Computational Mathematics with SageMath
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Inner Product Spaces
Standard dot product in R^n
Lecture – 37
Inner Product Part 1 with SageMath

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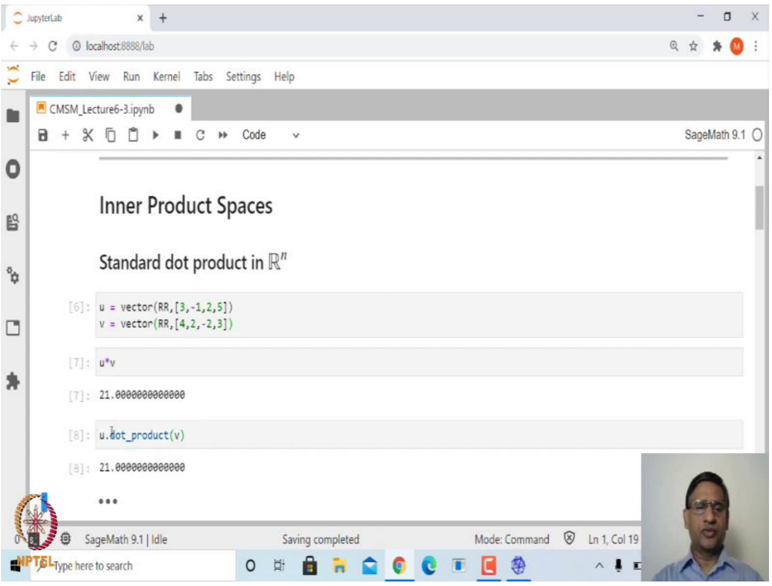

Welcome to the 37th lecture on Computational Mathematics with SageMath. In this lecture, we will explore Inner Product in SageMath.

So, first let us start with standard inner product on Rn. This we have seen earlier. For example, if you have two vectors u and v in R4, then you can define or find an inner product or dot product of these two vectors using star command.

If you say u star v, it will give you the dot product of u and v, or you can say u dot underscore product in the bracket v. u dot underscore product is a method inside SageMath in order to find the dot product.

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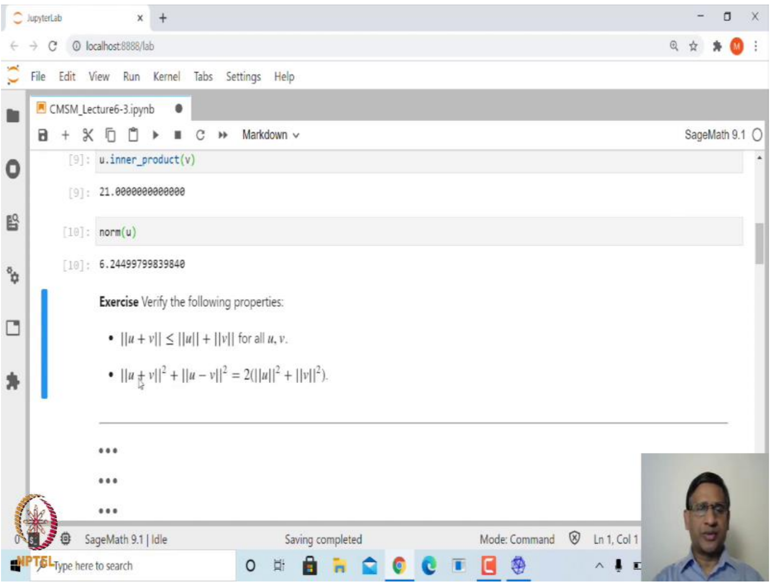

You could also mention u dot inner underscore product. By default in R n, inner underscore product will give you the standard dot product. Of course, you can also find norm of a vector that is length of a vector using norm command. You can also find norm which is in various forms like l1 norm or sup norm. I will leave this as an exercise try to verify these two properties of norm that is triangle inequality and this is what is known as parallelogram law. This is sum of the diagonals length of the diagonal square is going to be twice the length of squares of each side right.

Now, let us look at how we can define inner product on a vector space. We will work with vector space over R. If it is vector space over complex field, then definition will be slightly different.

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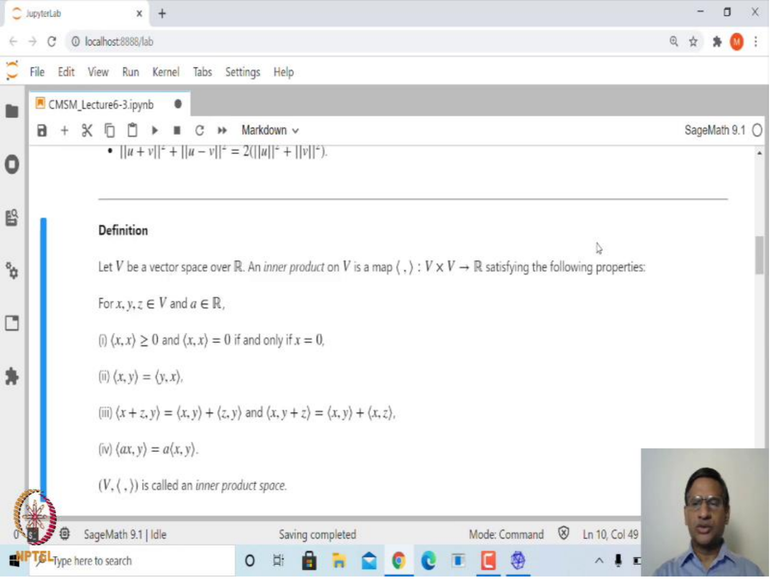

Suppose you have a vector space V over R, then an inner product on V is a map from V cross V to R satisfying the following properties.

The first property is if you take inner product x with x, that is always non-negative. Inner product x with x will be 0 if and only if x itself is 0. And it is commutative. So, inner product x with y will be same as inner product y with x. This is where the inner product defined on a vector space over complex field will differ. That will not satisfy this commutativity rather it will be slightly different.

Next property is associativity, or you can say it is linear in each of the component. So, if you take x plus z take inner product with y, this is same as inner product of x with y plus inner product of z with y and this is true for every x y z in V.

Similarly if you have x inner product with y plus z, this is x inner product with y plus x inner product with z.

Next one is if you take alpha times x inner product with y, this is same as alpha times inner product x with y for any real number alpha and for any two vectors x and y.

So, any map from V cross V to R which satisfies all these four properties is known as inner product on V, and then the vector space V along with this inner product is known as inner product space.

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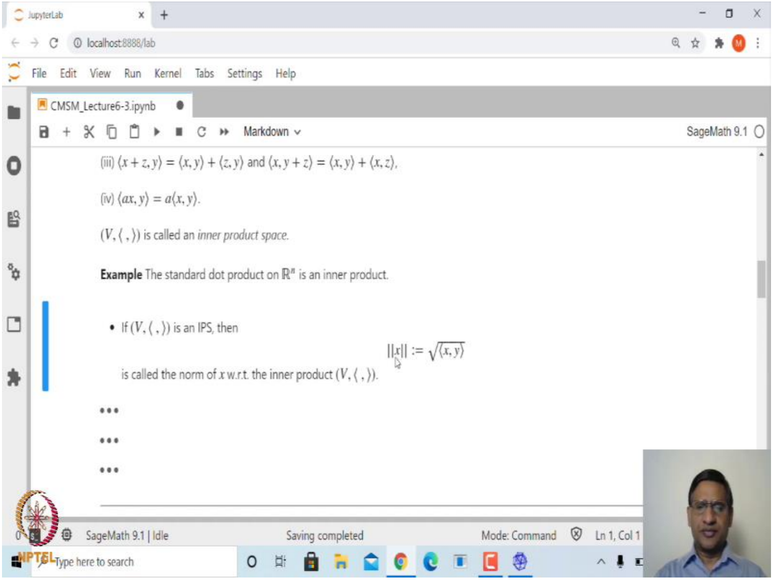

For example, if you look at R n with a dot product, you can check that the dot product in R n satisfies all these properties, and hence it is an inner product.

Let us look at few other examples.

But before that let me also define, once you have inner product defined on a vector space, then you can define length of any vector that is we denote by norm of x, we read this as norm x and this is defined as positive square root of inner product x with itself. It is not x with y; it is x with itself right.

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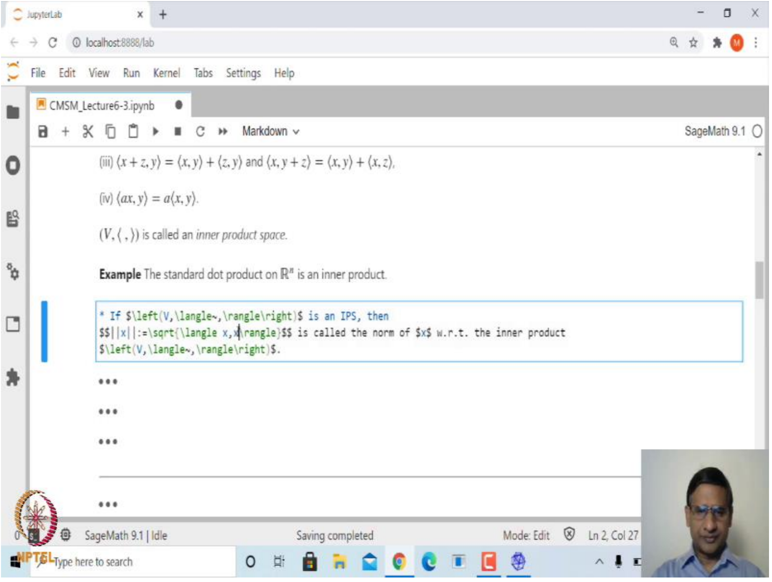

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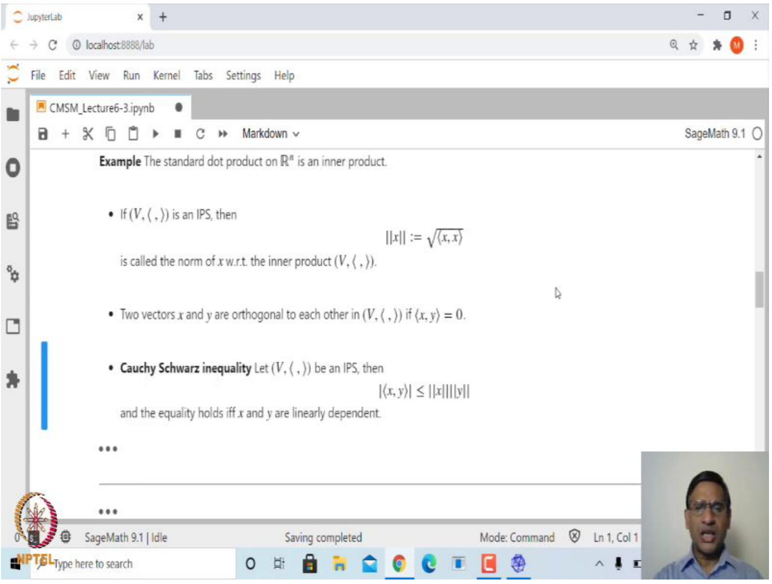

This is what is called norm of x with respect to this particular inner product right. So, here you should notice that, since norm of x with itself is non-negative, you can find its square root, and since length we want length it should be non-negative, therefore, we take non-negative square root. And of course, if I take norm of x square that is nothing but inner product x with itself.

So, in this case, this norm is induced by this inner product. Once you have inner product, then you can define norm. Of course, one can define norm function independently, and one can relate this.

Similarly, if you have two vectors x and y, we say that they are orthogonal to each other, if inner product x with y is 0. In that case, we say two vectors are orthogonal to each other. In terms of dot product you have x dot product with y is equal to 0, but that can be extended to any inner product space.

This norm and inner product are related by this very actually important inequality is which is known as Cauchy Schwarz inequality.

And what does it say? If you take V an inner product space and take any two vectors x and y, look at the mod of inner product x with y, this is always less than equal to norm of x into norm of y. And this equality will be true if x and y are linearly independent, so that is the inequality which is very powerful and very useful inequality at the same time very important.

Actually, this Cauchy Schwarz inequality gives you a way to defined angle between two vectors x and y. Let us look at this inequality. For example, let us assume that both x and y are nonzero, because if any one of them is 0, then it does not make sense to talk about angle between zero vector and some other vector. So, let us say let us assume that both x and y are nonzero. In that case if you divide this inequality on both sides by norm x norm y, this would mean that modulus of norm of modulus of inner product x with y upon norm of x into norm of y will be less than equal to 1. That is same as saying if I remove this this mod, then it would mean that minus 1 will be less than equal to inner product x with y upon norm of x into norm of y less than equal to 1. So, this particular quantity will lie between minus 1 and 1 for each x and y, nonzero x and y.

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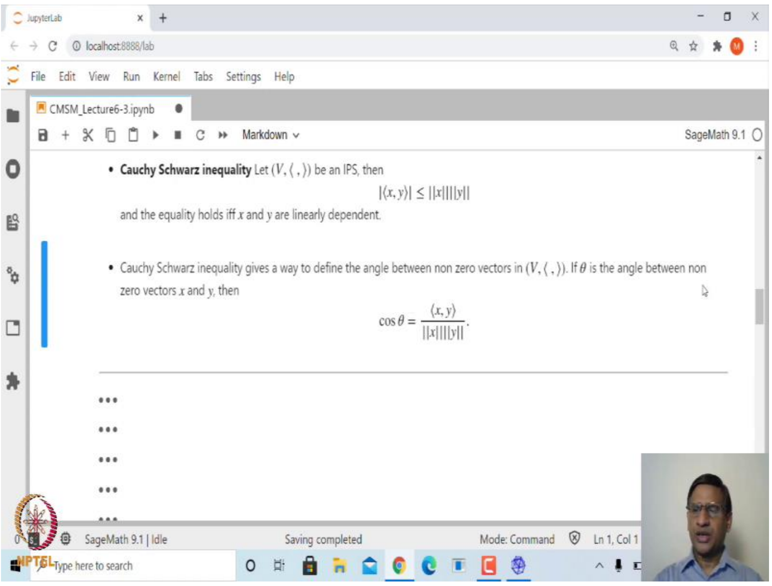

And you know a function which always lie between minus 1 and 1 and that is cos of theta. So, if theta is angle between two vectors x and y, then cos of theta is nothing but inner product x with y divided by norm of x into norm of y, in particular theta will be cos inverse of this quantity. So, this definition actually is motivated by this Cauchy Schwarz inequality.

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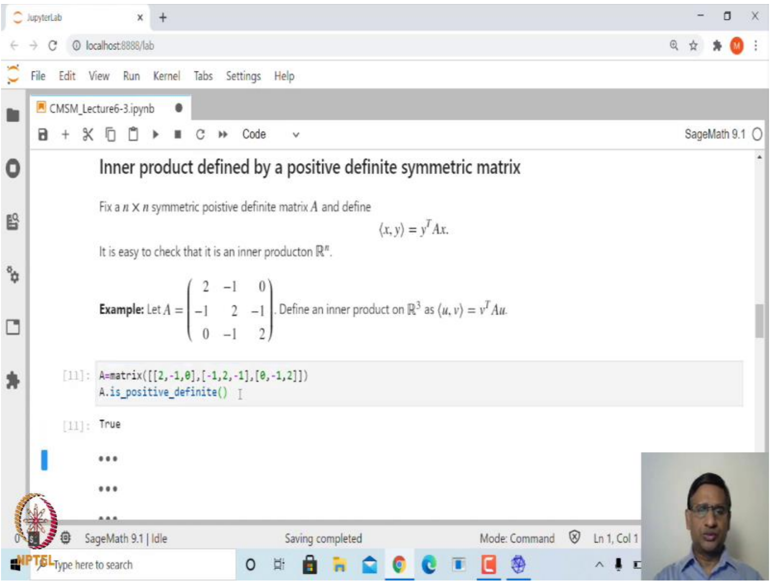

Let us look at some examples. So, we have one example of inner product on R n which is which is dot product.

Now, let us fix a positive definite real n cross n symmetric matrix A, and then on R n we can define x inner product as y transpose Ax.

Actually this y transpose A x can be thought of as dot product with y with Ax. You see, Ax will be a vector in Rn, y is also vector in R n. So, y dot product with Ax makes sense and that is nothing but this y transpose Ax as a matrix multiplication. One can show that this is an inner product on Rn, and in fact all the inner product in R n, or any product inner product in R n would appear in this form.

Let us look at an example. Suppose you take this matrix A, you can check that this matrix A is positive definite. There is inbuilt function to check positive definiteness of a matrix, of course, this is symmetric. Define this inner product u, v as v transpose Au.

So, let us look at this. A is this matrix and we can check whether A is positive definite. The answer is true. You can also check whether A is symmetric. A dot is symmetric, will give you true if it is symmetric matrix.

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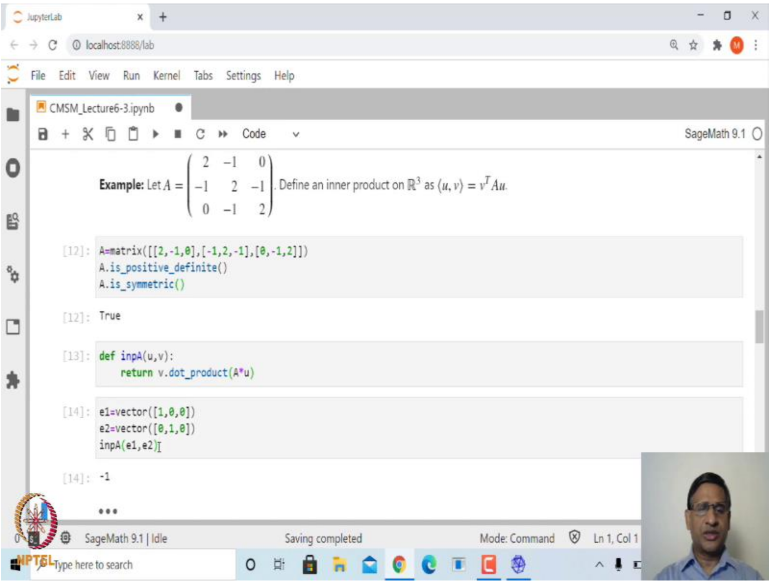

So, once you have this then let us define inner product. I am giving this name as inpA, A stands for this matrix of u and v is what should it return, it is just dot product of v with the matrix Au or the vector Au. So, that is the inner product.

Now, you can take any two vectors for example, let us take standard unit vectors e1 and e2, and let us find what is inner product of u1 with with u2. So, it is minus 1. With respect to standard dot product e1 and e2 are orthogonal. So, it would be 0, whereas, in this case, they are not orthogonal.

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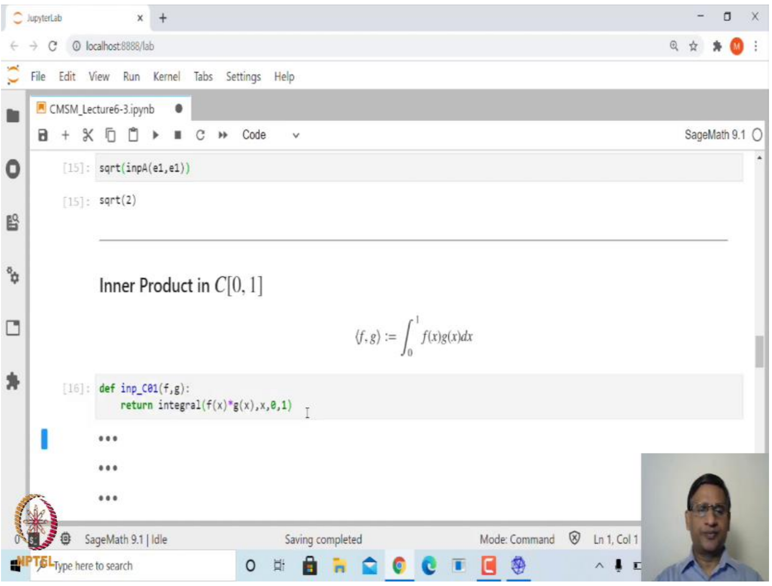

Similarly, you can find what is the length of this vector e1 with respect to this inner product, so that is the square root 2, and it is not 1.

So, you can have some vectors which are unit vector with respect to a standard dot product in R n, whereas with respect to some other inner product it may not be or it may not longer be unit vectors.

Let us take another example. Let us look at this vector space C[0, 1], the set of all continuous functions on closed and bound interval [0,1].

We have seen that this is a a vector space over R. So, we can define inner product of any two functions f and g in C[01] as integral from 0 to 1 of f(x) times g(x).

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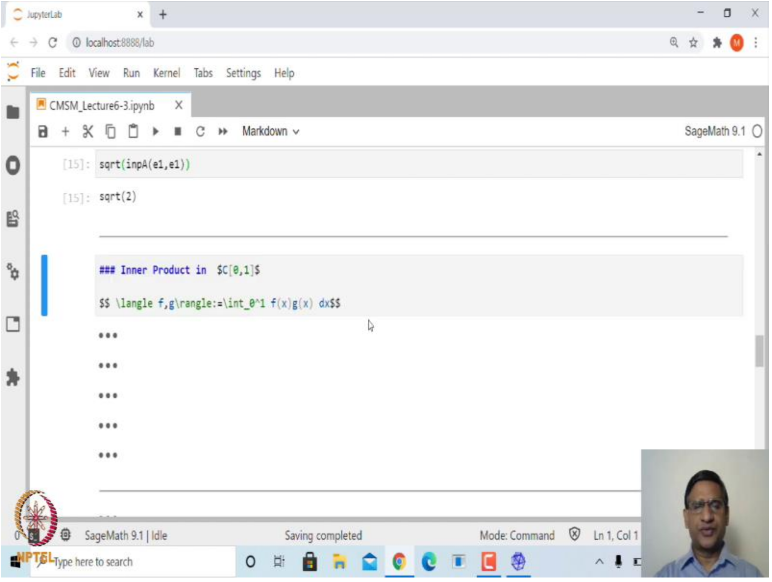

It is easy to check that this is an inner product. This will require little bit of calculus or real analysis especially when you want to show that if inner product of f(x) with itself is 0, then f itself is 0. That will use continuity of the function.

So, let us define this inner product. I am giving the name inp underscore C01 of f and g. And what should it return? It should return integral of f(x) g(x) and x going from 0 to 1. So that is the inner product.

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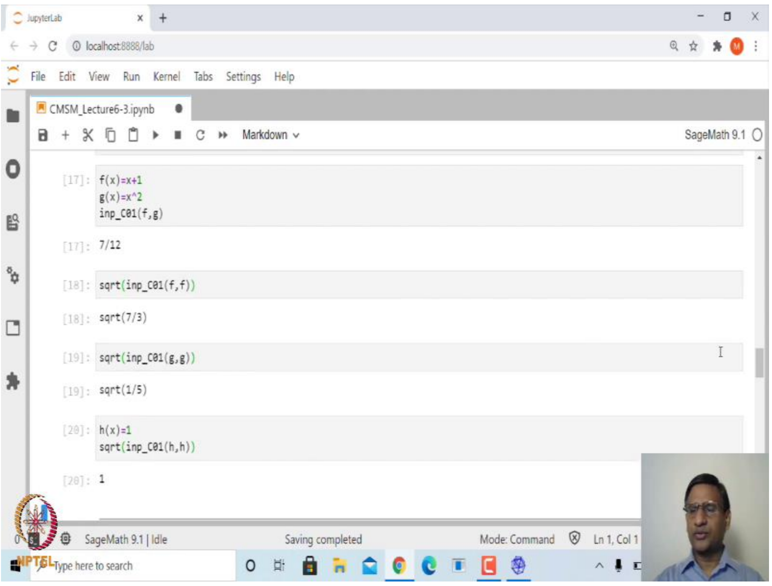

Now suppose we take two functions continuous functions f(x) equal to x plus 1, g(x) equal to x square, and then we can find inner product of f with g. So, it will find integral of x plus 1 times x square integral from 0 to 1. So, x plus 1 into x square is going to be x cube plus x square. And its integral will be x power 4 by 4 plus x to the power 3 by 3. And when you add this when you evaluate this, you will get 7 by 12.

Similarly, you can find length of any vector, in this case, length of let us say x plus 1 is square root of inner product underscore C01, f with itself this is square root 7 by 3. Similarly, you can find length of g.

In case you have this constant function 1, then its length is going to be 1. This is again 1. You can you can find out various concepts regarding this. So, may be try to verify Cauchy Schwarz inequality for this inner product.

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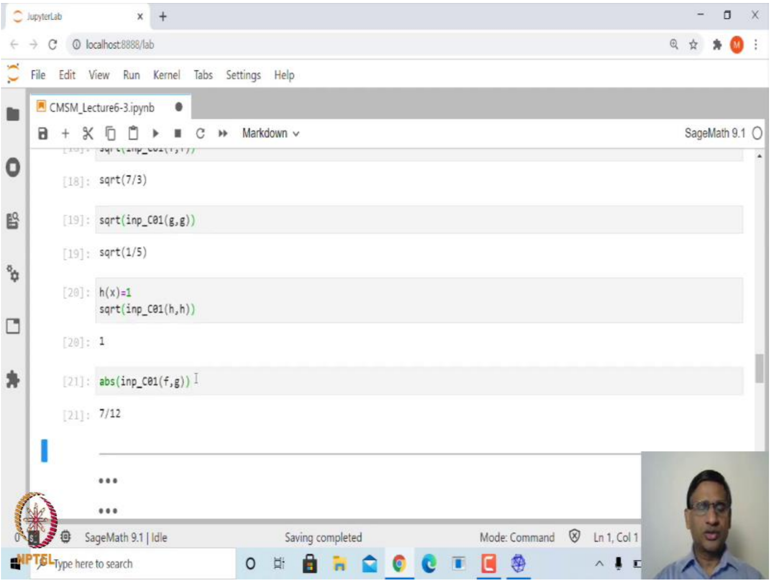

So, if I look at, say inner product, so inp underscore C01of f with g and take the modulus of this. This is inp take the modulus of this that is absolute value of this, absolute value of this. So, this is 7 by 12.

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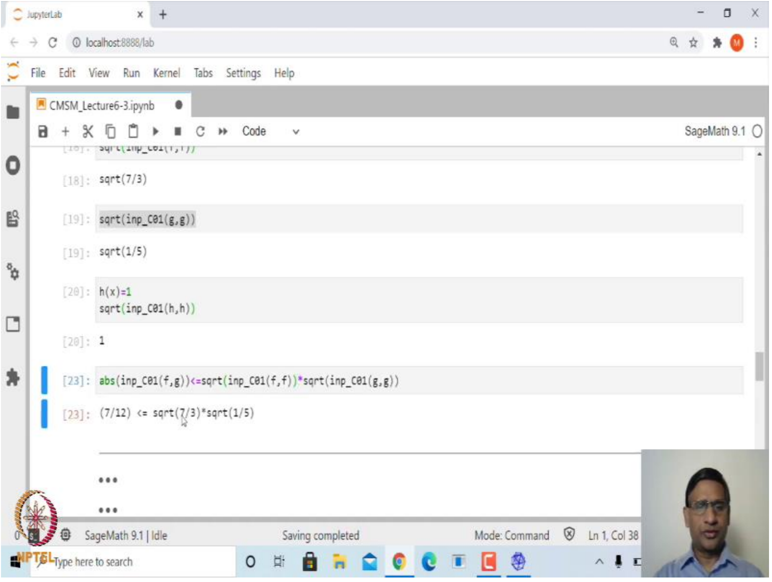

And is it less than equal to norm of f into norm of g that is what we want to check. So, this is f comma f.

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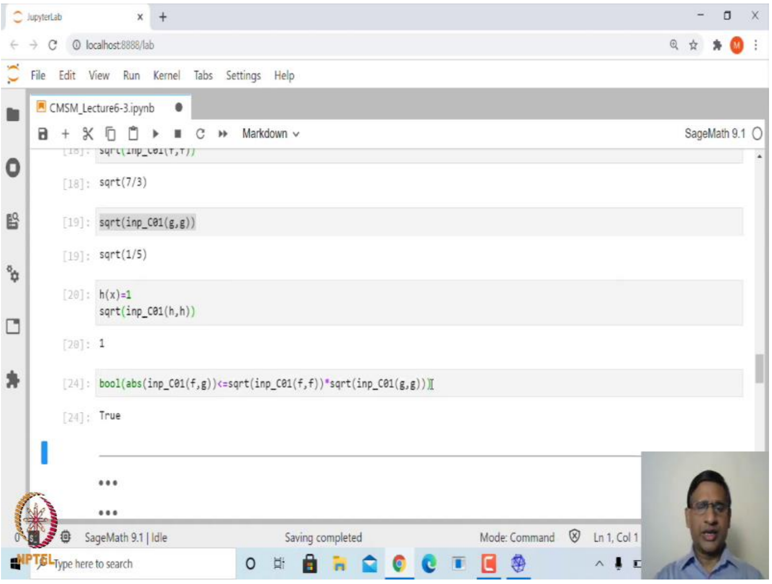

Let me say whether it let us give bool, the answer is true. So, if I have to check this inequality, you can say bool absolute value of this less than equal to this, the answer is true. So, it satisfies Cauchy Schwarz inequality. That is a verification of Cauchy Schwarz inequality.

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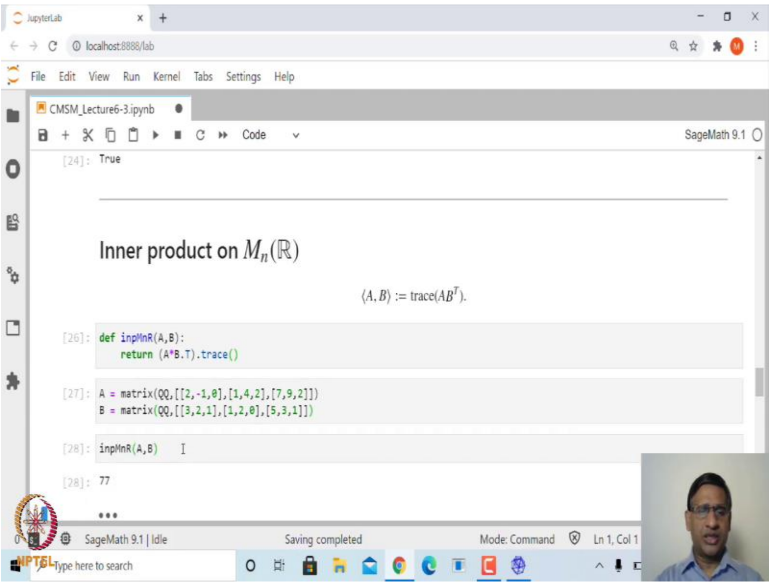

Next let us look at this M n R, set of all n cross n matrices over R. On this we can define inner product as inner product of A and B, two matrices A and B as trace of A into B transpose. So, let us define this. This is definition here, we are calling as inpMnR of A and B, it should return A into B transpose and then take trace of this.

So, for example, if I take two matrices A is equal to this matrix 3 by 3 and B is equal to this 3 by 3 matrix, we can find inner product of A and B. This is 77.

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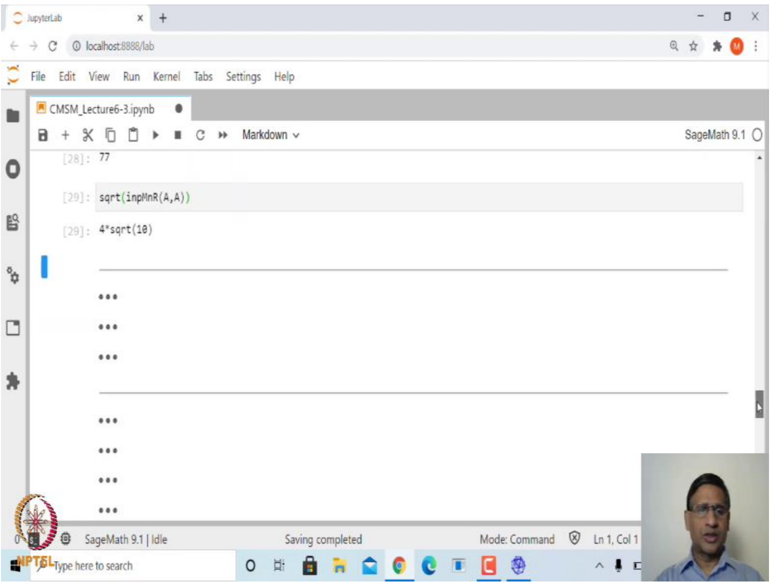

You can also find length of this matrix with respect to this now.

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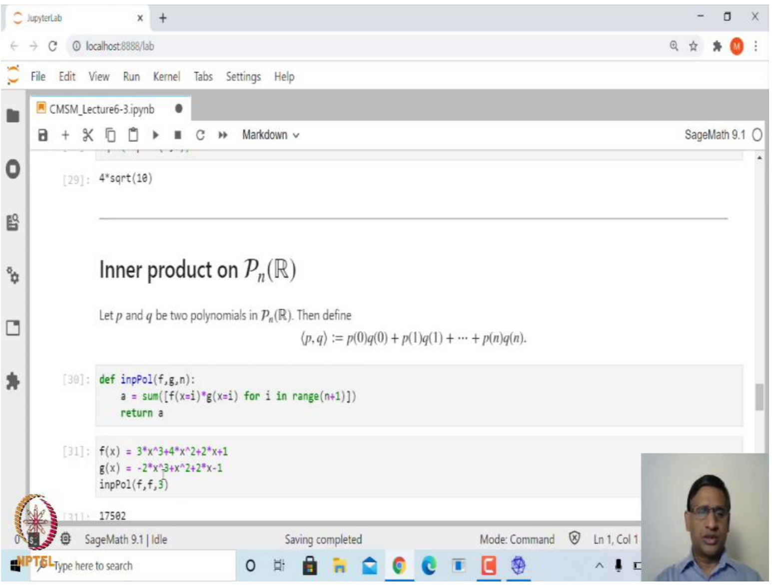

You can also define inner product on P n R, this is set of all polynomials of degree less than equal to n. One way to define is this take any two polynomials p and q of degree n over R. And define inner product of p with q as p(0) into q(0) plus p(1) into q(1) plus dot dot dot p(n) into q(n). And it is very easy to check that this is an inner product.

So, how do I define that? So, we will, say inpPol, that is the name I am giving and f and g. And this is n, stands for the degree here. Take f(x) at i, g(x) at i, and multiply these two f(i) and g(i) and run i over range n plus 1. Because it has to go from 0 to n. So, n should be included, so 0 to n plus 1, and then take the sum, that is the inner product.

Of course, you could check instead of 0, to n, whether we can take any n plus 1 real numbers. For example, let us say any n plus 1 real numbers a1, a2, a n plus 1. So, p(a1) times q(a1) plus p(a2) times q(a2) plus dot dot p(a n plus 1) q at n plus 1, whether that is also an inner product. that you can check right.

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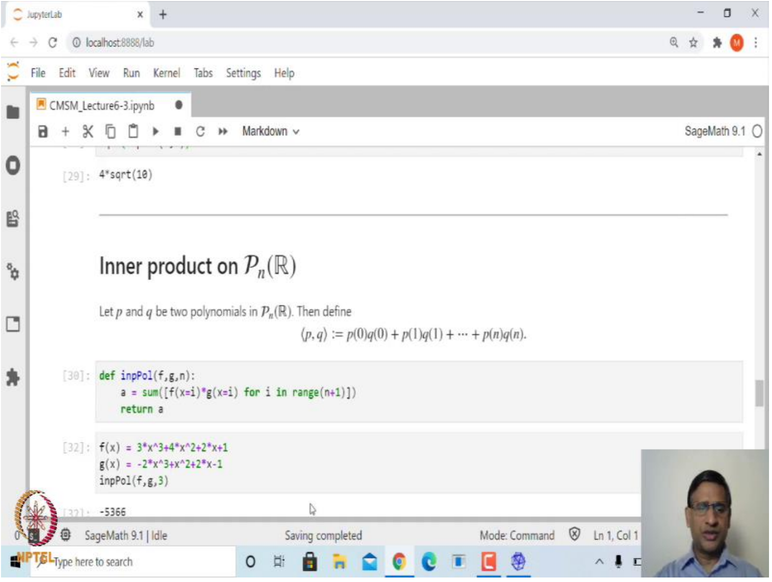

So, let us see, I have two polynomials of degree 3, and then we can find inner product of f with g using inpPol. Similarly, you can find length of each of this polynomial right.

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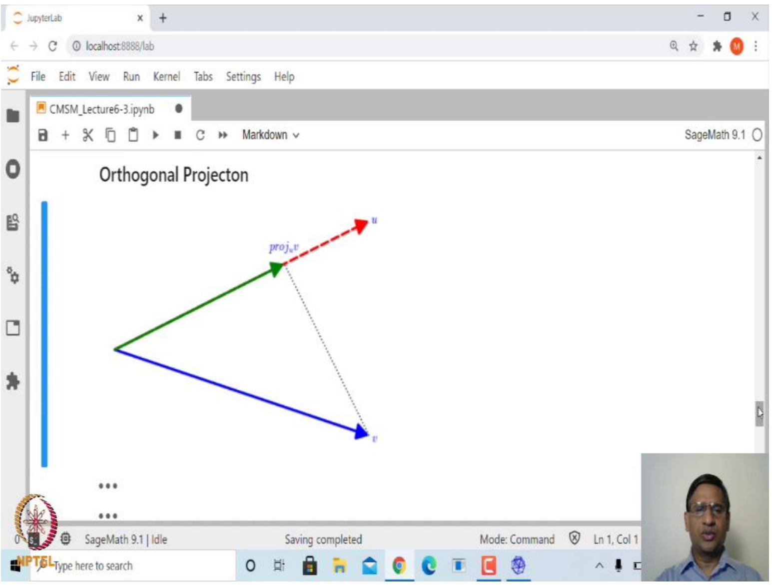

Once we have an inner product on a vector space, we also saw how to define orthogonal vectors, you say two vectors are orthogonal with respect to that inner product. Now, let us extend this idea of orthogonal projection, which we have done in case of dot product in R n.

So, if you have inner product space and you have a vector v and a vector u. You want to find orthogonal projection of v onto u, see the definition which we have looked at for dot product that dot product, will be replaced by inner product thats all.

So, you take perpendicular from v to this u, this line u, this is the origin, and then this particular vector, this green is called orthogonal projection of v onto u. And how do we obtain this vector? So, actually this will be a scalar multiple of u. So, this projection of v onto u will be some t times u. And since this is perpendicular to this vector, this vector dot product with u will be 0. And what is this, this vector? This vector is actually if you look at this t times u, plus this vector, which is v minus t times u, that is this vector. So, this v minus t times u will be orthogonal to u, and that will give you what it should be value of t.

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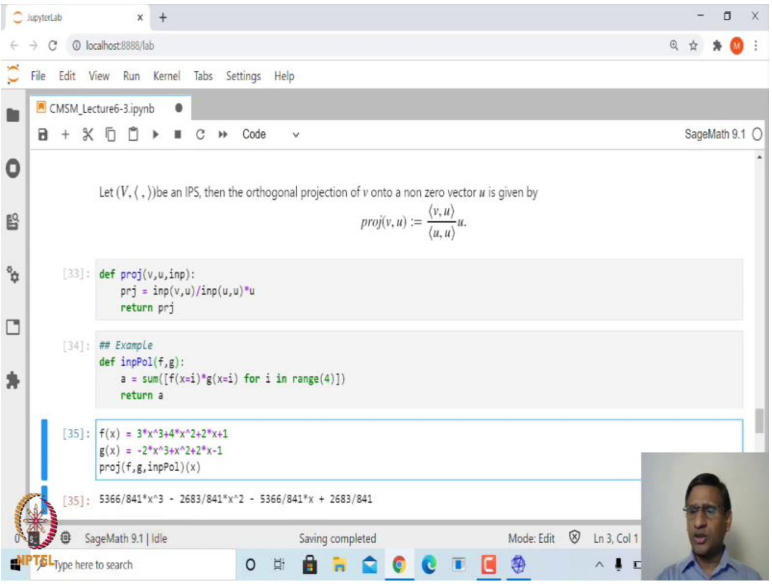

It turns out that scalar t, is inner product of v with u divided by norm of u square. Therefore, the projection of v onto u is given by inner product of v with u and divided by norm of u square times u.

So, let us define a function for orthogonal projection. I am defining this as projection proj v with u that is v onto u and with respect to the inner product. So that is the inner product here. Then what it should return? Inner product of v with u divided by norm of u square times u, and then return.

So, let us look at for example, if I have two polynomials, this is the inner product with respect to which we want to find orthogonal projection. This we have already defined. So, I do not need to to do it again. Since there we had given here n, then in the definition let me remove that n because here inner product I am taking as inner product v with u.

Of course, one can take care of that definition as well. So, take this inner product define this inPol of two polynomials f and g as this. This is a polynomial of degree 3. And f and g are two polynomials let us see what is orthogonal projection of f onto g, so that is the orthogonal projection.

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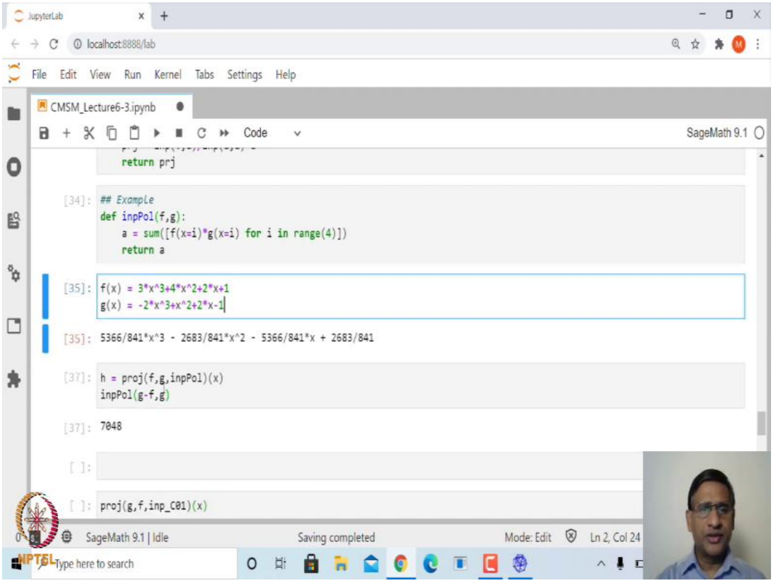

Now, you could also verify that. I will call this as let us say h which is orthogonal projection of f onto g. Let me do it in different cell. This is my h. And suppose we take g(x), or let me say g minus h g we are taking orthogonal projection of f onto g.

So, it will be g minus f, this inner product with inpPol with g this should be 0. So, this is let us see we have obtained the projection of f onto g with respect to this inner product. And we are taking, what is it? We, we are taking f.

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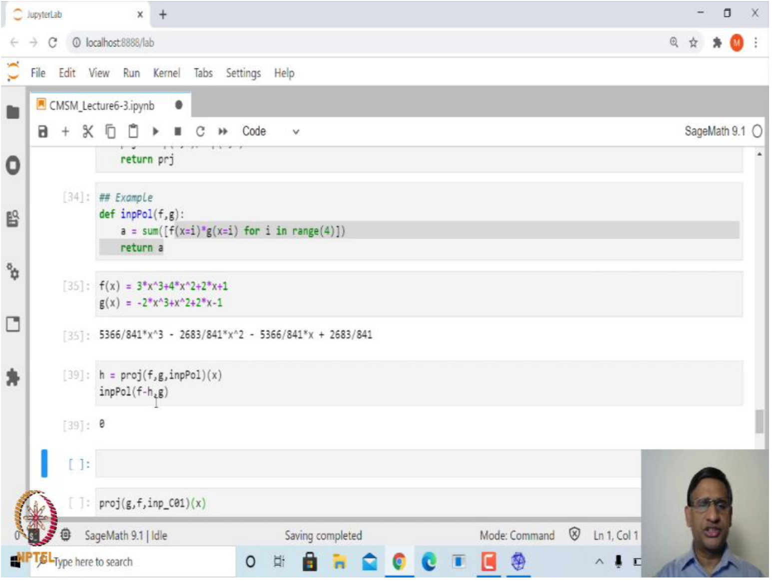

It should be f minus h, the inner product with, no, f minus h inner product with g this should be 0 correct. So, this is the verification What you have obtained h is an orthogonal projection of f onto g right.

Similarly, let us take the same polynomials f and g, we can define what is orthogonal projection of f onto g with respect to inner product that we have defined on C[0,1]. So, this polynomial f and g can also be thought can also be thought of as elements of C[0,1].

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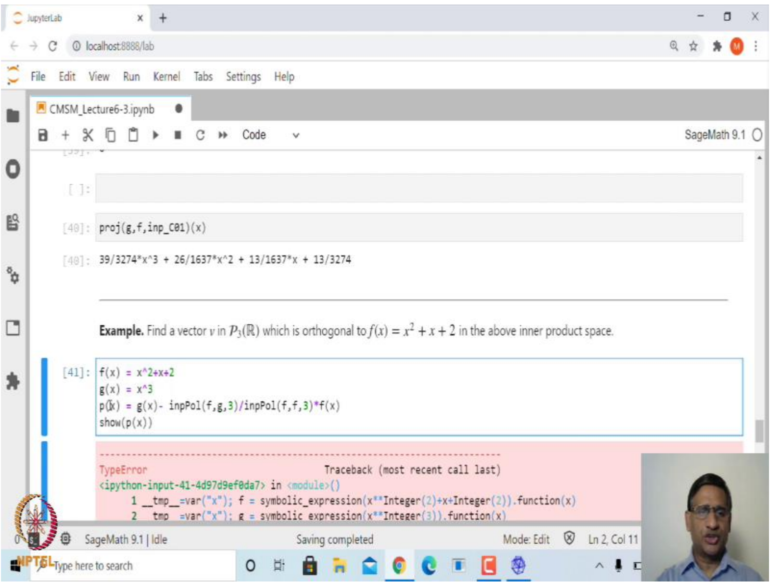

So if I take orthogonal projection of g onto f with respect to this inner product that is inner product which is integral from 0 to 1 f(x) g(x) that is this one right. Similarly, let us take another example. Suppose, we want to find a vector v in P3R that is a polynomial of degree less than equal to 3 which is orthogonal to this vector, right.

So, how do we obtain? If I want orthogonal to this vector, we can start with any vector in this vector space. So, let us say we start with x to the power 3, x cube, and take orthogonal projection of x cube onto x square plus x plus 2, and then take out orthogonal projection component, that will be a vector which is perpendicular to f(x) which is equal to x square plus x plus 2.

So, let us see that. We define f(x) is equal to x square plus x plus 2. g(x) as x cube, and define f(x) is equal to g(x) minus this is not 3.

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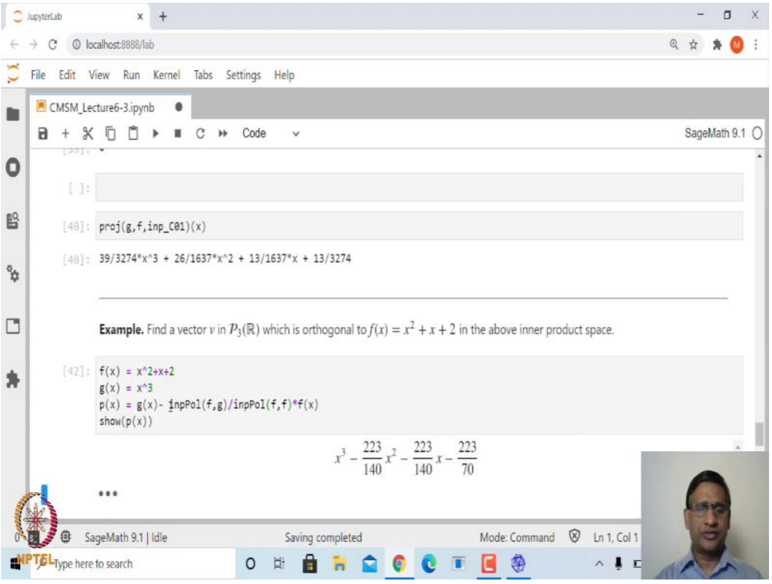

This g(x) minus orthogonal projection of f onto g. So here it should be inner product of g on to f.

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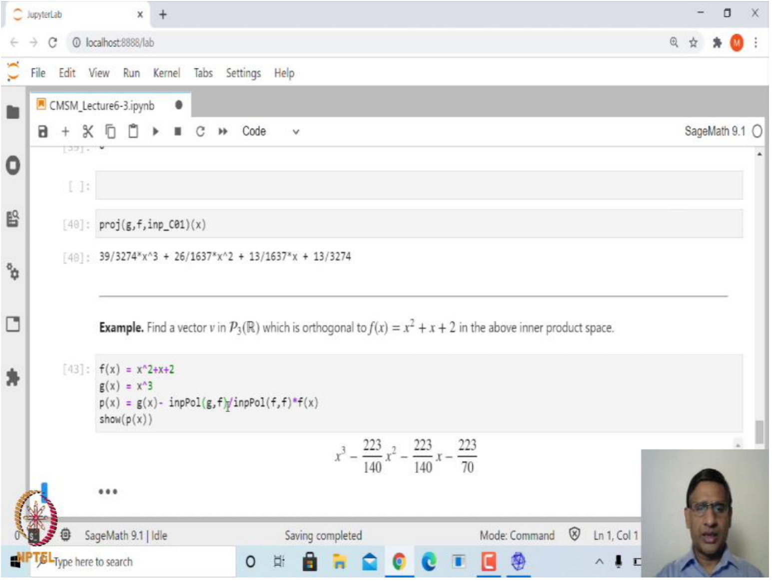

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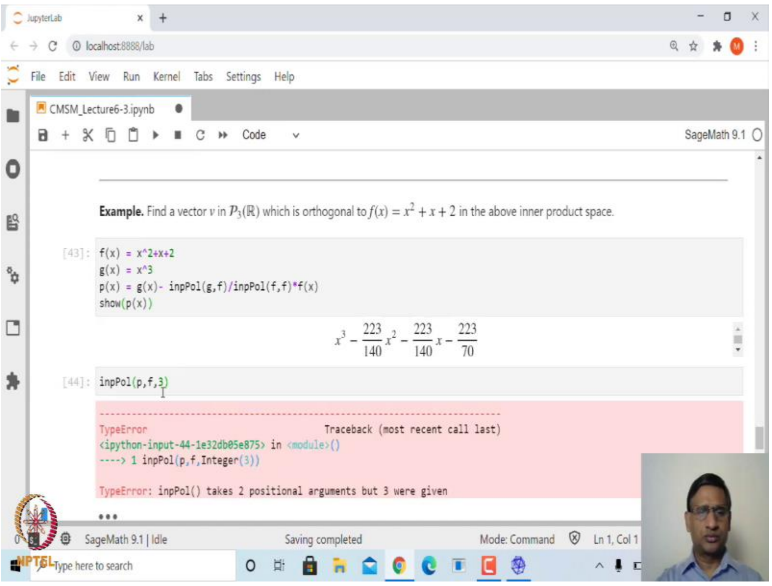

And now let us check whether p is perpendicular to f, hat is correct.

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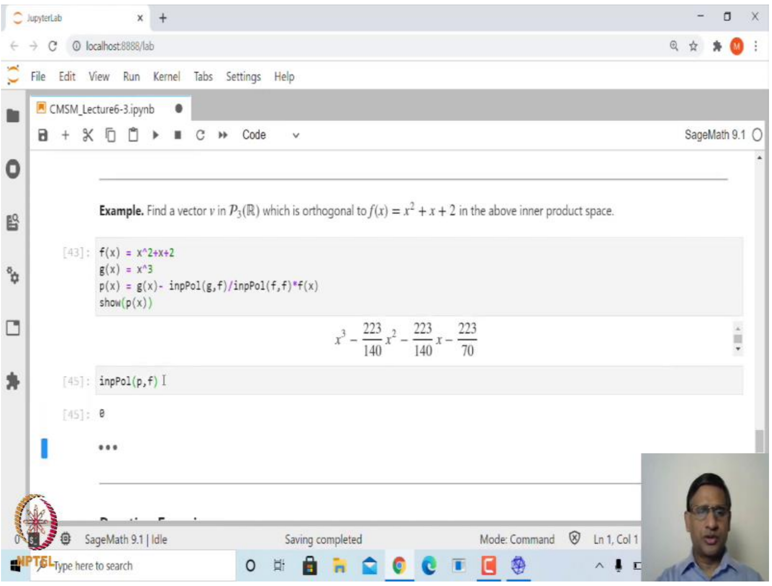

So, we have found a polynomial, which is orthogonal to this given polynomial x square plus x plus 2.

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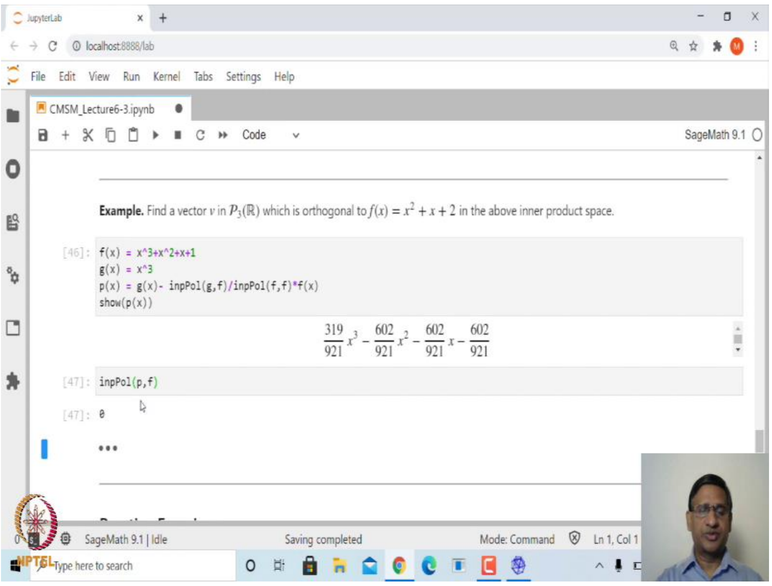

If I change this, for example, if I say x cube plus x square plus x plus let us say 1, and ake any g which is different from this f, this will work. So, now, again you can check whether p is perpendicular that is correct right.

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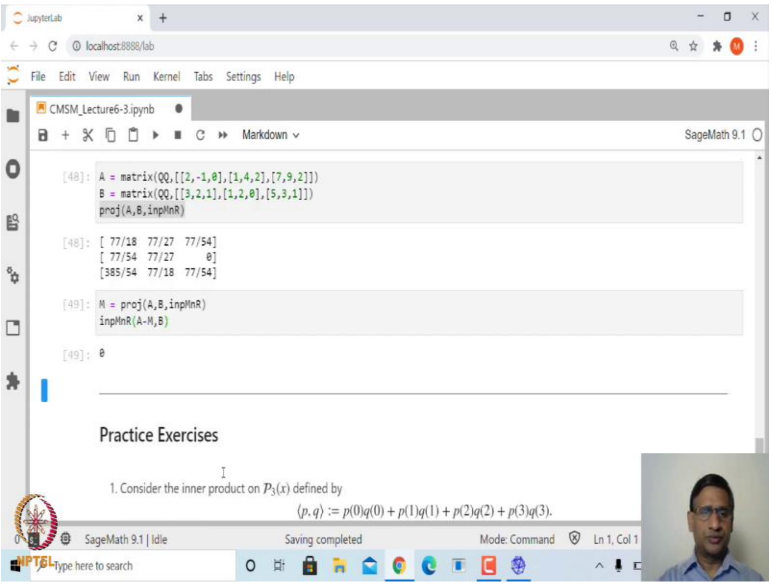

Similarly, you can find orthogonal projection of a matrix A onto B with respect to the inner product defined on M n R. That is the again you can check whether this orthogonal projection which we are obtained, let me call this as a matrix M. M is equal to this matrix.

Orthogonal projection of A onto B, and then we can check that A minus M this inner product with B this should be 0. The answer is true that is correct. Therefore, this is this is orthogonal projection of A onto B. This is the matrix which orthogonal projection of A onto B.

We can explore many more examples of inner product defined on finite dimensional vector space. So, all we need to do, is to define because it, as it may not be inbuilt already. So, you need to define it separately using def or you can use a lambda function for definition.

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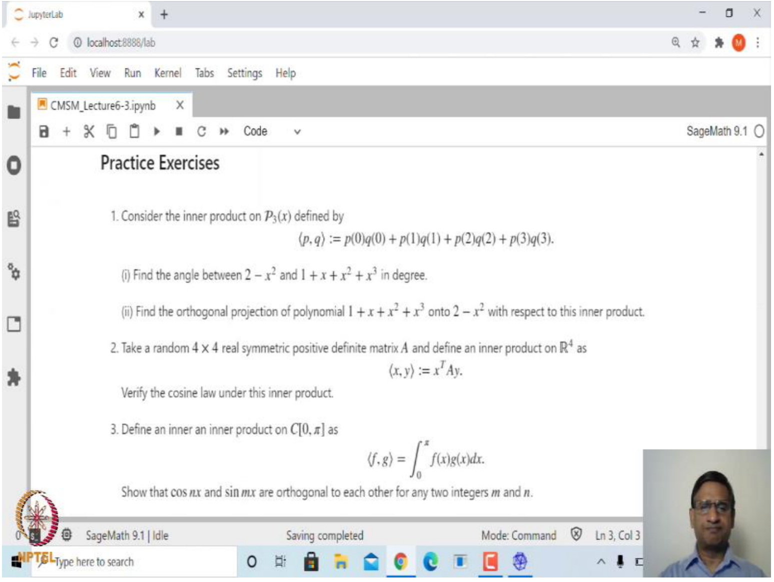

Now, let me stop here. I will leave some simple exercises. So, first one is this on P 3 x define this is inner product and find angle between these two vectors in degree. Next problem is find orthogonal projection of this onto this with respect to this inner product. So, this example we already did. This is just a different polynomials.

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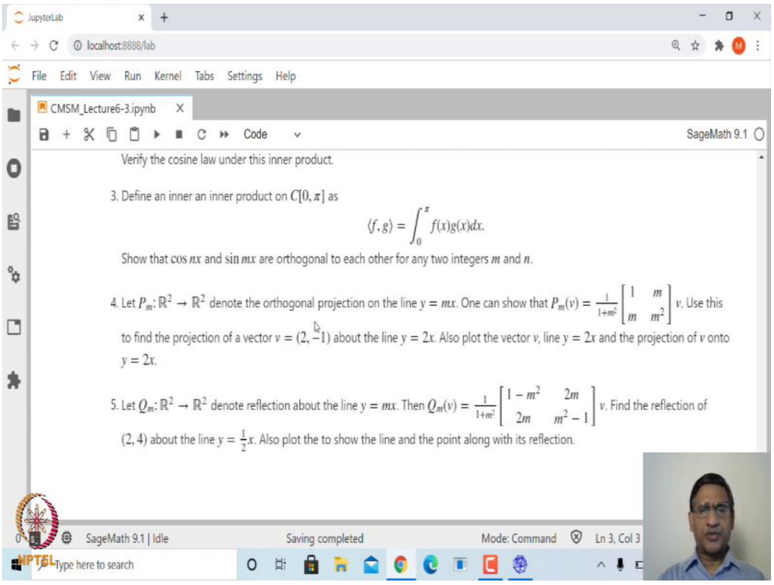

Similarly, define this inner product and verify the cosine law under this inner product.

Define inner product this 0 to pi now of f(x) g(x) on C[0,pi], and check whether the cos(n x) and sin(m x) are orthogonal to each other for any two integers m and n.

And the last one is, we can also define orthogonal projection onto any straight line. So, if I take a straight line y equal to m x, and then suppose we say that orthogonal projection is a map from R 2 to R 2, that map let us say P m can be given by this matrix multiplication.

So, P m will be a matrix or matrix of this P m is going to 1 upon 1 plus m square into 1 m m and m square. So, this is orthogonal projection. So, you could you could think of this line as a line passing through a vector. So, the orthogonal projection of a vector onto another vector that is what is required, but you can obtain this as explicit formula.

Similarly, you take, let us say reflection about any line y equal to m of x, we have seen reflection of a vector about x axis. So, if I have a vector x, y, if I reflect about x the line x-axis, then it is nothing but, x comma minus y.

And in this case, so if I denote this map as Q m which is a reflection about this line y equal to m of x, then this matrix of Q m will be given by this. And so, use this definition to find the reflection of a vector 2 comma 4 about the line y equal to half x.

And also try to plot graph of these all these points along with the line and the points which is reflection. Similarly, you should plot for the projection. So, thank you very much.

In next class, we will look at some more concept on inner product.