Computational Mathematics with SageMath
Prof. Ajit Kumar
Department of Mathematics
Institute of Chemical Technology, Mumbai

Lecture – 39
Orthogonal Decomposition with SageMath

Welcome to the 39th lecture on Computational Mathematics with SageMath. In this lecture we will continue with exploring more concepts in inner product spaces. Last time we looked at how to find an orthonormal basis using Gram Schmidt orthogonalization process.

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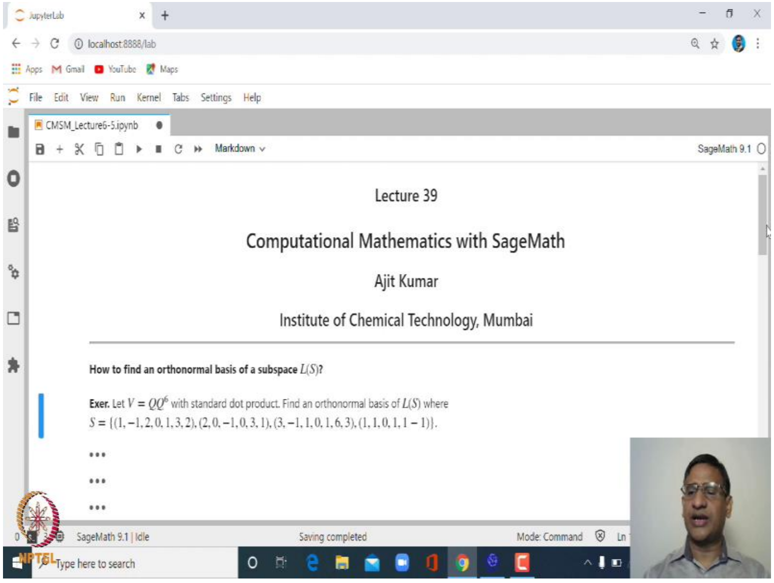

Now, suppose you are given any subspace which is linear span of a set of vectors. How do we find an orthonormal basis of L of S? This we already saw, that using inbuilt function,

you can find orthonormal basis. Or you can apply QR factorization to any matrix. It can be singular, it can be rectangular.

So, one can apply this QR factorization to L of S, that means, you can take S to be column matrix consisting of set of vectors from S. Or you can take linear span of S, you will get a basis matrix of the L of S, and then apply Gram Schmidt orthogonalization process to that. So, that you can do. So, I leave that as an exercise for you to do. For example, take QQ to the power 6 and take these 5 set of vectors S, define L of S and find its orthonormal basis.

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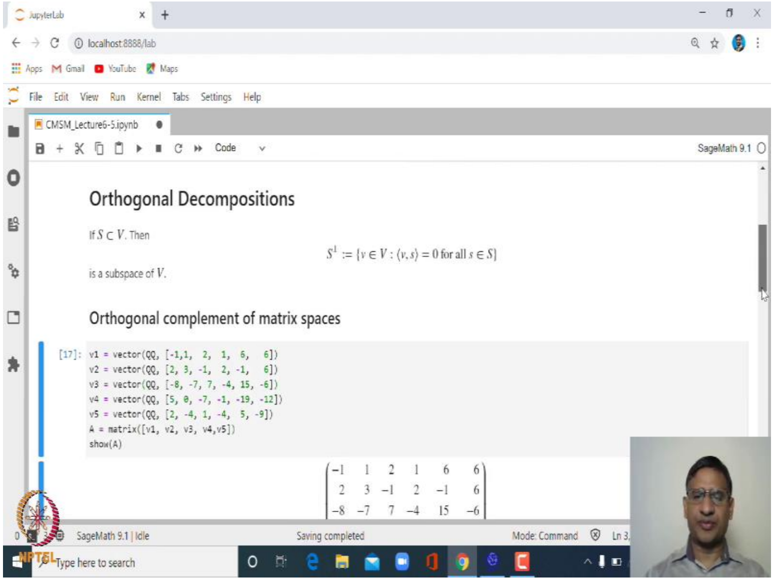

Let us look at what is meaning of orthogonal decomposition. This is again a very important concept. So, suppose you have, a subset of an inner product space V, with some inner product.

So, when we say vectors inner product space, it will come with some inner product. Then you can define set of all vectors in V which are orthogonal to every vectors in S, that is what is called S perp or orthogonal complement of S. And you can check that this is a subspace of V. S need not be a subspace. However, this will always be subspace of V.

So, for example, if I take S to be singleton 0 in V, then every vector will be perpendicular to 0 vector. Therefore, S perp will be the entire V right.

In case we take one dimensional sub space, a single vector v, then all those vectors which are perpendicular to v will actually form n minus 1 dimensional subspace , that one can check. That one can check using the rank nullity theorem.

Next let us look at how do we find orthogonal complement. S perp is known as orthogonal complement of S. In case S happened to be the subspace, orthogonal complement will also be subspace. So, suppose you have a matrix A, we have seen that to every matrix, there are 4 subspaces associated to that matrix. So, let us find out orthogonal complement of all these subspaces.

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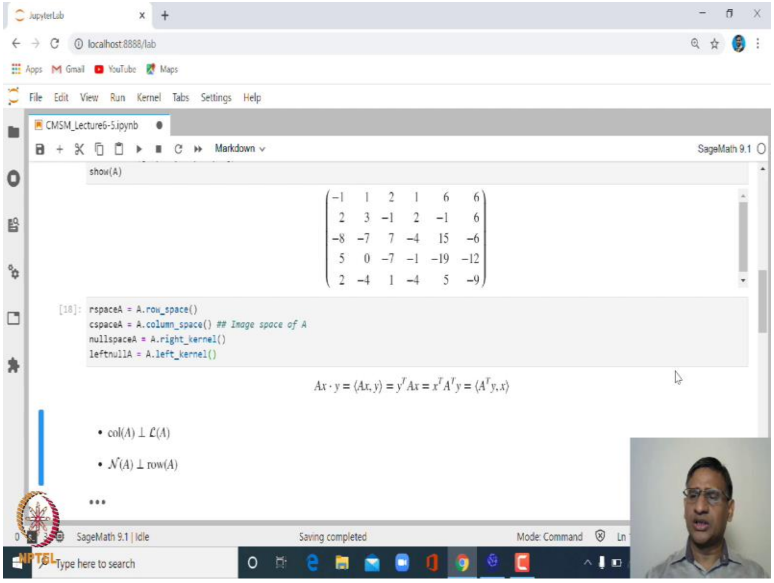

So, let us take this matrix which is which is a 5 cross 6 matrix. And, let us take subspaces of this A, namely the row space of A. I am denoting this by rspaceA, cspaceA, column space of A, is the same as image of A, null space of A which is obtained as A dot right kernel. And left null space which is actually obtained as A dot left kernel, which is nothing but right kernel of A transpose.

So, these are the subspaces. In order to find orthogonal complement of each of these subspaces, one concept will be quite easy. If I take dot product of Ax with any vector y, that is Ax inner product with y, in case of standard dot product it is y transpose Ax. Now, this you can check that this is actually a real number, this is a scalar. So, its transpose will be its itself.

So, if I take transpose of this, this is nothing but A transpose x transpose into A transpose y, which is same as saying A transpose y, x. So, if you look at a Ax inner product with y, is same as A transpose y inner product with x. In case A happens to be symmetric both inner product Ax with y is same as Ay with x. This is what is used in order to prove all these things etcetera.

Now, let us look at the result. The result says that if you look at the column space of A, this is going to be orthogonal to left kernel of A.

That is same as saying these two subspaces are orthogonal complement of each other. Similarly the orthogonal complement of null space of A is row space of A, in particular these two are orthogonal complement of each other.

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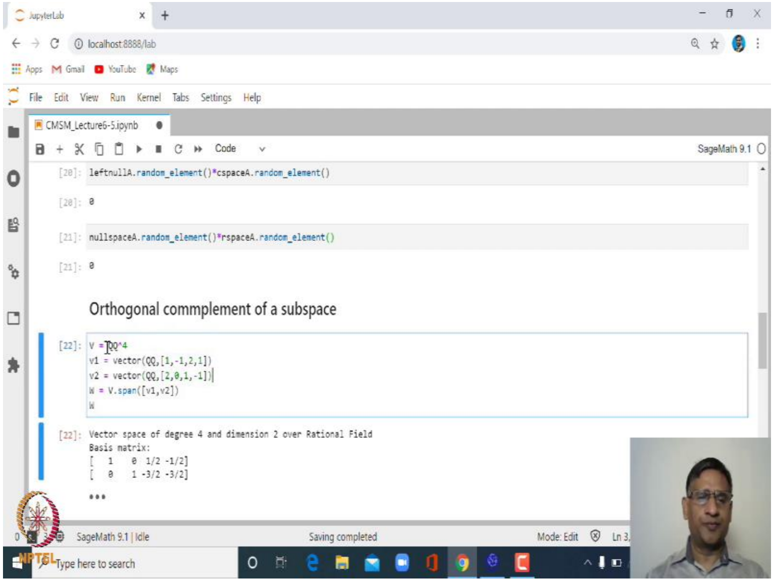

So, how do we verify this? In order to verify this, we can just take a random element in null space of A, and take a random element in column space of A. And multiply these two, is same as saying take dot product of these two. And, then see whether we are getting 0? If you run once more you will always get 0, or you can take some arbitrary vector and then try it out.

Similarly, you can take a random element of null space of A and take a random element of row space of A, and then check that the dot product of these two, will always be 0. So this actually shows that these two column space and left null space of A are orthogonal complement of each other. And similarly null space of A and row space of A are orthogonal complement of each other.

Now, in general suppose you have a subspace, let us say W, spanned by these two vectors v1 and v2 in Q 4. So, W is a subspace of V spanned by these two vectors v1 and v2. We want to find out what should be an orthogonal complement of this subspace. When you say this W, it also gives you basis. So, you can find a basis of this. Of course, these two in this case are linearly independent. So, this itself will form a basis of W.

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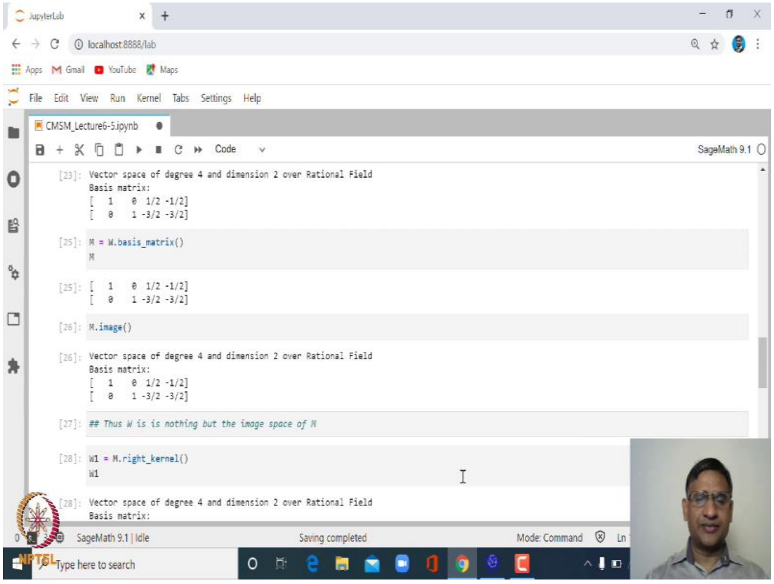

So, what you can do now? You can take this as a matrix. So, M is equal to W dot basis matrix, this will give you this matrix ,which you see here. Let me print what is M, this is the matrix. Now, if you try to find out what is image of this matrix M. This is a vector space of degree 4 dimension 2 over rational field with this basis.

So, you can see here this linear span of v1, v2, that is, W here is same as image of this basis matrix. Now we know that this is image of basis matrix, and we just now saw that if you have the matrices associated with a matrix, how to find its orthogonal complements. So, how do we find orthogonal complement of this?

This is nothing, but take a right kernel of M and that will give you orthogonal complement of this subspace W.

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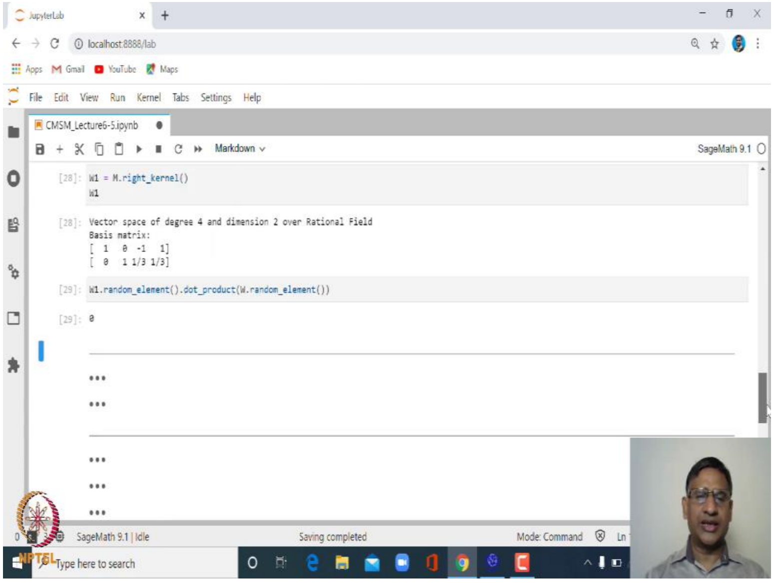

That is how we can find orthogonal complement of any subspace. You can check that any random element of W1 which is the right kernel of M dot product with any random element in W, it should be 0, right. So, we have found orthogonal complement of any subspace of an inner product space V.

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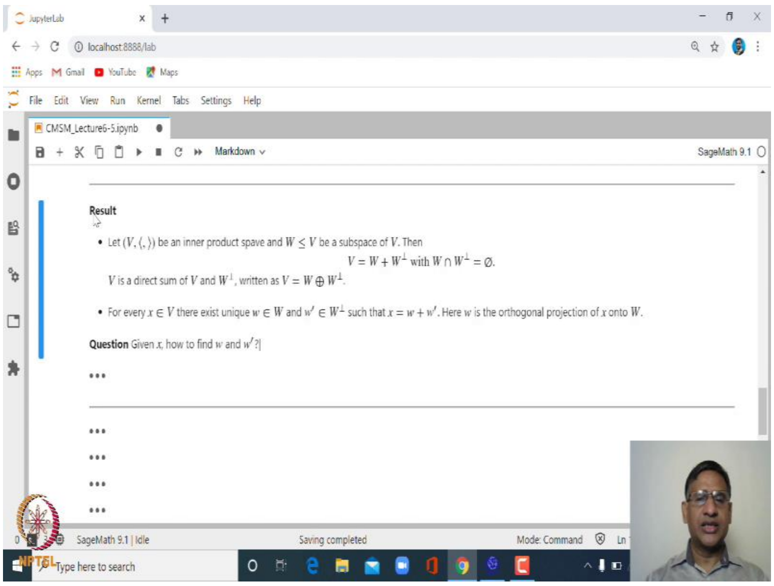

Next, let us look at. Suppose you are given a subspace W. Then we just now saw, how to find its orthogonal complement. You one can show that intersection of W with W perp, this is not an empty set, this is not empty set, it should be singleton 0.

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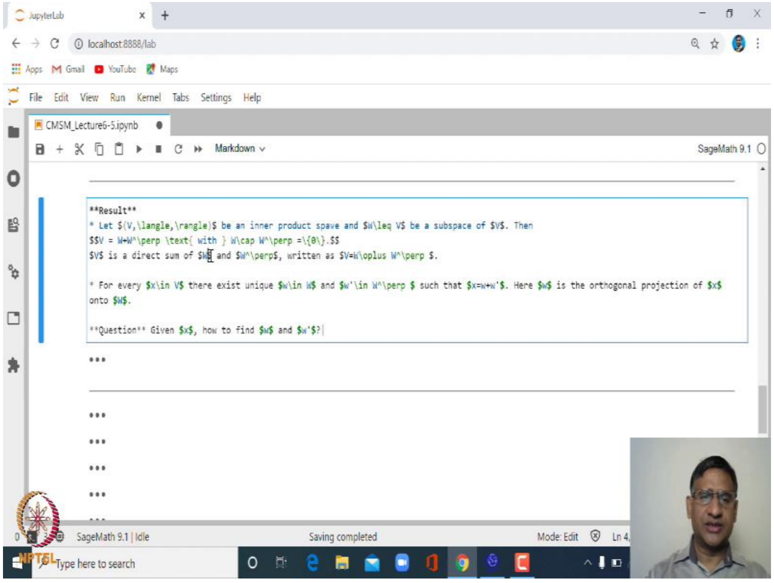

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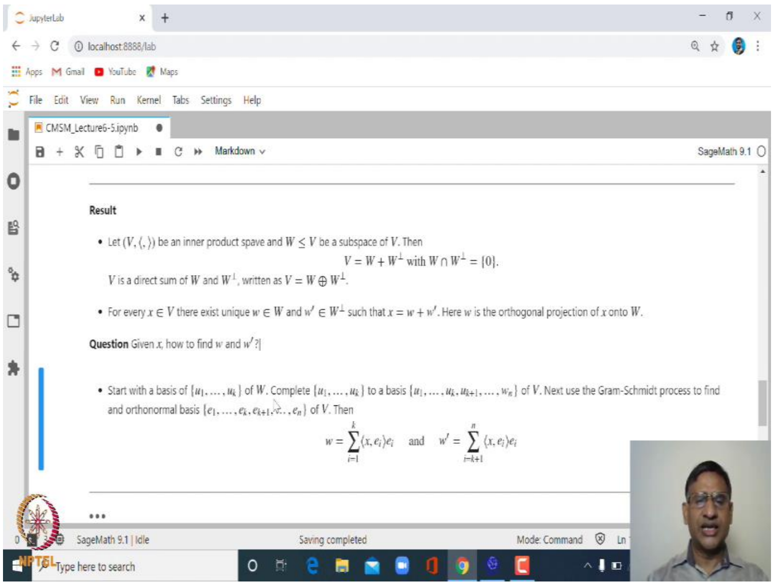

So let me call this as singleton 0, this is 0 subspace, only 0 is can be in both. It is quite easy to check, and can show that V is sum of W plus W perp and not only that, since this intersection is 0 space, this is known as direct sum.

So, V is a direct sum of W and W perp, that is, what is the result. This also means that if I take any vector x, then you can find a vector w in capital W and w dash in capital W perp such that x is w plus w dash and this w and w dash are unique. So, here w is known as orthogonal projection of x onto W.

So, not only we have defined what is meaning of orthogonal projection of a vector onto another vector, here you have a notion of finding orthogonal projection of a vector onto a subspace. So, now the question is how do we find this w and w dash given x. This is again quite easy. One way to do is, you start with the basis u1, u2, ..., uk of W. Since this is a linearly independent set of vectors, it can be completed to a basis u1, u2, ..., uk, u k plus 1 to un.

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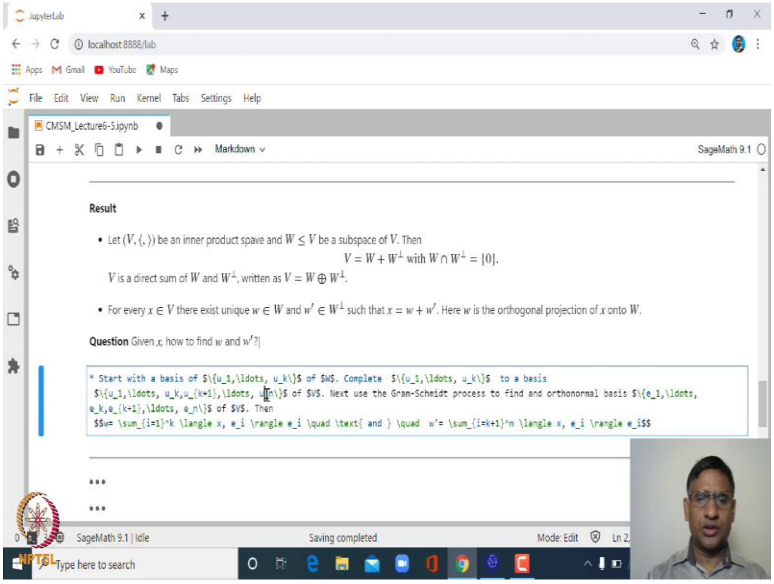

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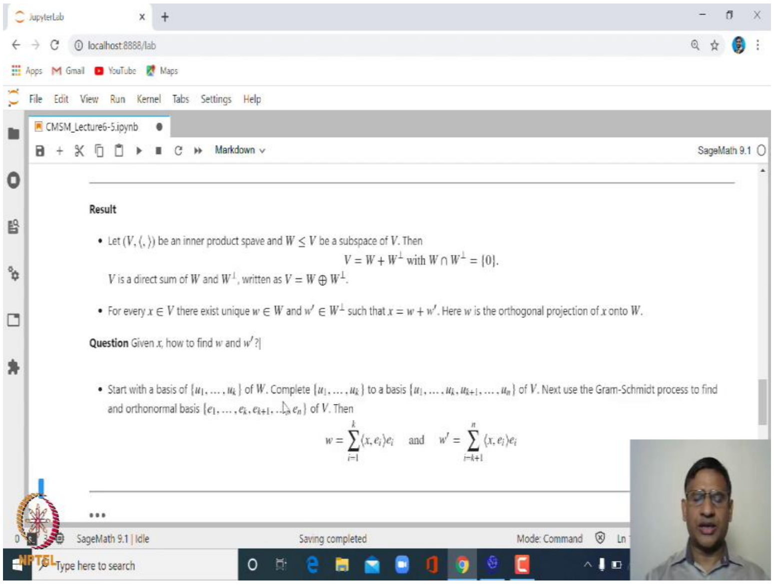

So, last one should be un of V. So, you can complete these two, to a basis of V and then apply the Gram Schmidt process. When you apply the Gram Schmidt process, all these vectors, let us call these vectors we have obtained as e1, e2, ek. These e1, e2, ek will be actually lie in the linear span of u1, u2, uk, that is in W, and these will be in orthogonal complement.

We know that in this case x can be written as summation, let us say, summation xi ei and xi is nothing but x inner product with ei. Then you just take out only the component which are in the first part, for which i going from 1 to k. And w dash as the the components starting from k plus 1 to n. That is how you can obtain w and w dash from this.

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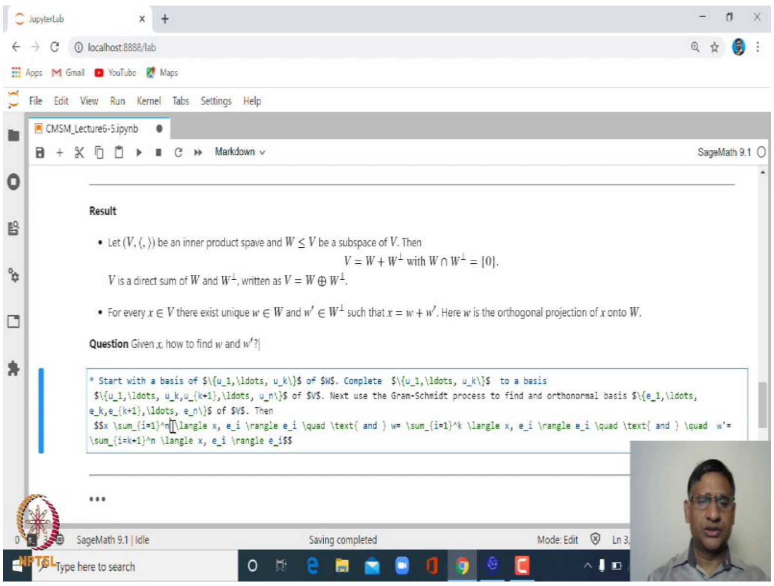

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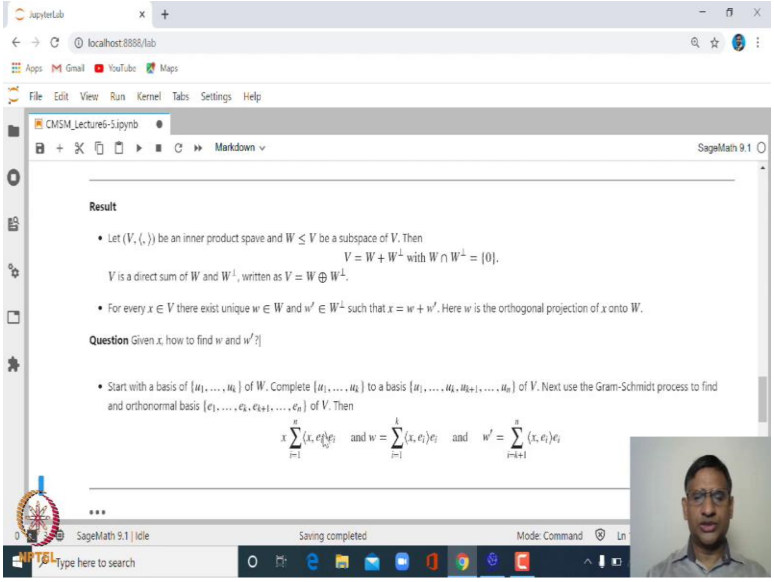

Actually I should I have also written, what is this x. In this case x is going to be this with i going from 1 to n now.

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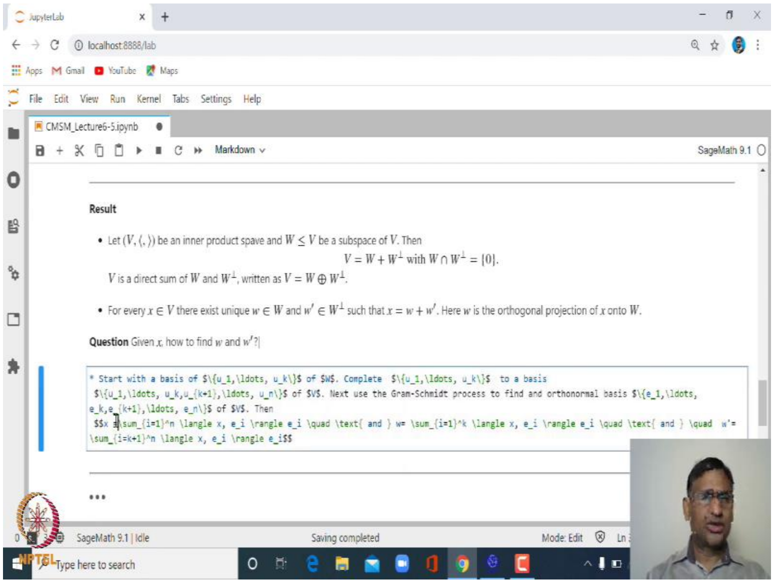

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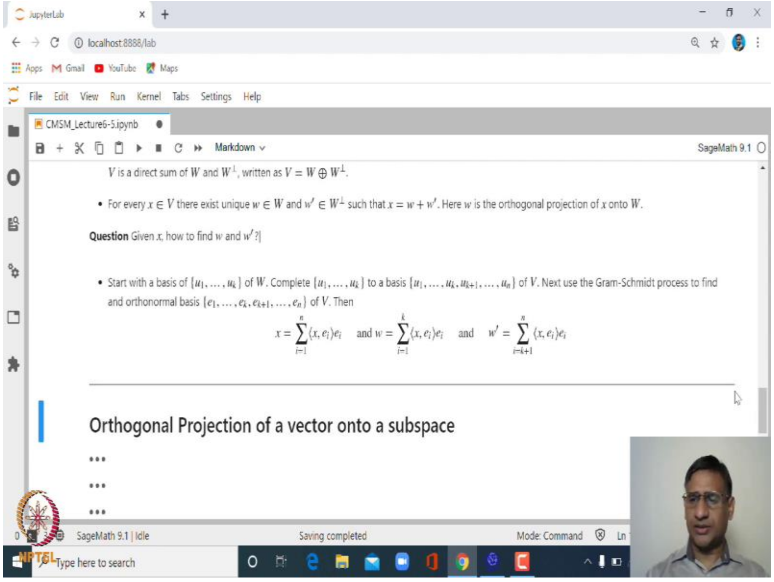

So, this x is equal to this and then in that case define w and w dash like this. So, that is one way of obtaining. However, you can also obtain this in a slightly different way.

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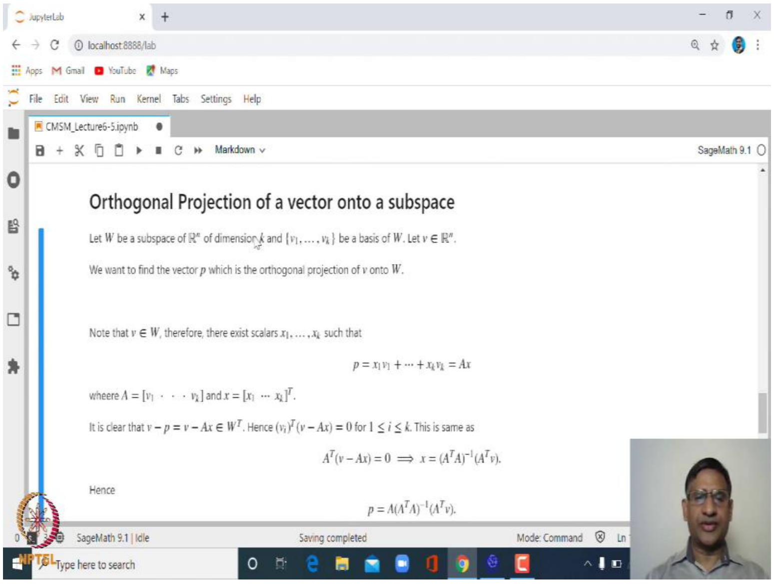

How do we do that? So, let us assume that W is a subspace of R n and of dimension k and v1 to vk be a basis of W. This need not be an orthonormal basis. Now, take any vector v in Rn, we want to find a vector p which is orthogonal projection of v onto W. This is what we want to find. We want to find p.

Now, if you look at this p, p will lie in the subspace W, p is orthogonal projection of v onto W. So, p will lie in W. Therefore, I can write p as x1 v1 plus x2 v2 plus dot dot dot xk vk. Now, we have seen that this I can write as A times x, where A is column vectors v1, v2, vk and x is a column. Sorry A is the matrix whose first column is v1,

the second column is v2 dot dot dot last column is vk and x is column vector x1, x2, xk. So, that is what it says. Now, if you look at, since p is orthogonal projection of v onto W, if I look at v minus p, that should be perpendicular to W. That means, v minus, and what is p, p is A times x. So, v minus Ax will be perpendicular to W. That is what it it says. So, v minus Ax lies in W perp.

Now, this this is same as saying, if I take any vector in W, this will be perpendicular to v minus Ax. In particular, every vector v1, v2, vk will be perpendicular to v minus Ax. And so, that that can be translated into A transpose v minus Ax is equal to 0. And when you further work it out, what we will get is A transpose v is equal to A transpose Ax.

Therefore, in case A transpose A is invertible, what you have is, x equals to A transpose A inverse times A transpose v. But, A transpose A is invertible because A is a matrix whose columns are v1, v2, vk which are linearly independent. So, A transpose A will be k cross k matrix and which will be invertible. One can show that rank of A transpose A is same as rank of A, and rank of A is k. Therefore, rank of A transpose A will be again k and hence it will be invertible.

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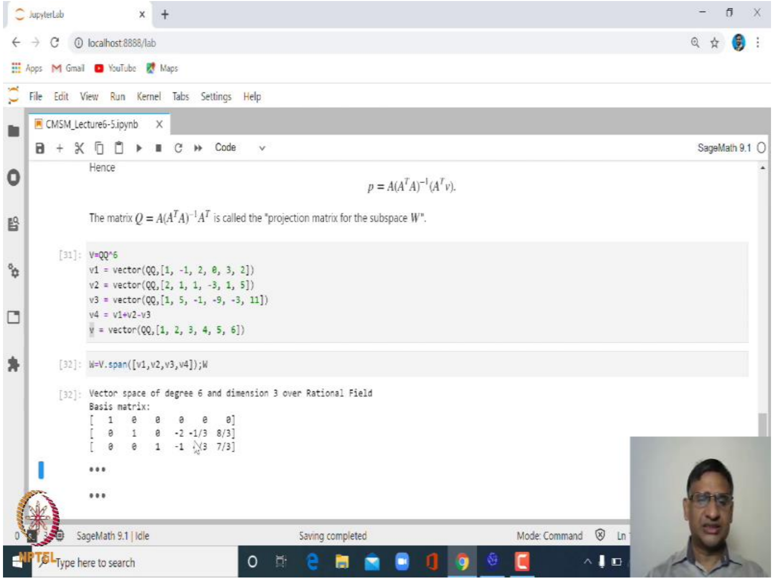

Therefore, we have obtained an expression for orthogonal projection of a vector v onto W. What is vector p? Vector p is Ax, and what is x, x is A transpose A inverse times a transpose v. Therefore, p is A times A transpose A inverse A transpose v.

So, we have obtained an expression for orthogonal projection of a vector onto a subspace W. What is the this here, all you need to do is, obtain this matrix A which is column vectors of basis vectors of W.

Let us look at an example. Take V to be QQ to the power 6 and let us take a vector v, which is vector 1, 2, 3, 4, 5, 6, we want to find orthogonal projection of this vector v onto W.

So, how do we do that? So, let us define linear span of this vector v1, v2, v4 here v1, v2, v4 are not linearly independent. In case you can see here v4 is v1 plus v2 minus v3. So, this is a basis elements. This is a 3 dimensional subspace W. So, these 3 rows are the basis vectors.

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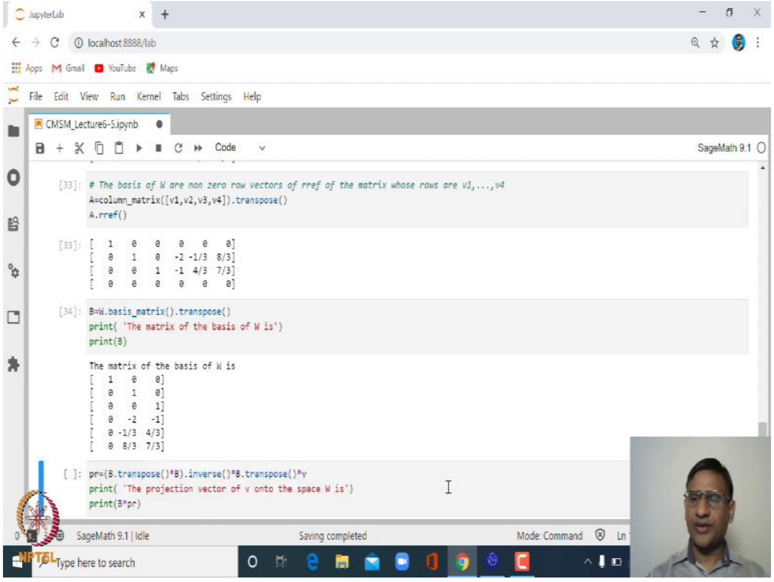

Now, what do we do? We have looked at, these can be obtained using RREF. This we have already done so, I am not going to get into this.

Next what do we do? We define B to be the basis matrix of this, and take transpose of that, because we have to have the column matrix.

So, take this B, this is nothing, but basis vector of W. This is the column of basis vector of W, that is a B. Now, how do we find the orthogonal projection? All we need to do is, let us define B transpose B take the inverse of that and multiply by B transpose into v. And, then orthogonal projection is going to be B into this x. So, this is your x, for which we a formula, that we found out. So that is the orthogonal projection of v on to capital W.

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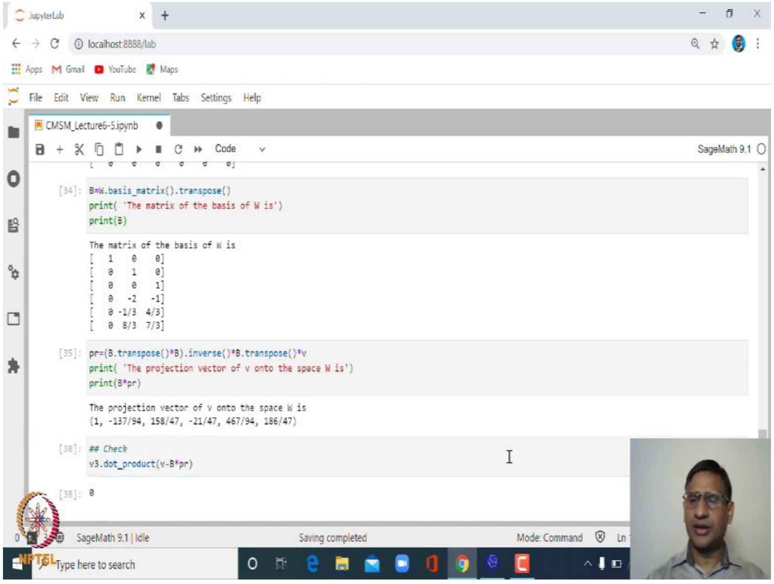

So, that is how we can find the orthogonal projection of a vector onto subspace right. And of course, you can check that v1 is perpendicular to v minus orthogonal projection of v onto this W. Not only v1, you can check v2, you can check v3, all of them should be perpendicular. In particular you can take any random vector in W and check that is a perpendicular to this orthogonal projection.

This is a very important concept, and in the next class we will see an application of this as least square problem. We have already seen least square problem using calculus, but now we will see that least square problem boils down to actually finding orthogonal projection.

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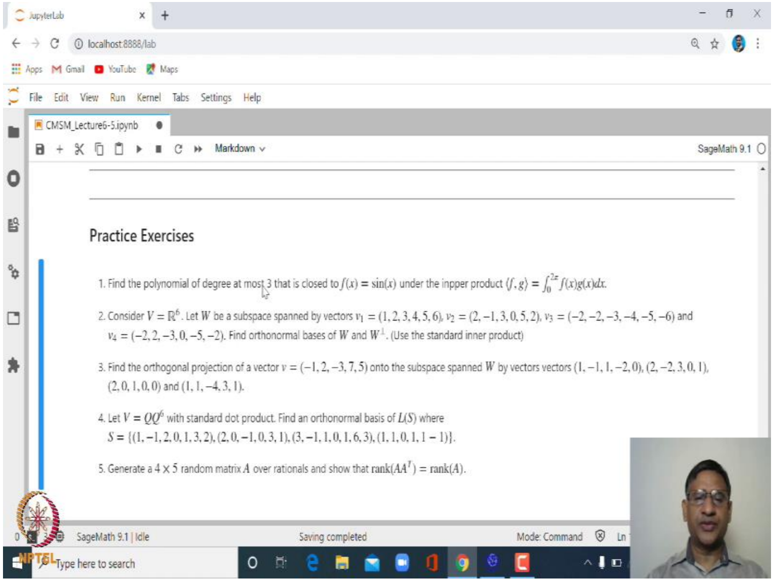

Let me leave you with these few simple exercises. So, these are the exercises.

Find the polynomial of degree at most three that is close to the sin x function under this inner product. Now, what it means is that if you look at the sin x function, and you want at most degree 3. So, if we look at subspace spanned by let us a constant polynomial degree 1, degree 2, degree 3, that will be a subspace. Then, this vector which is closest to this is going to be orthogonal projection of this function onto subspace spanned by 1, x, x square, x cube. That is what it means. So, that is how you can solve this problem.

Next consider this R to the power 6, take the subspace it is spanned by v1, v2, v3, v4. Then find an orthonormal basis of W and W perp. So, you can use standard inner product.

Then find orthogonal projection of a vector this onto subspace W. And this is the the problem, which I started with. Take V to be Q to the power 6, QQ 6 with respect to the standard dot product. Find an orthonormal basis of L of S and you can generate any 4 cross 5 random matrix over rational. And show that the rank of A into A transpose is same as rank of A. This proof is again very simple but here it is just a verification.

Thank you very much. We will see you in the next class.