is just to check what you have obtained is correct.

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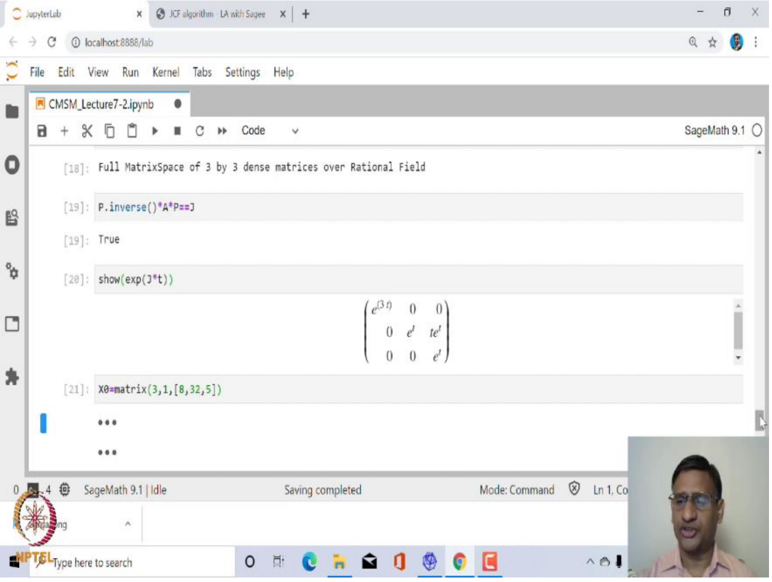

And then you can ask it to find what is exponential of this Jordan canonical form. So, here this is the, this is the diagonal entry. So, here it will be just e to the power 3t and this is a exponential of Jordan block which is 1, 1, 0, 1.

So, you see here e to the power, e to the power t, t times e to the power t. If it were 3 cross 3, then it will, next it will come as, t square by 2factorial into e to the power t and some things like that. So, you can, you can take some, let us say 5 by 5 Jordan block with respect with, with some diagonal entries, let us say 2, 2, 2 and diagonal, next to the upper diagonal as 1, 1, 1 and then find out its exponential, right?

So, this is the initial vector, that is X 0 is equal to 8, 32, 5 that is the X 0. So, X0, X 1 at 0 is 8, X 2 at 0 is 32, X 3 at 0 is 5, right?

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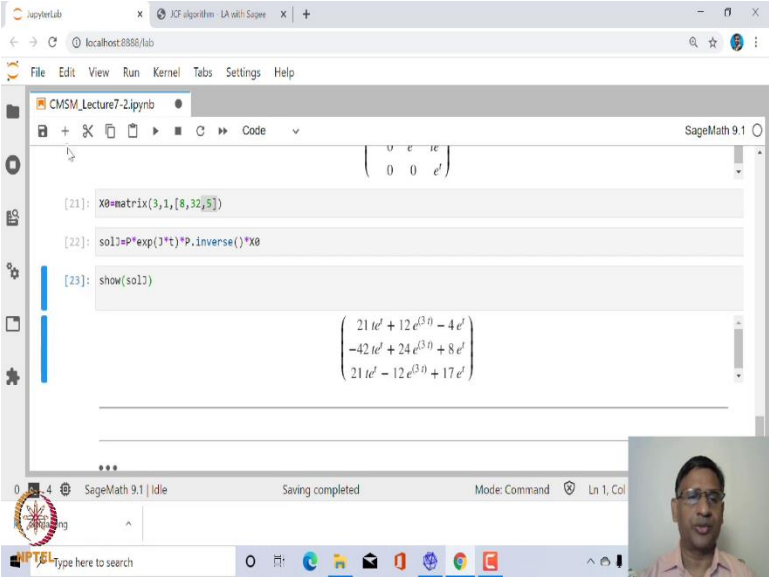

Now, we know what, how to solve this. So, we already have the solution is given in this form as X is equal to P into e to the power J, J times t into P inverse X 0. So, that was solution, this is what you get, right? So, if you compare, you can see here, both these are same. x1 dash t is same as what the first component here, and so on. So, this is how we can solve system of ordinary linear differential equations using eigenvalues, eigenvectors. In case it is diagonalizable; it is quite easy, if it is not diagonalizable, then one can use Jordan canonical form.

This Jordan canonical form again is very important concept. I did not look at the theory of this Jordan canonical form, it is slightly involved, but however, it has this application, and once we know Jordan canonical form of a matrix, which we can find out using SageMath, we can make use of this quite nicely, right?

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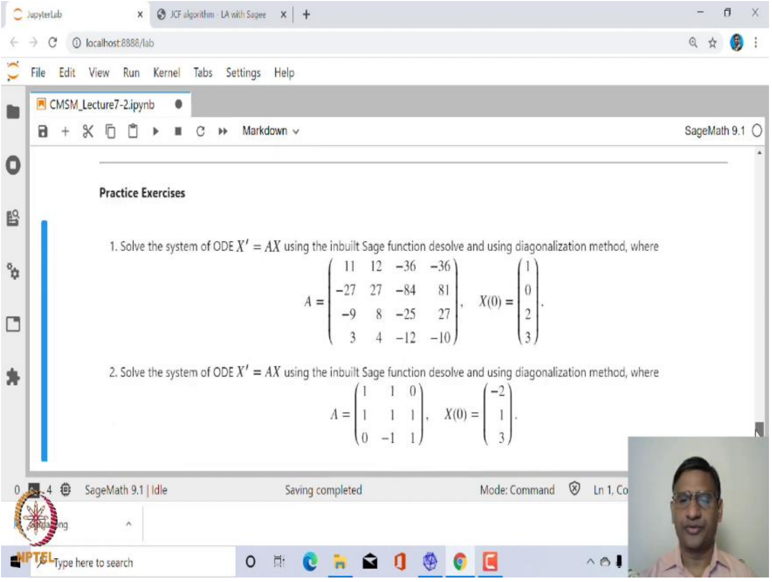

So, let me just leave you with couple of exercises. One is solving this system of ordinary differential equations; this is 4 equations in 4 variables. So, in both these cases, you should try to solve this system using inbuilt function, desolve, desolve underscore system, and also use this diagonalization method. So, whether it, if it is diagonalizable, you can use diagonalizability of A, if it is not, you have to use Jordan canonical form, right?

So, let me stop here. So, we will look at some more applications of linear algebra in next class.

Thank you very much.