**Computational Mathematics with SageMath**
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**Inner Product Spaces cont…**
**Lecture – 38**
**Inner Product Part 2 with SageMath**

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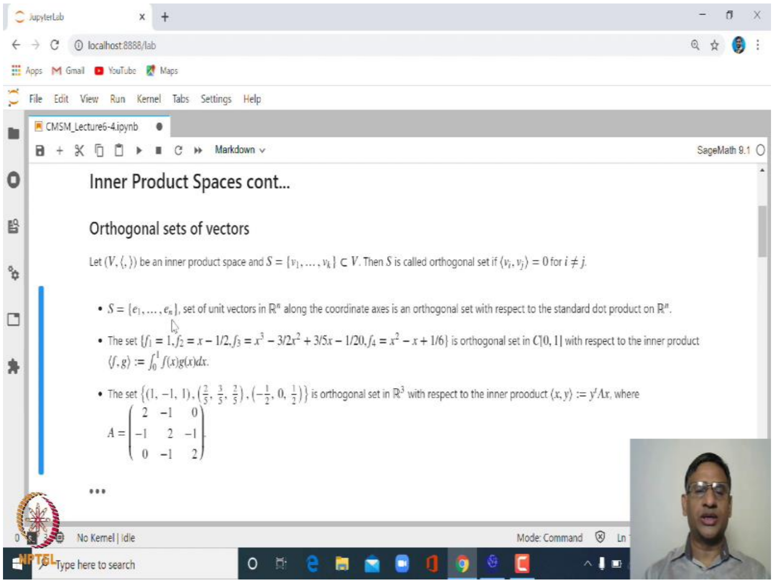


Welcome to the 38th lecture on Computational Mathematics with SageMath. Let us continue exploring some more concepts in Inner Product Spaces.

Let us define, what is meaning of an orthogonal set of vectors in an inner product space. Suppose V is an inner product space and S, a subset of V containing vector v1, v2, vk, this could be infinite subset as well. Then we say that S is an orthogonal set if inner product of vi with vj is 0 whenever i is not equal to j. That simply means, if you take any two distinct elements in S they should be orthogonal. Such a set is called an orthogonal set.

For example, if you look at the standard unit vectors e1, e2, en with respect to the usual dot product they are orthogonal set of vectors. In fact, each of this vector has norm 1. Such a set is also known as orthonormal set.

Similarly, f1 is equal to 1, f2 is equal to x minus half, f3 is equal to x to the power 3 minus 3 by 2 x square plus 3 by 5x minus 1 by 20 and f4 is equal to x square minus x plus 1 by 6.

You can check that this is an orthogonal set with respect to the inner product defined on C[0, 1], which is inner product f g equals to integral of f (x) into g(x) from 0 to 1.

So, it is easy to check that this is an orthogonal set.

Similarly, you look at set of these three vectors 1, minus 1, 1, 2 by 5, 3 by 5, 2 by 5 minus half, 0, half and define inner product on R3 as inner product x, y is equal to y transpose Ax, where A is this matrix. This is symmetric positive definite matrix. This example we have dealt with earlier. So, you can check that this given set of vectors are orthogonal.

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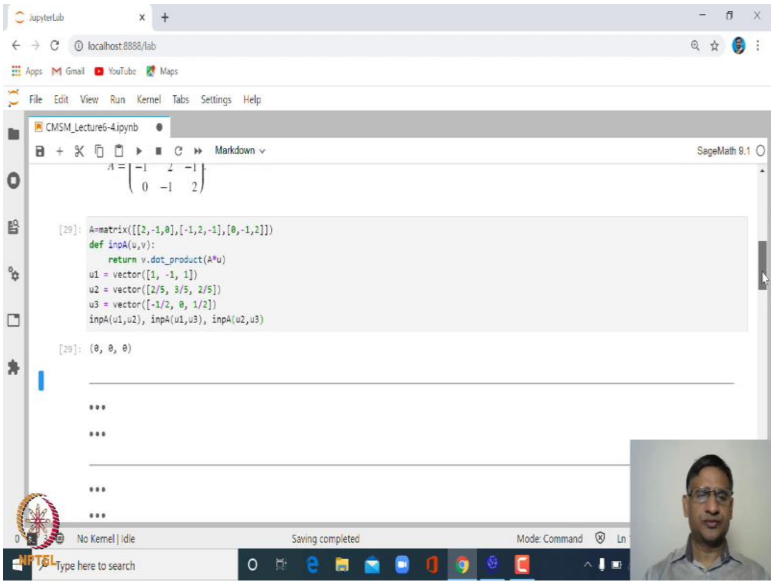


Let us check that. Define a matrix A and inner product this we have done already and two three vectors u1, u2, u3 and let us check inner product a of u1 with u2, u1 with u3 and u2 with u3, all of this should give you answer 0, That is, correct. So these three vectors with respect to this inner product are orthogonal to each other and therefore, this set of vectors is an orthogonal set. Similarly, you can check the previous one. I will leave that as an exercise.

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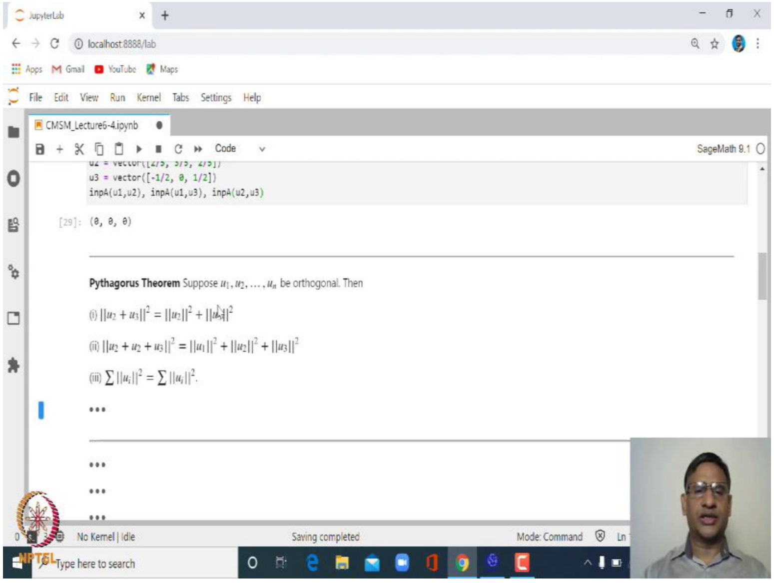


You can also verify the Pythagoras theorem.

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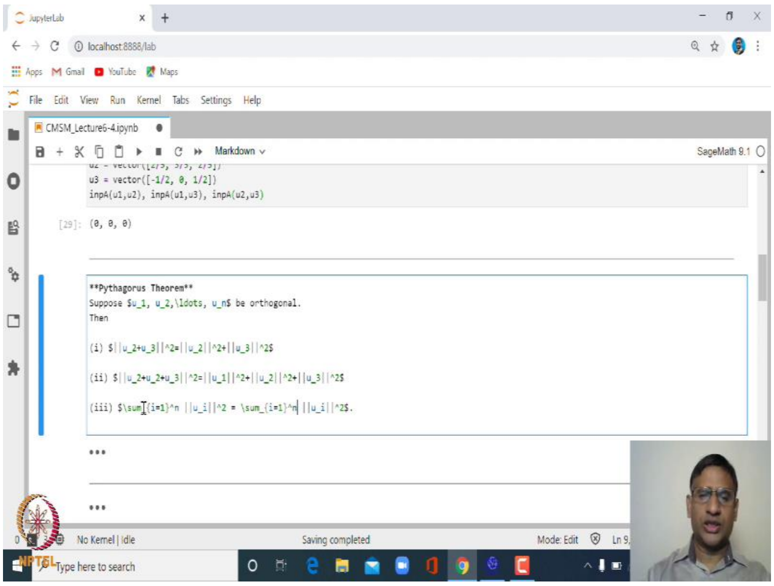


Suppose, you have vectors u1, u2, un which are orthogonal to each other and then if you look at for example, u2 plus u3 the norm square this is same as norm of u1 square plus norm of u2 square. This is, if u1, u2 are orthogonal, u1, u2 forms a right angle triangle.

This says that hypotenuse square is equal to base square plus height square and you can extend this to three vectors, u1, u2, u3 norm square is equal to u1 norm square plus u2 norm square plus u3 norm square. You can extend this to any set of finite vectors. This is only for the finite set of vectors.  Let me put here i is equal to 1 to n. So, also i is equal to 1 to n. This n is any natural number.

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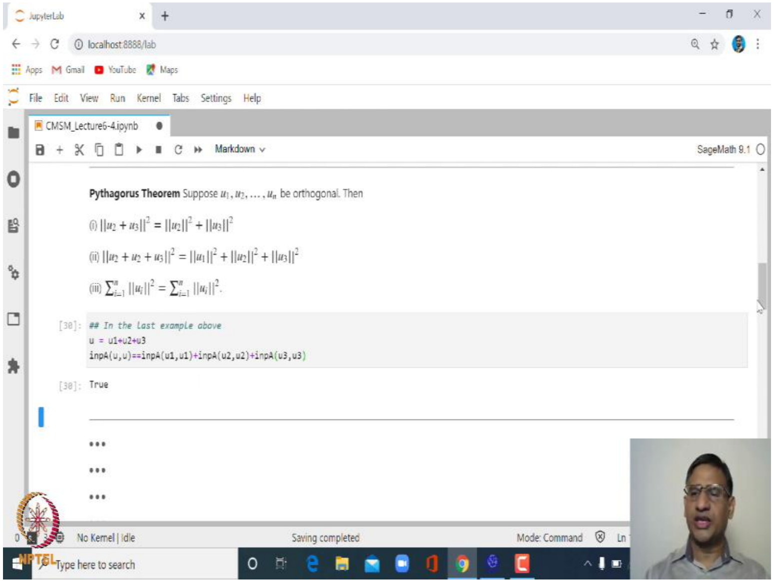


So, let us try to verify this Pythagoras theorem for the above example where you have u1, u2, u3 and in R 3 and this is the inner product. Let us define u to be  u1 plus u2 plus u3 and define the norm of u square is it equal to norm of u1 square plus norm of u2 square plus norm of u3 square,  you should get answer true, that is  correct. So this we have verified for three vectors, but you can generalize this to any finitely many orthogonal set of vectors.

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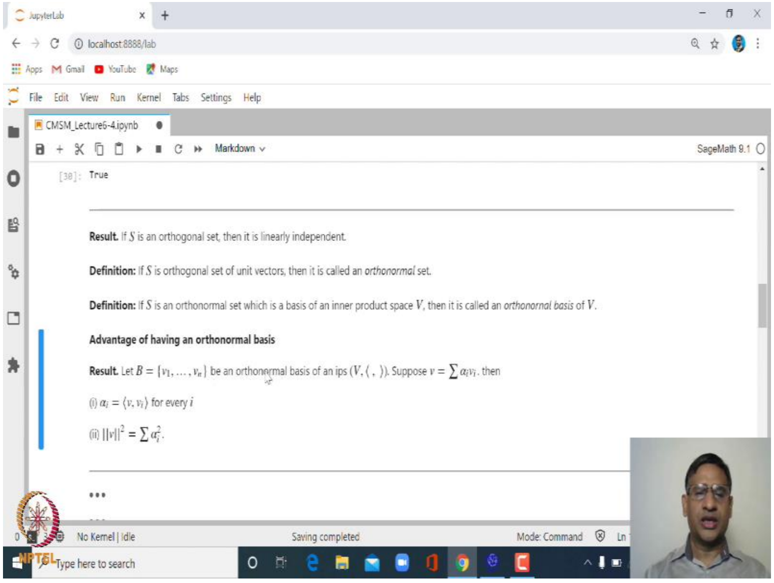


Now, let us look at, suppose you have S which is an orthogonal set then one can show that this is a linearly independent set of vectors. This is S is linearly independent.

So, any orthogonal set is linearly independent. In case, we have orthogonal set of unit vectors, it is going to be orthogonal and unit length, we call such a set as orthonormal set.

And, in case you have an orthonormal set which is also a basis of the inner product space V then such a basis we call as orthonormal basis.

Now, what is advantage of having an orthonormal basis? Let us just look at. Suppose you have a basis B containing v1, v2, vn, an orthonormal basis of an inner product space V.

Then, if you take any vector v, we know that v can be written as linear combination of v1, v2, vn and in that if v is summation alpha i v i then alpha i can be obtained as inner product of v with vi. That is quite easy to verify because you can just take inner product with v on both both sides. And, then you take inner product with vi on both sides on the right hand side v i inner product with vj will be 0 when i is not equal to j. So, only surviving term will be alpha i and alpha i is v inner product with v i and this is true for every i.

Not only that, norm of v square will be summation alpha i square. So, that is the advantage. In case we have an orthonormal basis then finding coordinate of any vector with respect to that basis is quite easy. All you need to do is, you need to take inner product of the vector v with ith coordinate of the basis vector. So, that is the advantage. This is what exactly we had for standard basis in R n. If you have any vector x which is summation xi ei, then xi is nothing but x inner product with ei.

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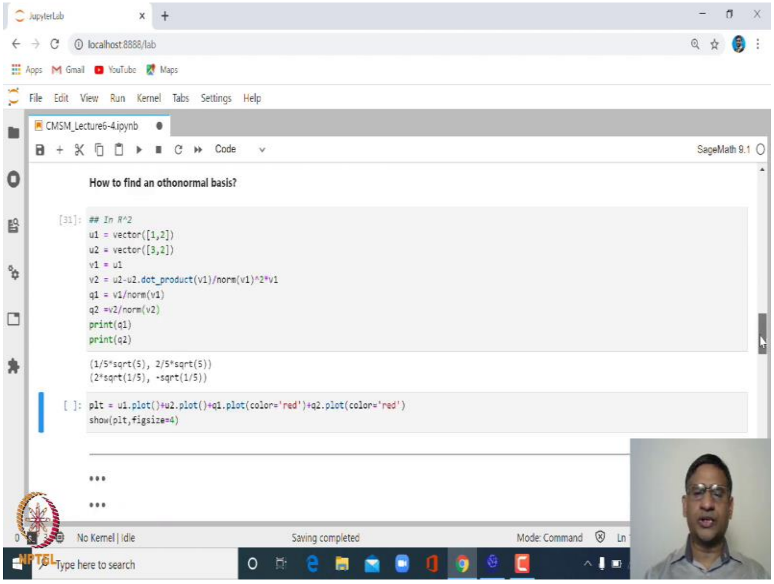


So, now the question is how does one find an orthonormal basis? Having orthonormal basis helps, but can we find an orthonormal basis. We have seen how to find a basis starting with any nonzero vector we can extend that to a basis of a finite dimensional vector space.

However, how does one find an orthonormal basis? So, this is again easy. So, let us start with R 2. So, in R 2 suppose you start two linearly independent vectors that is easy to find and then what we can do is we have seen that when we take orthogonal projection of a vector onto another vector that give gives rise to an vector which is perpendicular to a given vector, right.

We will use that idea. So, we have two vectors, let us say, u1 and u2. Then define v1 is equal, to let us say, u1 and for v2 what you do? From u2 you take out the orthogonal projection of u2 onto v1. Take out orthogonal projection of u2 onto v1. And, then let us define q1 to be v1 upon norm v1 and q2 to be v2 upon norm v2. You can check that these two vectors are orthogonal to each other.

This we have already checked. Only thing here we did was we made this as unit vector.

Now, let us try to plot graph of these two vectors u1, u2 which was given to vectors, and q1, q2 which we have obtained as orthonormal vectors.

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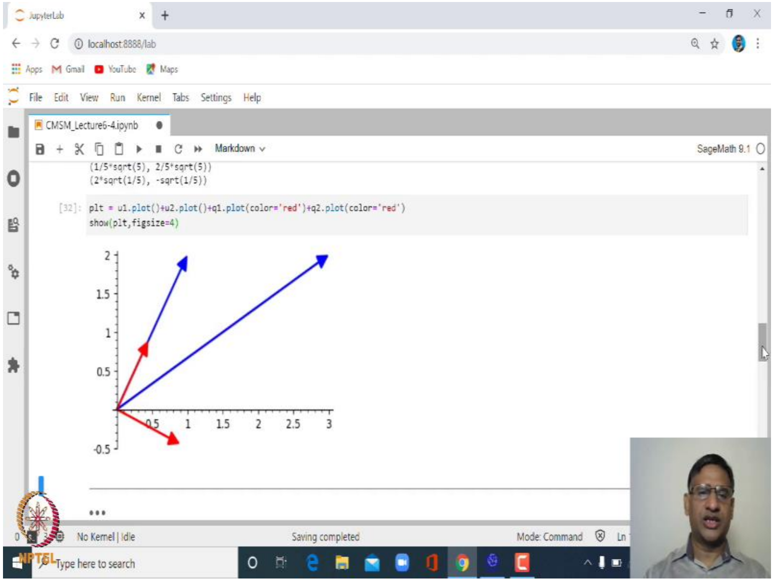


So, you can see here, this is actually u1, this is u2 and v1 we took as u1 divided by norm u1. So, it is unit vector along this direction and if you take the perpendicular of u2 on to u1 and then take out that orthogonal component this is what you get. This is your q2, right.

So, this is how we can generate two or two set of vectors which are orthogonal to each other and also they are of unit length. Once you generate orthogonal set of vectors, then you just need to divide by its length to make it orthonormal.

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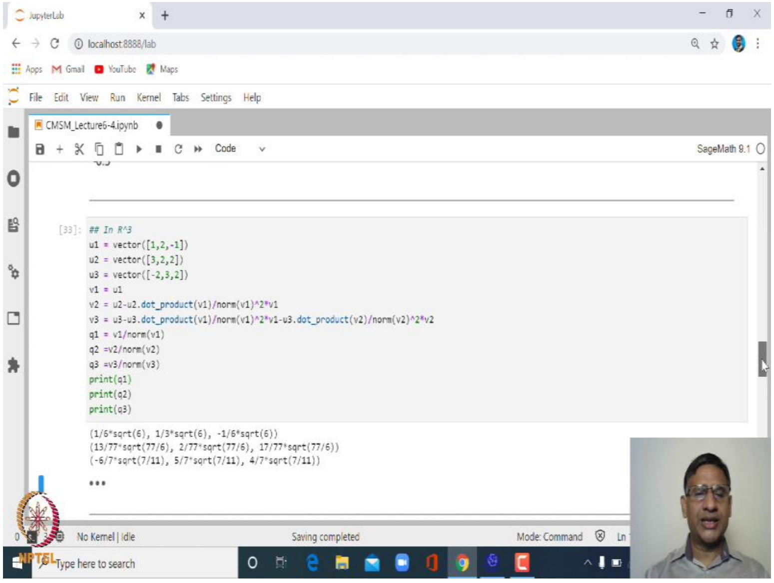


So, let us extend this in R 3. Suppose I have three vectors u1, u2, u3 in R3. Now we will do the same thing again. We start with v1 is equal to u1 and v2 is is a vector, how is it obtained?

Again, it is in exactly in a similar way that we obtained in the previous case. So, from u2, you take out the orthogonal component of orthogonal projection of u2 onto v1. How do you define v3?

To define v3, from u3 you subtract orthogonal projection of u3 onto v1 and also subtract orthogonal projection of u3 on to v2. And, then define q1 is equal to norm v1 upon v1 upon norm v1 q2 to be v2 upon norm v2 and q3 to be v3 upon norm v3.

Let us look at, what are the these three vectors. You can check that these three vectors are orthogonal to each other and they are of unit length.

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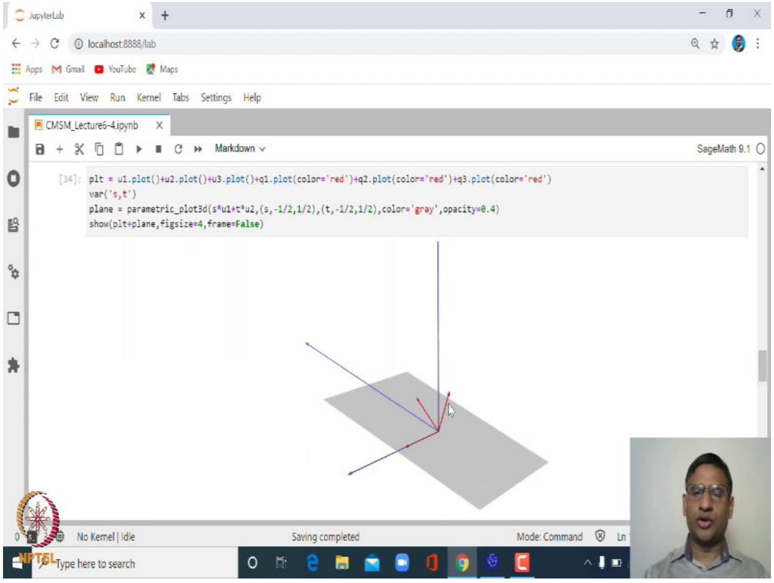


Now, again let us plot graph of these vectors, the graph of u1, u2, u3 along with the orthonormal set of vectors that we have obtained.

So, in this case you can you can just check. Let me rotate little bit this. So, this may be u1, this may be u2 and you have taken orthogonal projection of a u2 upon u2 on to u1 and then you have taken out. So, this is your v1, this is your v1 upon norm v1 and this is v2 upon norm v2, this is v3 upon norm v3. So, this is q1, q2, q3.

And, so, what you can see here, this is the plane which is shown, is plane spanned by u1 and u2 and the vector q3 is perpendicular to this plane. So, this is how you obtain three vectors which are orthonormal in R 3.

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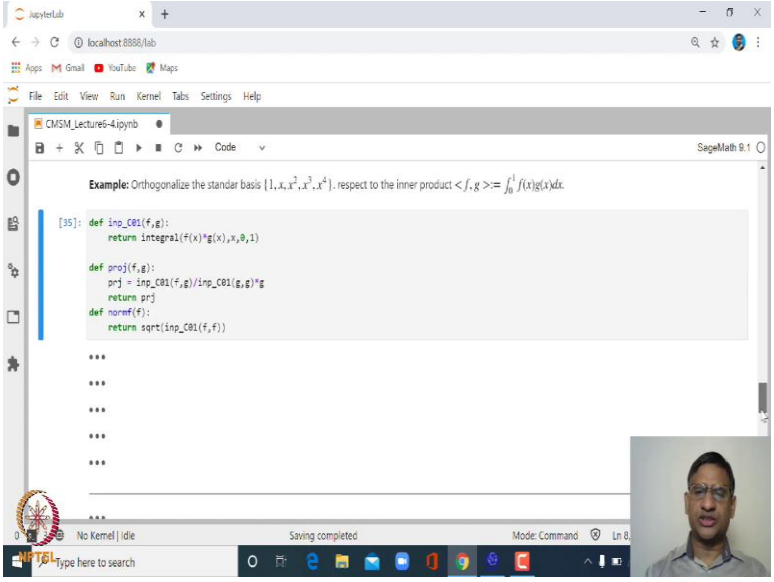


And, you can extend this idea to more vectors. So, for example, if I take four vectors 1, x, x square, x cube and x to the power 4 in the set of all polynomials of degree less than equal to 4 with respect to this inner product, we can extend this idea. So, first let us define inner product and the projection and the norm of vectors with respect to this inner product.

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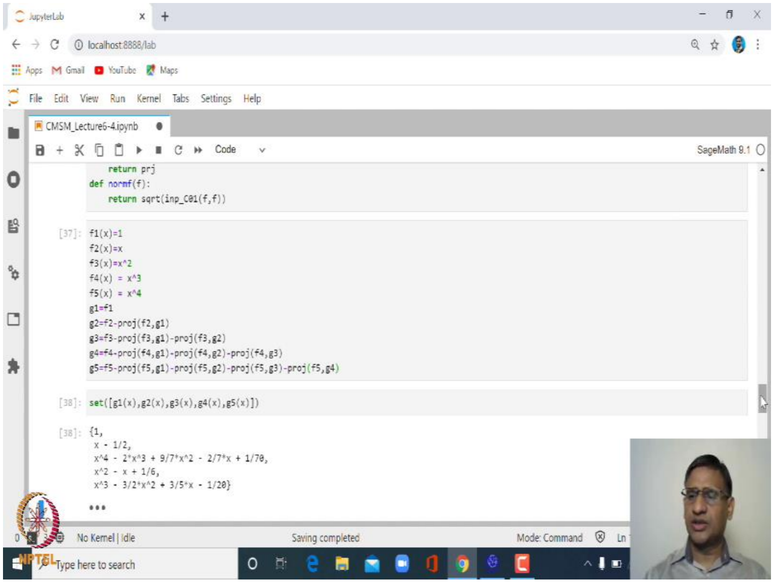

And, then let us define f1, f2, f3, f4, f5 these are five set of vectors. This forms a basis of of P 4 R. And, then again the idea is similar. Define g equal to f1, g2 is equal to f2 minus orthogonal projection of f2 onto g1, g3 again in the same way, g4 and g5.

So, in general what is gk? gk is defined as, from fk you take out the orthogonal projection of fk onto all the previous orthogonal set of vectors that have been obtained. So, let us execute this and let us also try to plot graph of this function. Let me first show what are these functions which are obtained 1, x minus half, x to power 4 minus x cube and so on. These are the the vectors. One can check that these are orthogonal set of vectors with respect to this inner product.

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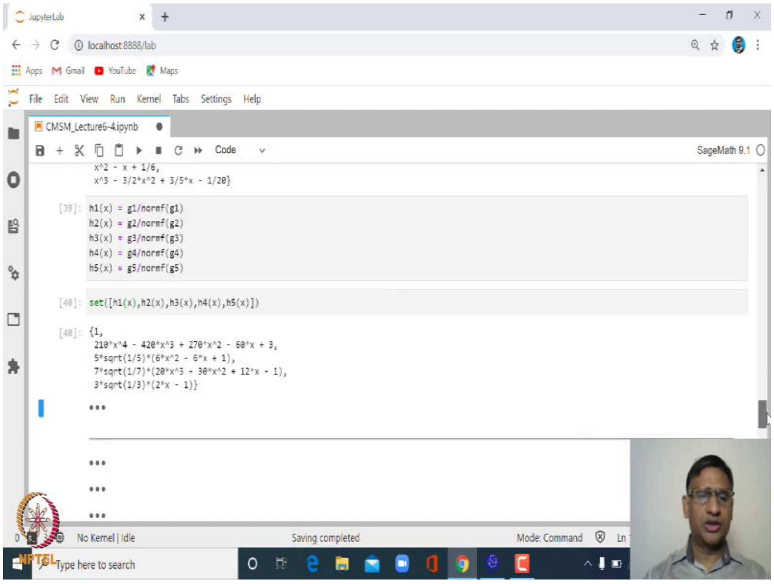


Let us define now unit vectors. So, divide each of this g1, g2, g5 by its norm and then these are the vectors we have got. So, this is an orthonormal set of vectors.

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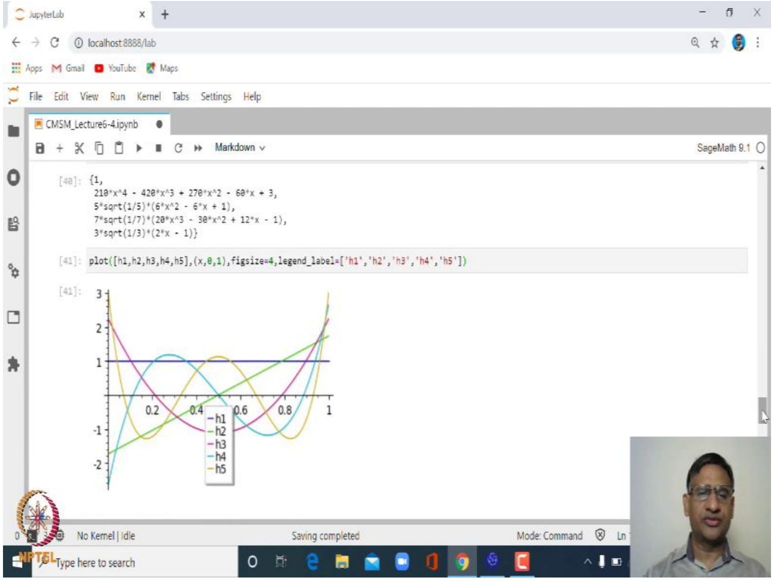


Let us also plot graph of all these functions. So, is this is straight line is h1 which is a constant function 1; h2 will be some kind of a straight line and h3, h4, h5. This is how these vectors, these orthonormal set of vectors look like.

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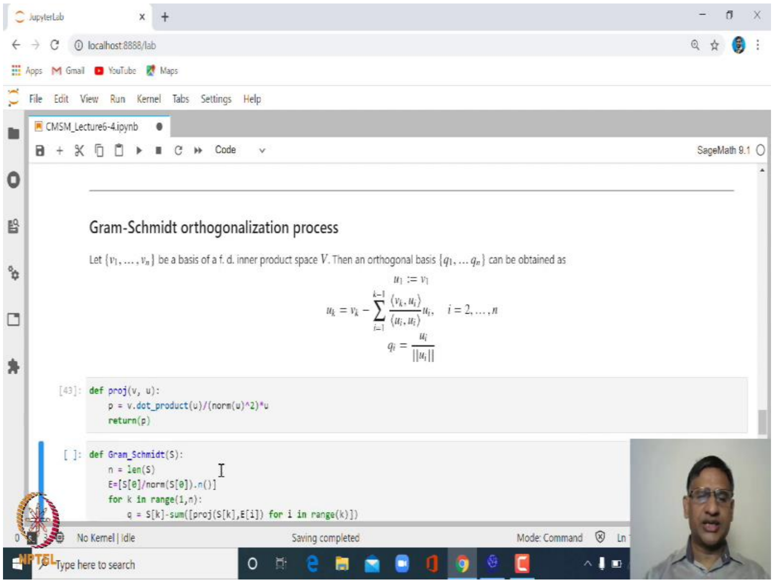


Now, you can actually extend this to any finite dimensional inner product space. this process is what is known as Gram-Schmidt process. So, what is this?

It says that, if you start with any set of vectors v1, v2, vn, a linearly independent set of vectors which forms a basis.  in particular take any basis v1, v2, vn of the finite dimensional

inner product space V and then then you can find an orthonormal basis q1, q2, qn. And, how do we obtain?  The process is exactly similar. So, you start with u1 is equal to v1 and then define uk to be vk minus orthogonal projection of vk upon ui  i going from 1 to k minus 1.

So, what we are doing? We are throwing out orthogonal projection of vk on to all the previous orthogonal set of vectors that we have obtained and then defined qi to be ui upon norm ui. This is what is called Gram-Schmidt orthoganalization process. This is quite simple.  We already did it for 2 vector, 3 vectors, 4 vectors and 5 vectors.

Let us create an user defined function for this, create a Sage subroutine for this. So, how do we do that? So, let us work with standard inner product in Rn.

We have the projection of a vector v onto u,  which is v dot u divided by norm of u square into u. So, that is the orthogonal projection.

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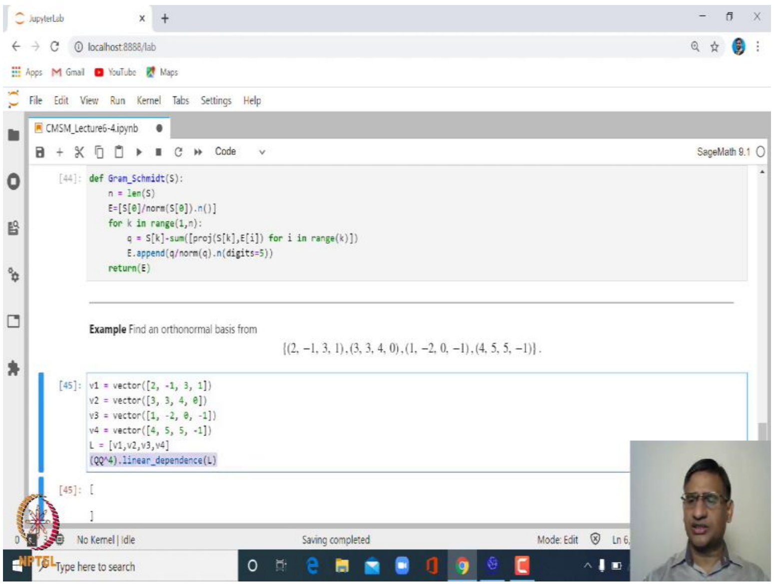


Next let us create Gram-Schmidt orthoganalization process subroutine. What we are doing? We have to pass a set of vectors which are linearly independent.  Define n to be the length of this set, that is number of vectors in S. Then initialize E; E is going to be the set of orthonormal vectors. So, first let us define unit vector which is the first vector in S, initialize that as first vector in E. And then what we do? Run a loop for k going from at 1 to n.

That means, it will start from actually the second vector in S. So, what is it? Define q2 to be Sk which is kth element in S from that you take out the orthogonal projection of Sk onto Ei, where i going from 1 to range k; that means, k minus 1. So, that is what is this loop. And, then append inside a the unit vector, that is q upon norm q. Here just I am saying that the display only 5 digits because this is going to convert into float and then return E. So, that is a very simple user defined function for Gram-Schmidt process.

And, of course, if you want to extend this to arbitrary inner product space, you may include the inner product with respect to which this orthogonal projection has to be taken.

So, that is quite easy. Now let us let us look at an example.

So, let us take these four vectors in in Q4 or in R4 and then let us define these vectors v1, v2, v3, v4 and L is equal to list of v1, v2, v3, v4. We can check that this set of vectors v1, v2, v3, v4 are linearly independent and hence it forms a basis of Q4.

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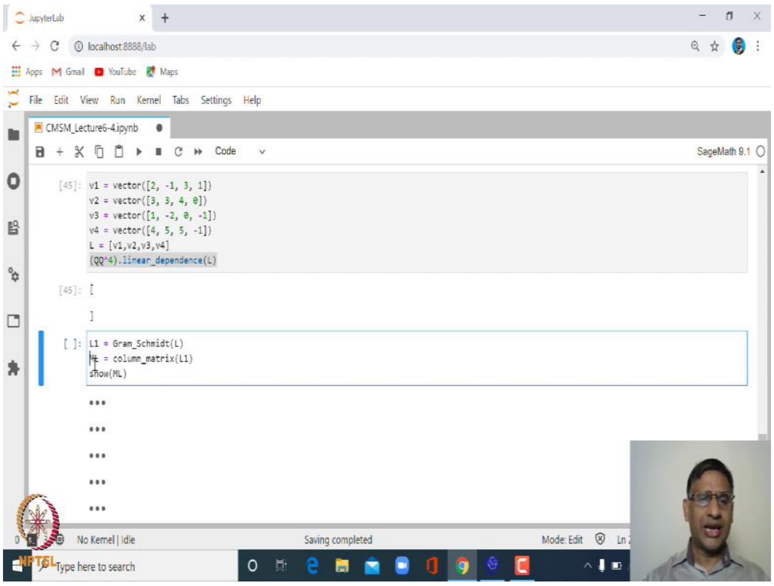

Now, let us call this Gram-Schmidt function which we have created and apply this to set of vectors L.

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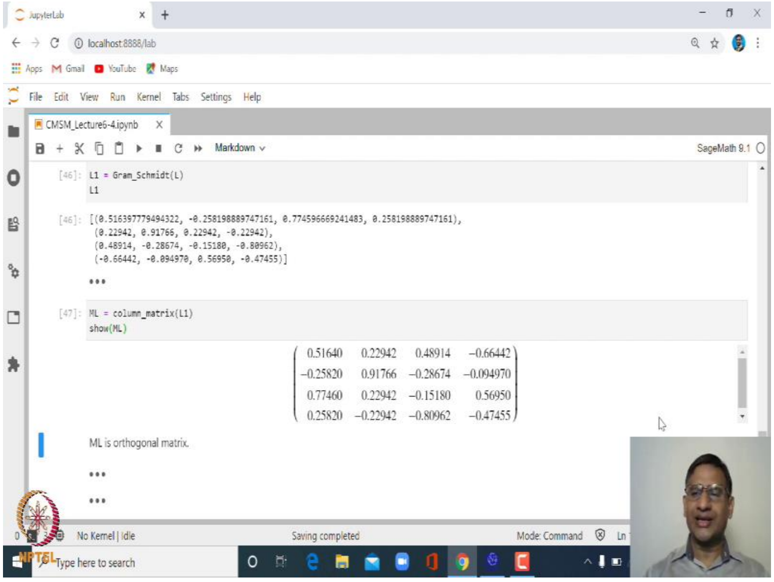


And, what we obtained is again a list of vectors L1. This is this is what we obtained. And, then let us create a matrix out of this. So, this the matrix, the first column is first element in this L1, second column is second element and so on.

And, this matrix, if you look at this matrix, this is actually a matrix whose columns are orthonormal set of vectors. Such a matrix is known as orthogonal matrix.

So, any matrix Q in which all the columns are orthonormal set of vectors, then we say that such a matrix is orthogonal matrix.

And, in case you have orthogonal matrix and if you multiply this matrix with its transpose, you will get identity matrix. So, of course, this should be square matrix here.

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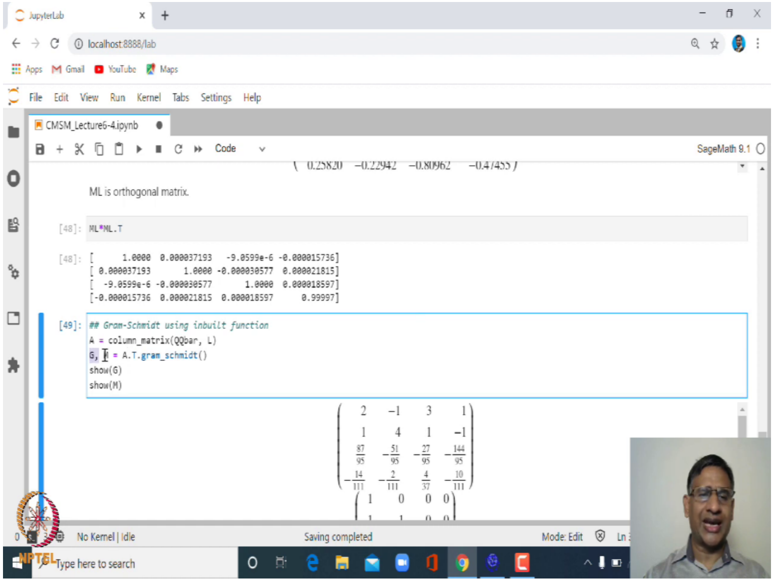


So, let us check that. We can very easily check here this, ML into ML transpose, ML times ML dot T, this should give identity matrix.

Of course, here this these things are almost close to 0, so, off diagonal entries of the order 10 to the power minus 6, because here there are lot of round off error.

It has some error, but you can see here, this is diagonal entries 1 1 1 this is also close to 1.

Next let us look at, there is a inbuilt function to find Gram-Schmidt orthoganalization process. Suppose you define a matrix which is a column matrix of a set of linearly independent vectors.

So, we have already defined what is L. L is set of linearly independent vectors v1, v2, v3, v4, in this case and define A to be column matrix over QQ bar.

So, it is extended rational field you can define over RDF and CDF. Then on A transpse, let us apply Gram-Schmidt gram underscore Schmidt and this gives you actually two matrices.

This gives you two matrices, output is G and M, I have stored this in G and M.

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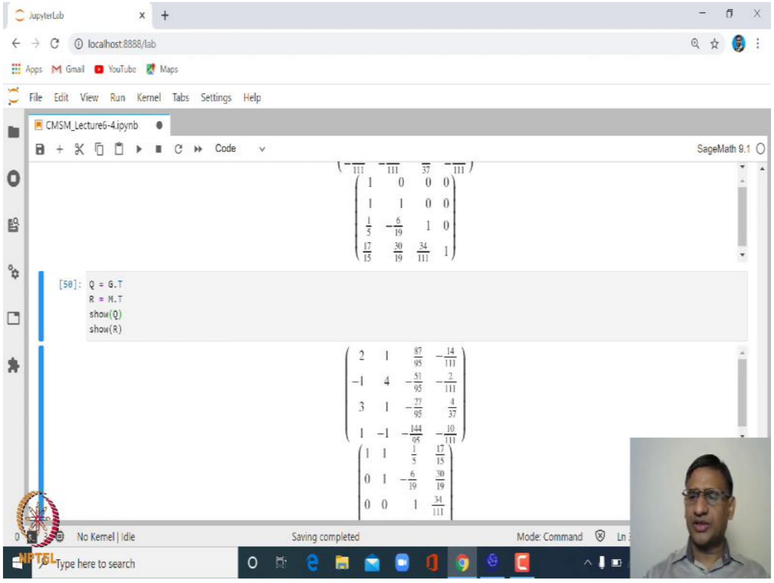


So, in this case, this is G and this is M.  M is lower triangular matrix.  If you look at  and if I take the transpose of each one of this. Let us say transpose of this defined in Q and transpose of this defined in R and then this is what we get.

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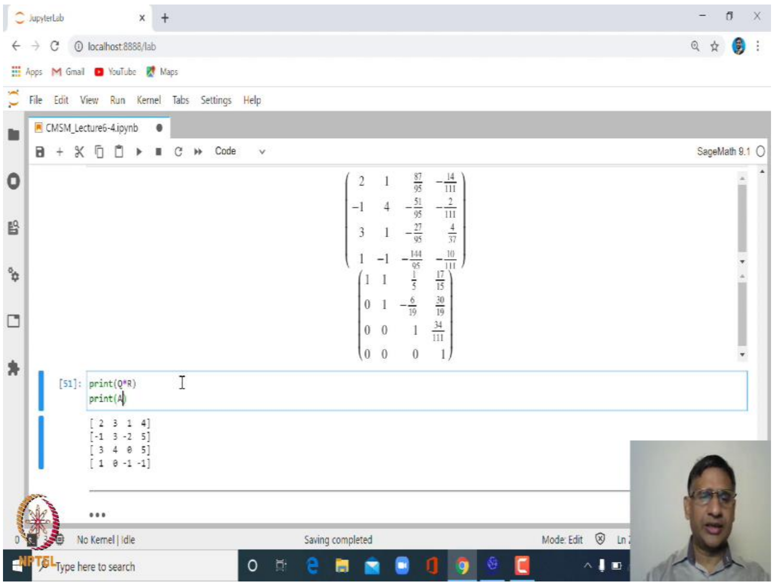


And not only that if you try to multiply these two  Q and   R,  multiply Q and R,  this is what we got using QR using Gram-Schmidt,  and then when you multiply this you will get this matrix A. This is what the matrix A if you if you look.  Let me just also print, what is this matrix A. So, print A.

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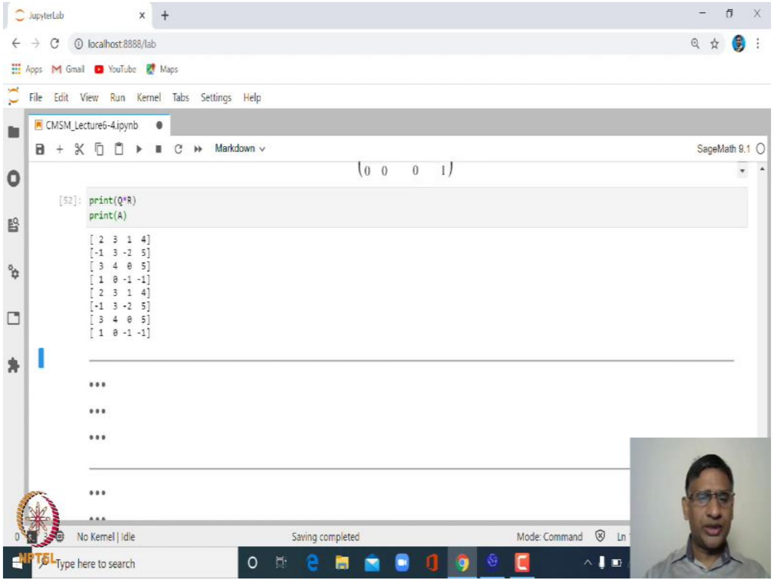


So, that both are the same. So, Q into R is A, this also known as actually QR factorization. Here Q, though it is not an orthogonal orthogonal matrix, but this is set of orthogonal vectors not orthonormal.

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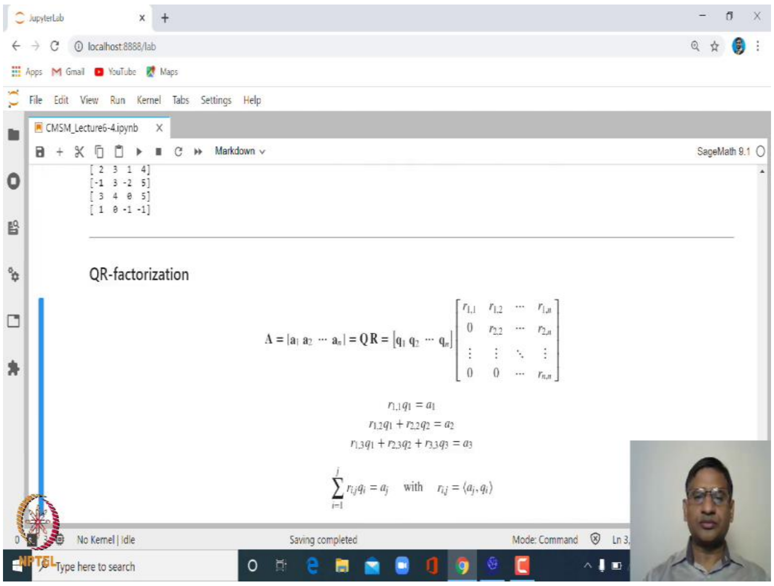


However, one can define in what is called QR factorization and then QR factorization can be obtained using Gram-Schmidt process. So, let us see what is it? So, it it says that if you have any matrix let us say A, it could be even square matrix any rectangular matrix or even non invertible matrix.

Let us write A as column a1, a2, an then it says that there exists an orthogonal matrix Q and an upper triangular matrix R such that A can be written as Q into R.

Now, if I say Q is given by column matrix q1 to qn and R is given by r11, r12, r1n, 0, r22 and so on. Then you can easily multiply these two matrices Q and R.

And, then equate both sides. What you will get? When you multiply this, by first column, you will get r11 times q 1 is equal to a1 and if q1 is orthonormal you will get what should be r11.

r11 will be nothing, but norm of a1 upon norm of a1 actually. Similarly, multiply this by second column, what will get is r12 times q1 plus r22 times q2 is equal to a2. Since q1, q2 are orthogonal

if you take inner product of both the sides with q1 you will get value of r21 and that will be inner product of a2 by q1. Similarly, if I take inner product of this whole thing by q2 then this will vanish, we will get r22. So, and you can continue this process.

So, in general what you have is summation rij times qi, i going from 1 to j, is nothing, but aj. That is, j-th column of A is product of this multiplied by j-th column of this R. And, from this you can take inner product of the both the sides with qi, you will get rij. So, you can obtain rij. Once you have obtained this q1, q2, qn, which we already did in case of Gram-Schmidt process.

This rij can be obtained using this formula and this is what is known as QR factorization, which is a very useful concept.

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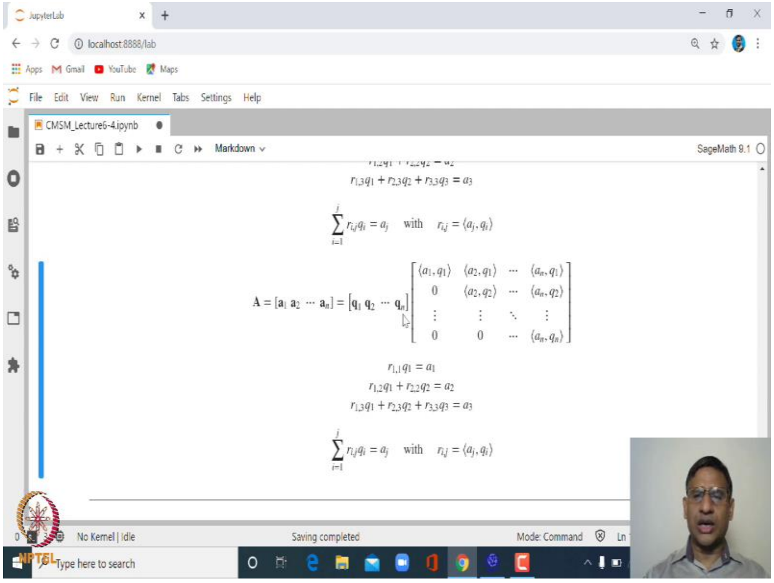


Let us see what is it. Once you have obtained, this is just a repetition. So, this once you have just obtained A is equal to QR, then rij is given by this. Let me just get rid of this part, this is just a repetition.

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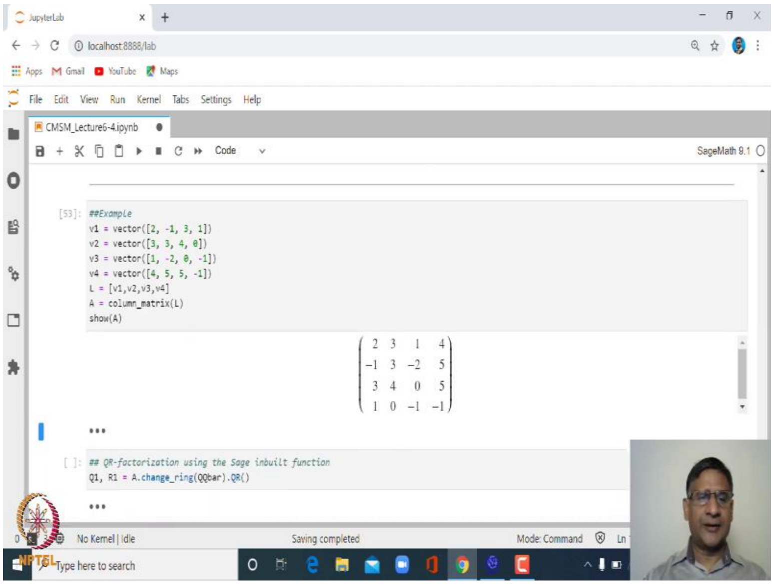


Then next let us look at how we can obtain this QR factorization.

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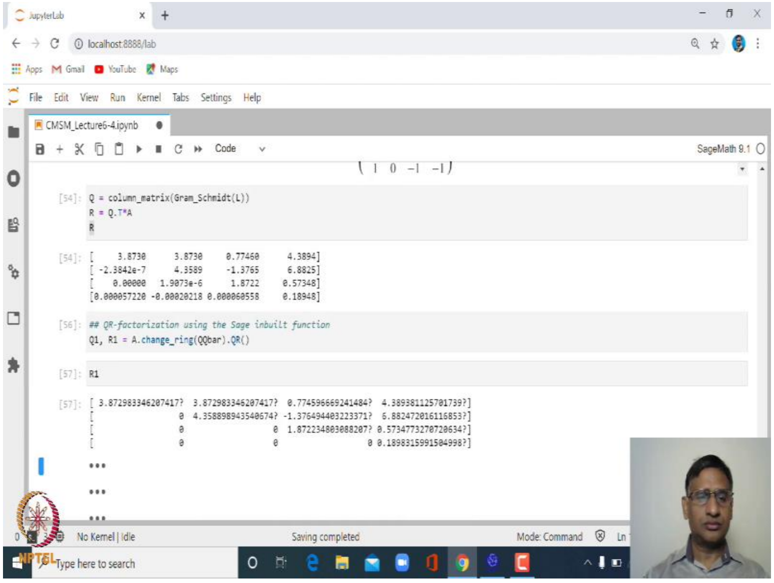

This is the again the same set of vectors and A is column matrix. Then we can obtain this Gram Schmidt process.

So, this is the column matrix, the Gram Schmidt orthoganalization process. This is what we did already earlier. And then let us define R is equal to Q transpose A. You see that Q is orthogonal matrix. So, Q into Q transpose will be identity. Therefore, Q transpose is nothing, but inverse of Q. So once we have obtained Q, we can obtain R by taking the Q inverse times the matrix A. You can also obtain this QR factorization using inbuilt function this is known as QR. So, define a matrix A which is over QQ bar and then obtained QR factorization of this and this again we will give you Q and R. So, store this in Q1 and R1 and if you multiply Q1 and R1. Let us see what is R1. This is R1 which is upper triangular matrix and diagonal entries will be non-negative in this case.

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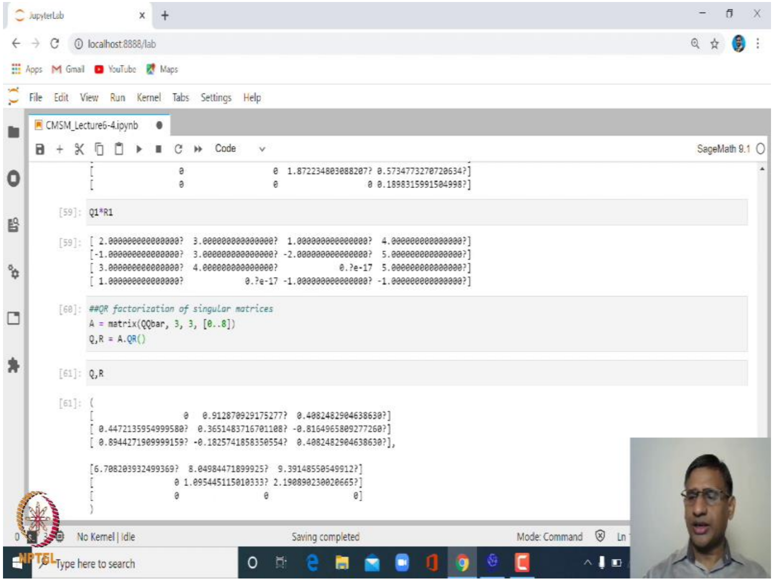


And, then let us you can check that Q1 times R1. Let us check Q1 times R1 this would give you the original matrix. Of course, it gives you in decimal because these computations are numerical computations. So for example, question mark says that it is a kind of approximation.

So, you can find QR factorization of any matrix. For example, even you can have a singular matrix. So, here what is it? This is 3 cross 3 matrix starting from 0 to 9. So, 0 1 2 3 to 0 1 2 in first row second row is 2 3 4, third row is 5 6 7. You can find QR factorization of that as well. So, in this case you can see what are Q and R. So, let me print Q and R, in this case this is what you get, right. So, it is not necessary that the matrix A has to be singular, non-singular matrix.

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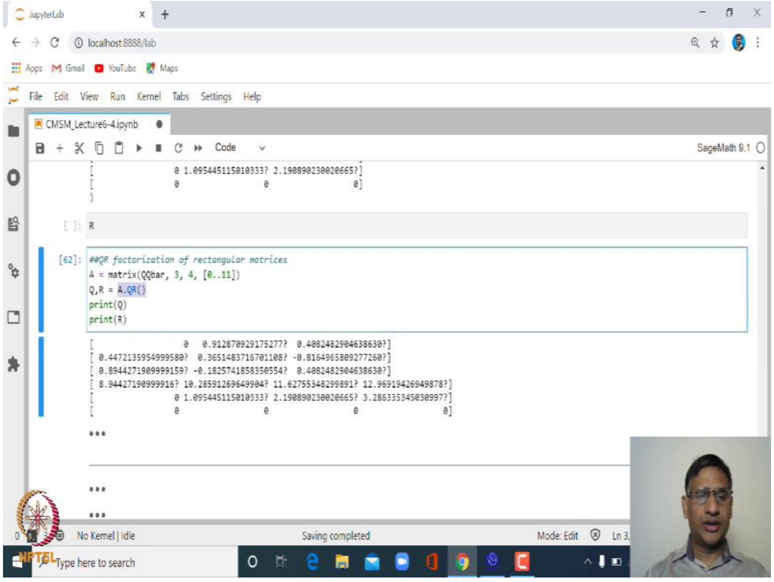


Now, you can also have a matrix which is rectangular matrix. This is I should write rectangular, you can have rectangular matrix, need not be square matrix.

So, if I have rectangular matrix, let us say this is 3 cross 4 matrix.  Again you can find its QR factorization using inbuilt function A dot QR.

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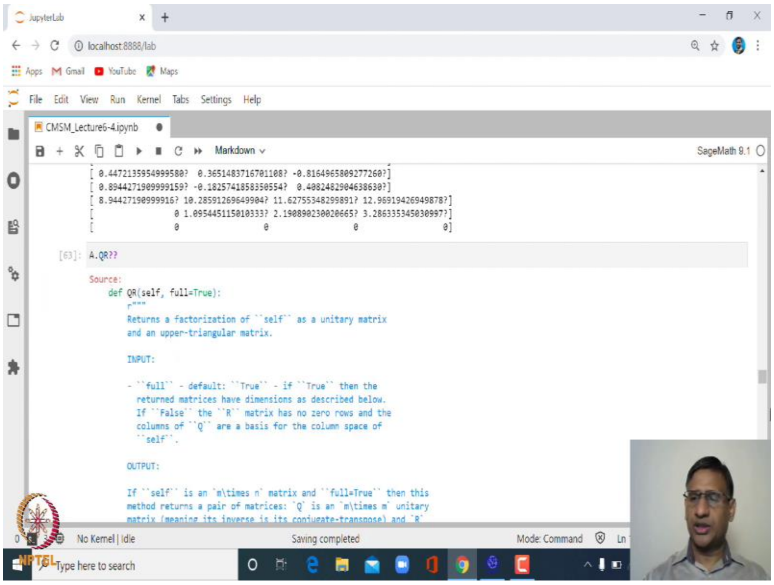


Next you can take help on QR factorization. So, if you have a matrix A and then say A dot QR double question mark or single question mark, you will see a help document.

And, that help document you can go through along with the examples and then see how you can make use of various options. This is the QR factorizations source code.

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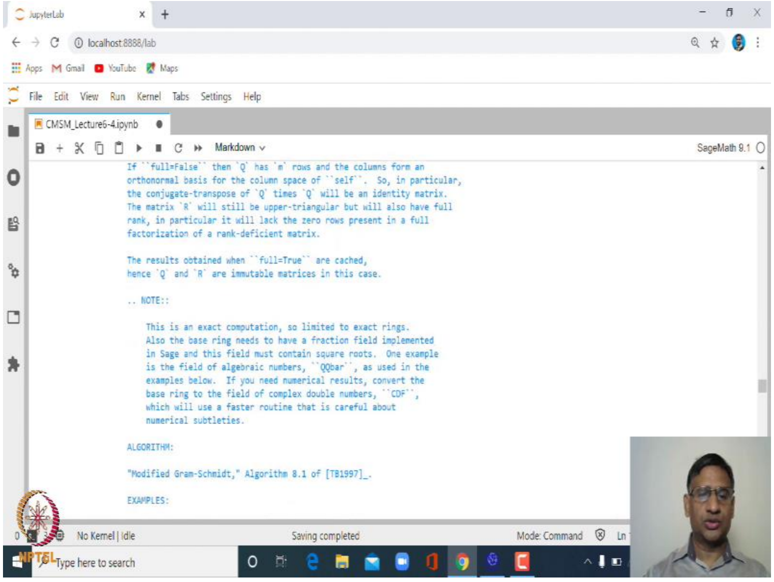


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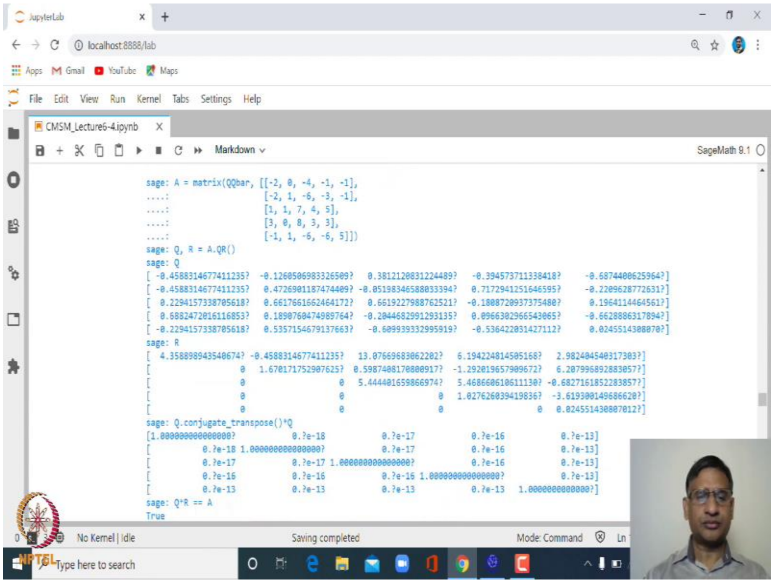


So, it also gives you the source code and here there are several examples. You can see here this is  bigger example. It is actually 5 cross 5 matrix.

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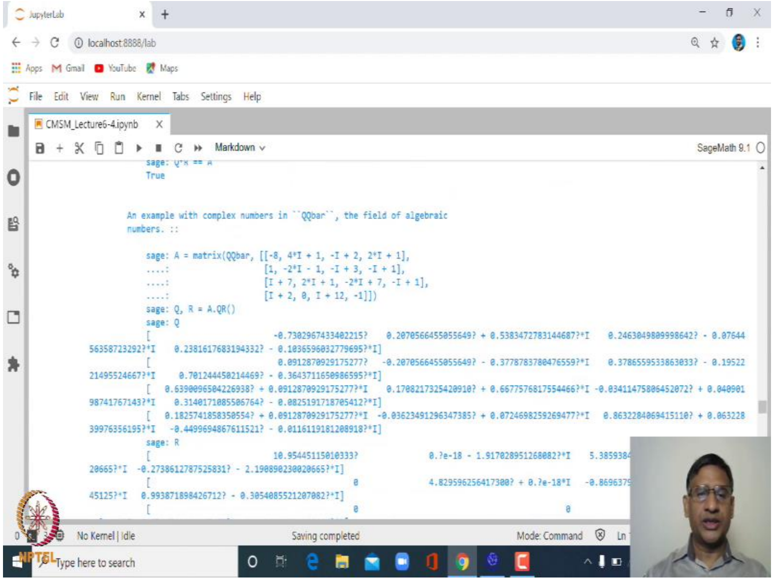


And, it can also find QR factorization of complex matrices. So, that is what it it says and so on.

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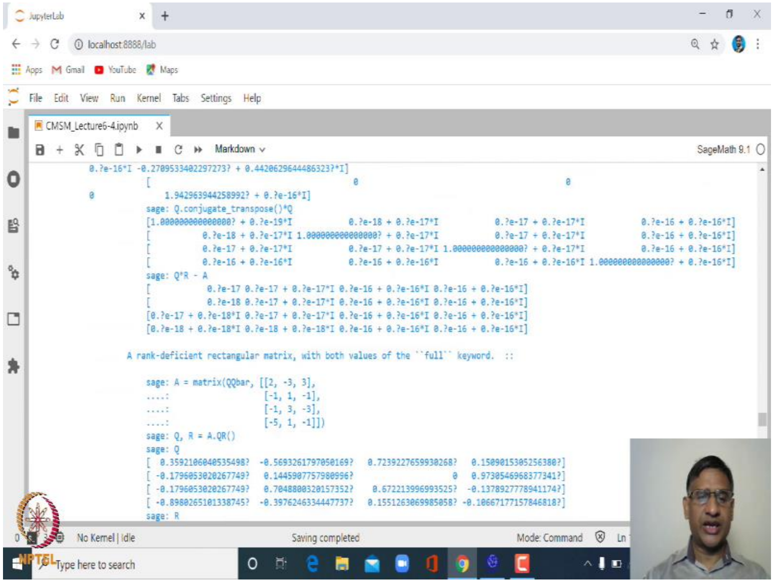


There are different examples.

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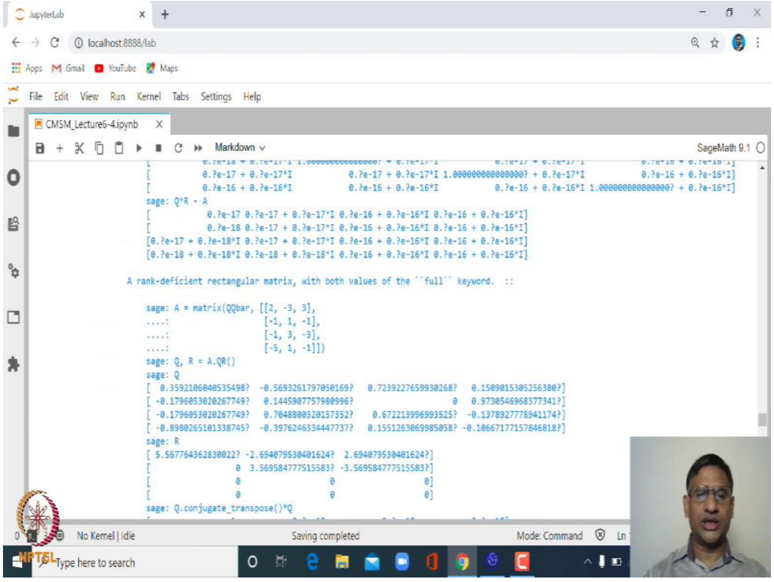

One can one can go through this list of examples.

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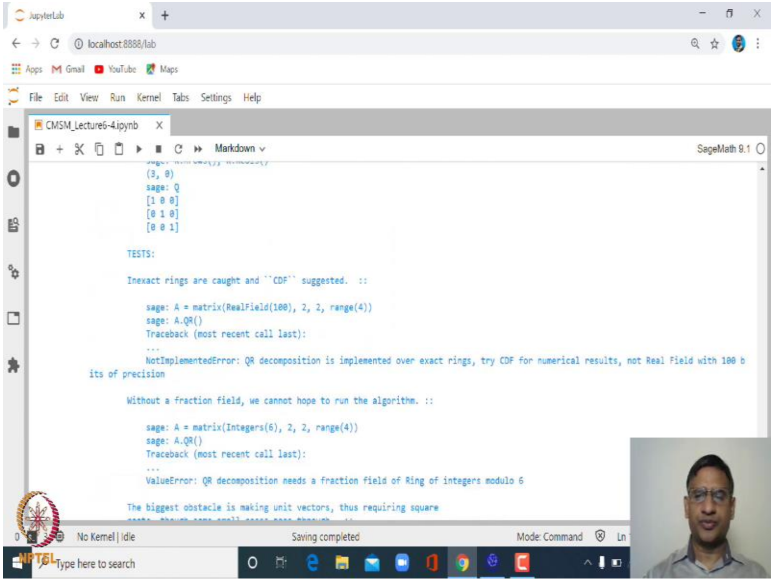

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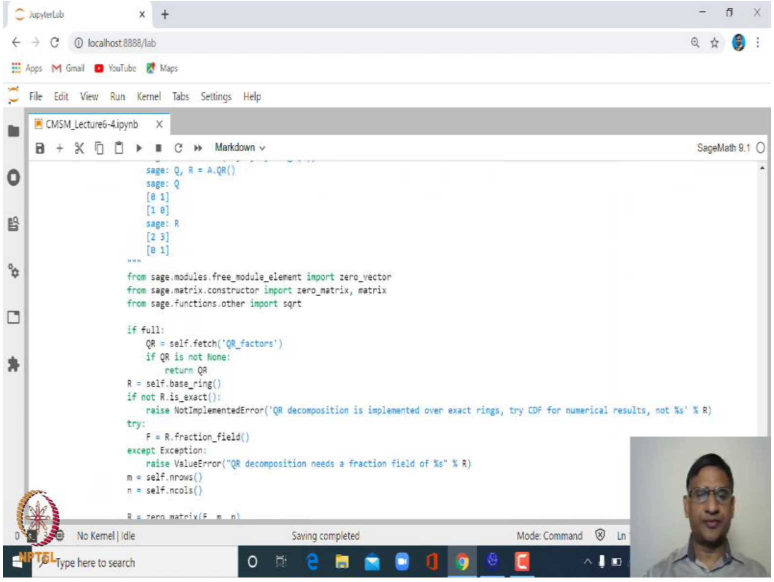

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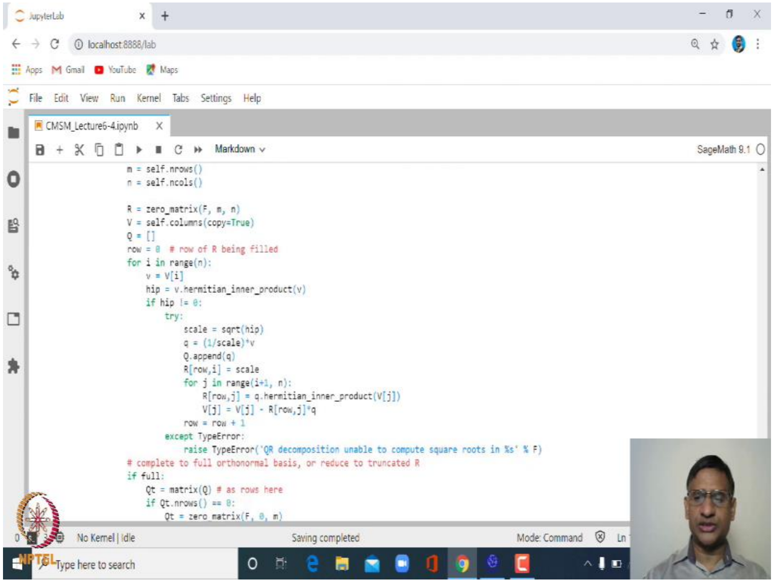

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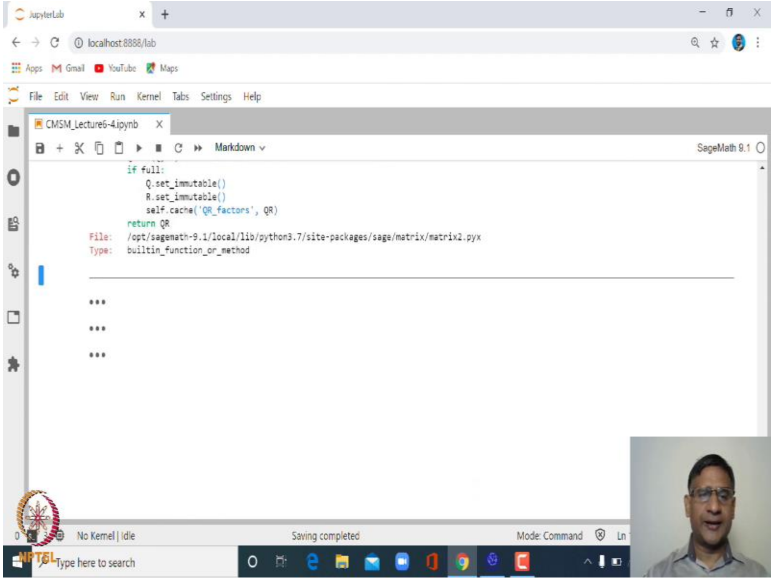


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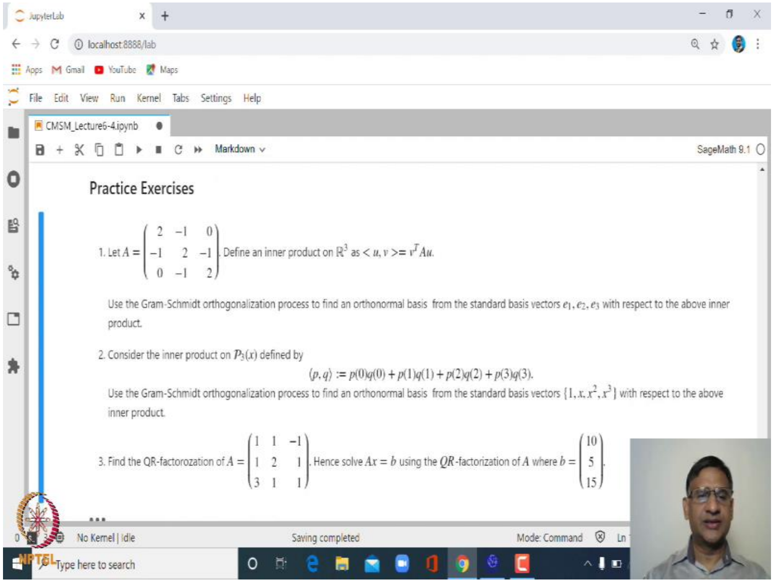


At the end let me leave you with few exercises. These are quite simple exercises.

So, first take a matrix A. Define inner product with respect to this matrix and then use Gram-Schmidt orthoganalization process to find an orthonormal basis with respect to a standard basis e1, e2, e3.

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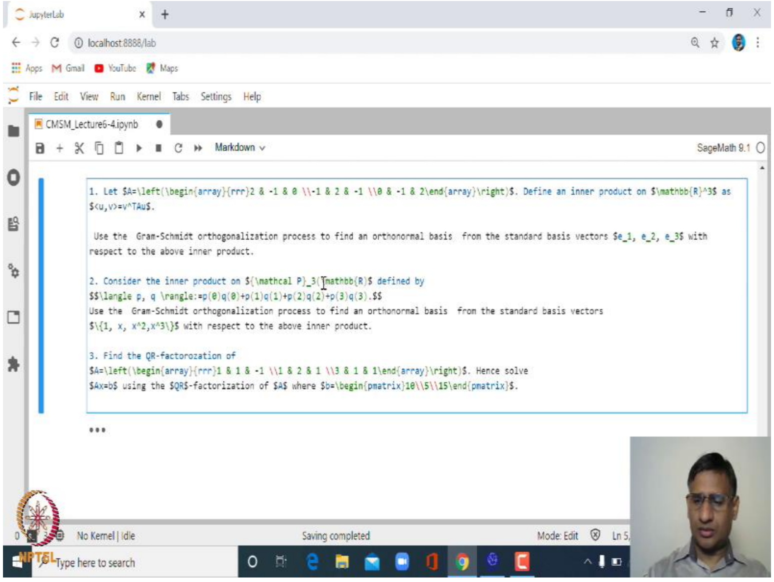


Similarly, define an inner product on P3, this I should write P 3 X, Instead of X let me write here. P 3 R and with respect to standard basis 1 to x cube. Again find an orthonormal set of basis and then find QR factorization of this and hence solve this Ax equal to b.

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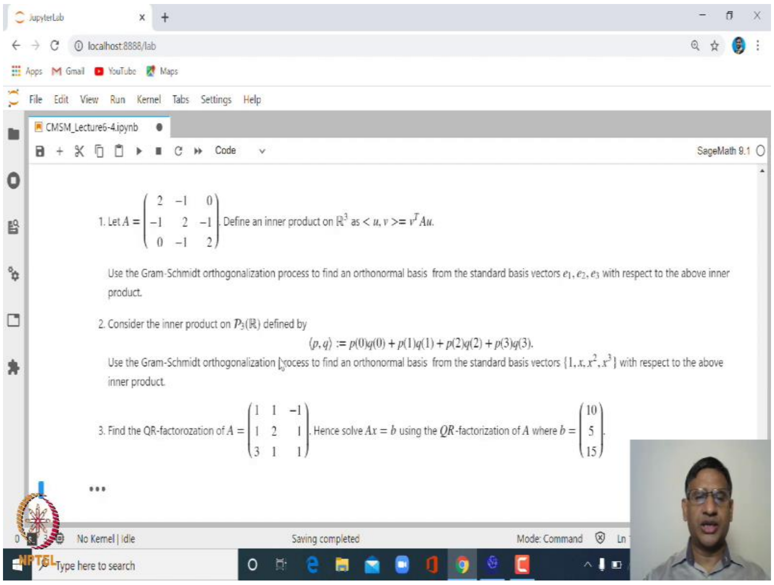


See once you have QR factorization of a matrix then this system Ax equal to b can be written as QR times x is equal to b. Now, Q is invertible. So, if you multiply both sides by Q inverse, then we will get R times x is equal to Q transpose b. Then what you have R times x and R is upper triangular matrix. So, actually to solve this system of linear equations,

it has boiled down to solving upper triangular set of linear system which is quite easy to solve using back substitution.

Of course, this may be more expensive than a Gaussian elimination method. However, once you know the QR factorization of a this coefficient matrix, then it becomes easier to solve.

So, these are set of exercises. We will look at some more concepts in next lecture.