Computational Mathematics with SageMath
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Eigenvalues and Eigenvectors cont..
Lecture – 36
Eigenvalues and Eigenvectors Part 2 with SageMath

Welcome to the 36th lecture on Computational Mathematics with SageMath. We shall continue exploring some more concepts on Eigenvalues and Eigenvectors.

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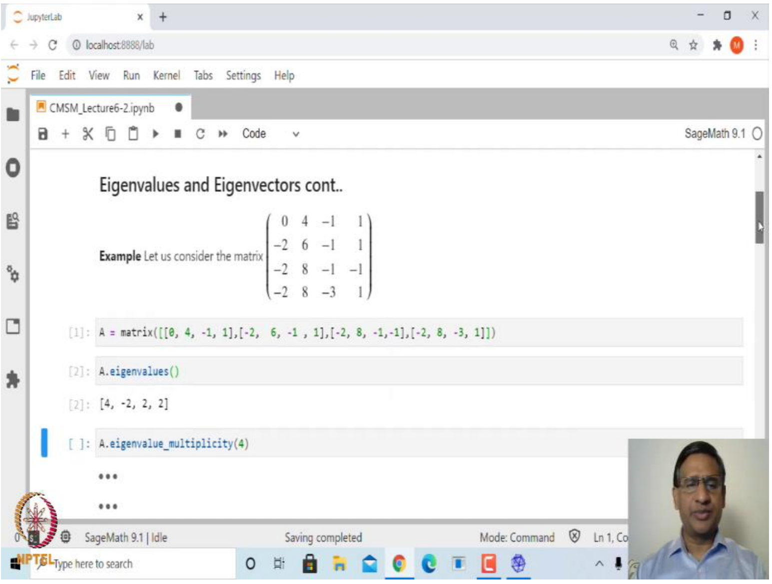

So, let us start with an example. Suppose you are given a matrix A, which is a 4 by 4 matrix. Let us declare this in SageMath. Next let us find out what are eigenvalues of A. You can see here the eigenvalues of A are 4 minus 2, 2 and 2.

So, 4 and minus 2 are appearing with algebraic multiplicity 1 and 2, has algebraic multiplicity 2. You can also find algebraic multiplicity of any eigenvalue using inbuilt method eigenvalue underscore multiplicity. So, for example, if I look at algebraic multiplicity of 4 it should give me answer 1.

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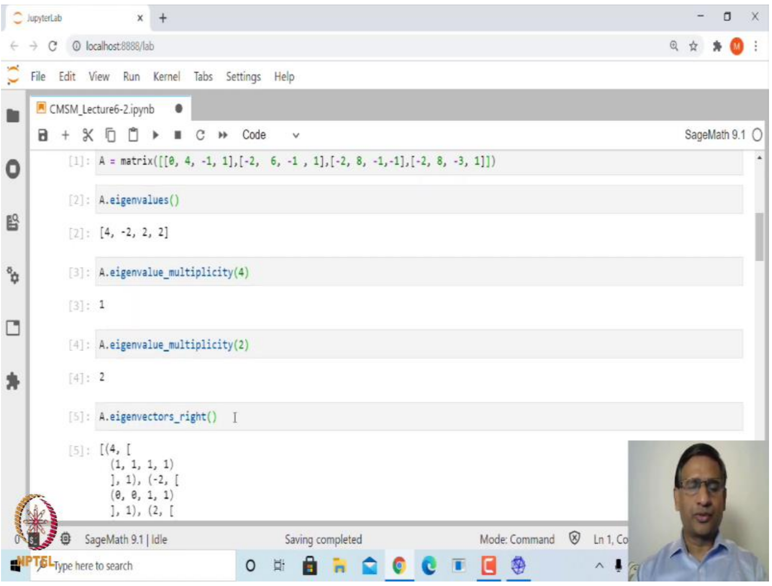

Similarly, if I look at algebraic multiplicity of 2 it should give me 2.

Next let us find out what are eigenvectors of A.

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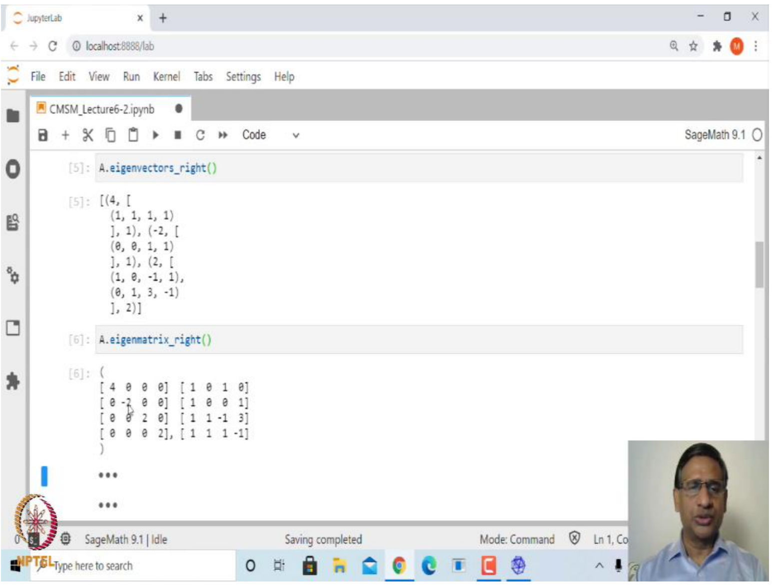

So, eigenvectors of A respect to eigenvalue 4 is 1 1 1 1 and it is having multiplicity 1. Eigenvector with respect to eigenvalue minus 2 is 0 0 1 1 and that is also of multiplicity 1. Eigenvectors with respect to eigenvalue 2, there are 2 eigenvectors with respect to eigenvalue 2. Namely, 1 0 minus 1 1 and 0 1 3 minus 1 and this is of multiplicity 2.

We can, in fact, find out this using another method called eigenmatrix underscore right. So, when you say A dot eigenmatrix underscore right it gives you actually 2 matrices. One is this first matrix which is nothing but the diagonal matrix with diagonal entries as eigenvalues. So, 4 minus 2 and 2 and 2, these are the 4 eigenvalues. So, that is the first output in this method. And the second one is nothing but the column vectors of eigenvectors corresponding to the eigenvalues. So, with respect to eigenvalue 4, the eigenvector is 1 1 1 1 and that is what you see in the first column of this matrix. Similarly with respect to eigenvalue minus 2, the eigenvector is 0 0 1 1, that is the second column and similarly with respect to eigenvalue 2, there are 2 eigenvectors one is 1 0 minus 1 1 which is the third column and 0 1 3 minus 1 which is the fourth column.

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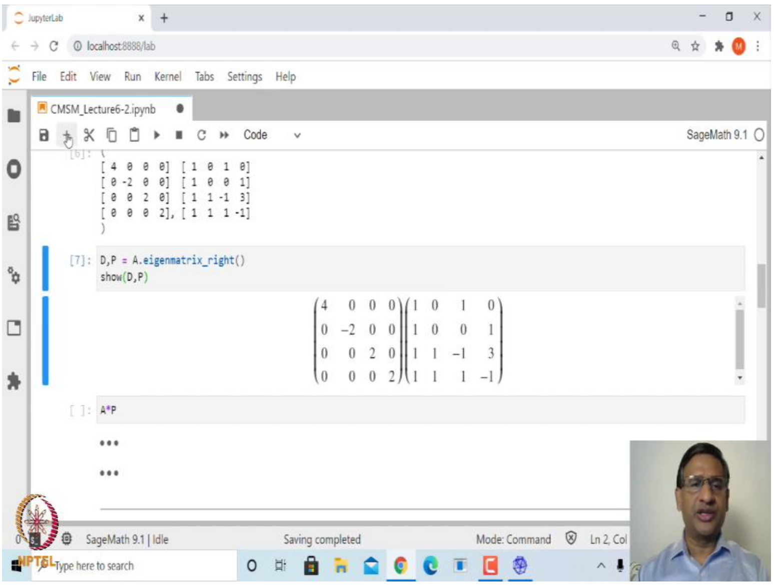

Let us let us store these outputs of A dot eigenmatrix underscore right in D and P. So, D is going to be this diagonal matrix of eigenvalues and P is the matrix of eigenvectors. So, these are this is D this is P. Right, now, if you look at this P, and you can check whether this P is invertible?

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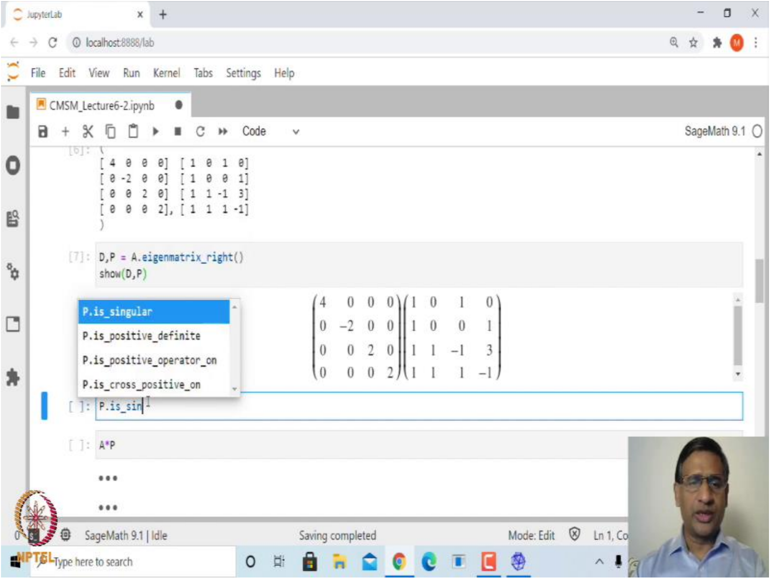

So, I can say P dot is underscore singular.

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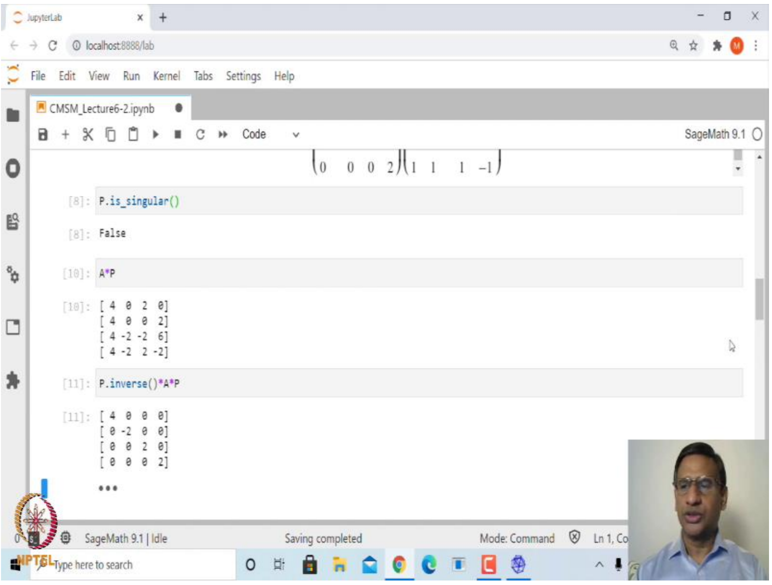

I should get answer false. This is not a singular matrix, this is a nonsingular matrix, in particular P is invertible. Now if P is invertible, is same as saying all these columns are linearly independent. So, what it means is, here these 4 eigenvectors that we have obtained for A, they are linearly independent. And in particular it will form a basis, that is what is called an eigen basis of A. Right, So, let us find out what is A times P, A times P is this matrix of eigenvectors A times P. What is a times P? The first column is 4, 4, 4, 4, which is nothing, but 4 times the first column and 4 is the eigenvalue. So, it is 4 times eigenvector with respect to the 4, the second column is 0 0 minus 2 minus 2 which is minus 2 times the second column of P. Similarly the third column is 2 0 minus 2 minus 2, which is 2 times the the third column of P. So, that is A times P.

Now, let us multiply A times P by P inverse on the left hand side. So, P inverse A times P, then what you get? You get this diagonal matrix D which is 4 minus 2 2 2 ok. That was the diagonal matrix. So, here A dot eigenmatrix underscore right results in 2 matrices, D and P, such that P inverse A P is nothing but D. This is what is called diagonalizability of matrix A.

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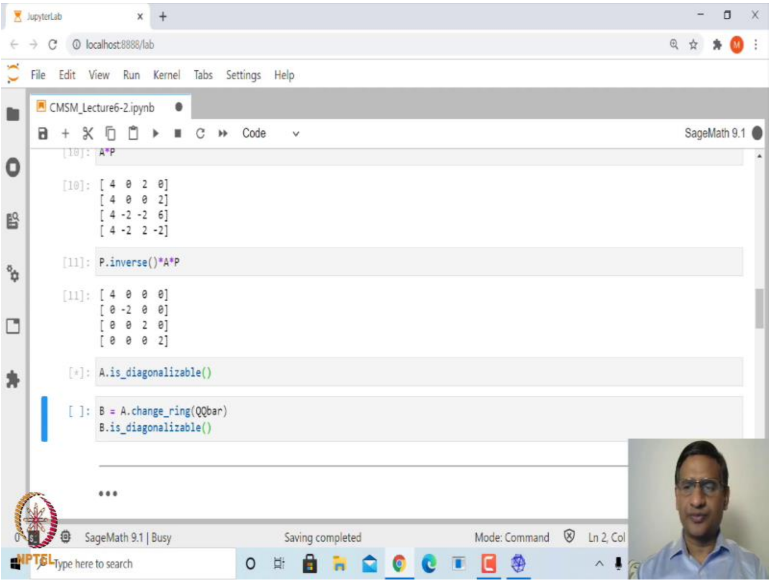

So, this matrix is diagonalizable and there is a inbuilt function to check whether something is diagonalizable or not. You can use is underscore diagonalizable method. So, for example, if I say A dot is underscore diagonalizable, then you will get an error. Let us see, this gives you an error.

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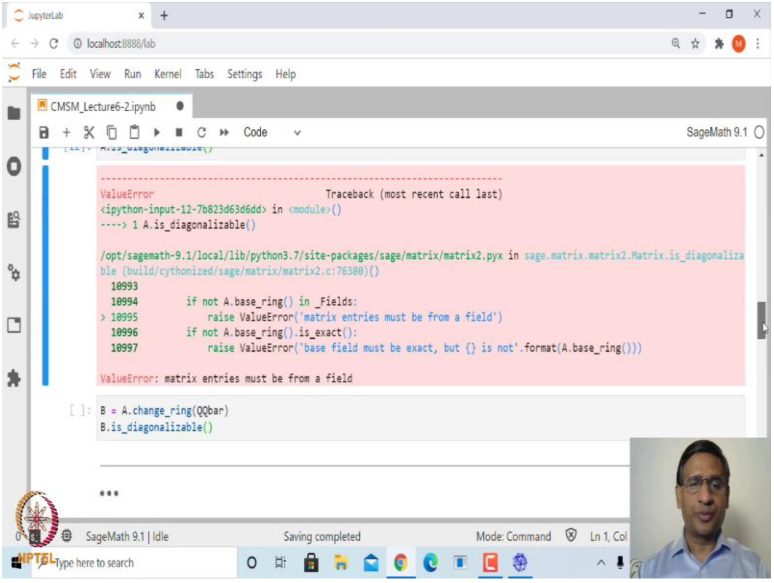

What it says? Matrix entries must be from a field right. Whereas, here the matrix entries by default, it may be thinking as a integers, which is not in a field. It is possible that the matrix is diagonalizable, but your underlying field is not the appropriate one then you will get false answer or you will get error. Now, let us change the field of A, that is change the ring, change underscore ring QQ bar. QQ bar stands for extended rational field.

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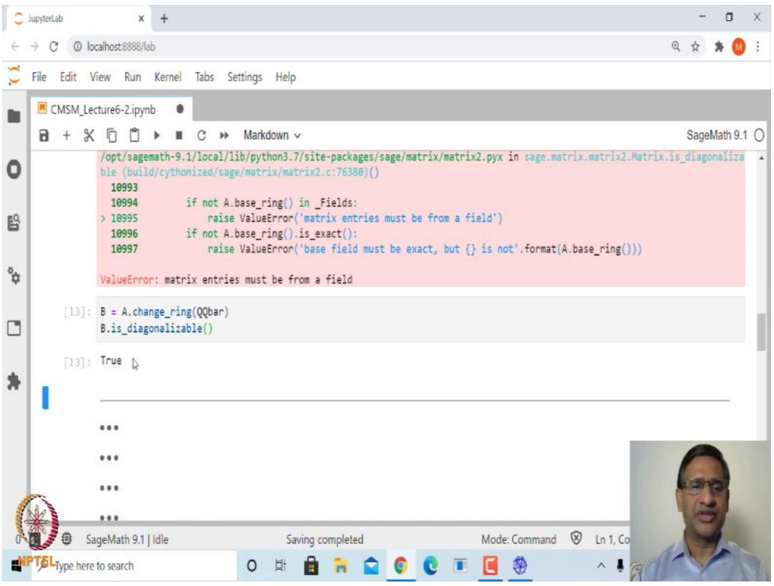

And then ask whether this matrix B is diagonalizable? The answer is true, we already know that this matrix is diagonalizable. This P is called diagonalizing matrix. So, here P is diagonalizing matrix and this matrix is diagonalizable right.

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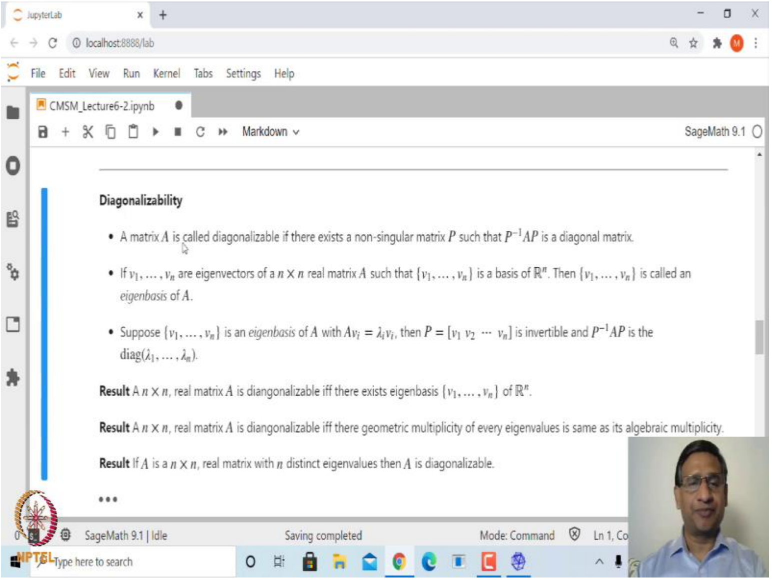

Now let us define formally what is meaning of diagonalizability. So, a square matrix A is called diagonalizable if there exist a nonsingular matrix P such that P inverse A P is diagonal matrix, ok. Already we saw that the matrix A is diagonalizable and P turns out to be the matrix of eigenvectors.

Now, the next result is that, this is a definition. If you have a vectors v1, v2, vn, which are eigenvectors of A with respect to eigenvalues lambda1, lambda2, lambda n. And also this v,1 v2, vn forms a basis of R^n. Then this basis, that is v1, v2, vn is known as eigen basis. So, eigen basis is nothing but a basis in which every vector is an eigenvector.

Now, suppose you have an eigen basis v1, v2, vn of A, then you define a matrix P to be the column matrix v1, v2, vn. That is, take the first column as v1, the second column as v2 and so on. Since this v1, v2, vn are linearly independent, P will be inevitable.

So, now, if you look at P inverse, A times P, that you can show that it is nothing but the diagonal matrix whose diagonal entries are eigenvalues with respect to eigenvectors v1, v2, vn. So, that is the very simple result, and it is very easy to prove also. You can look at any standard book on linear algebra.

The next result says that, if you have A, n cross n real matrix then it is diagonalizable, if and only if it has an eigenbasis. Thats what the previous one says, right. If you have an eigenbasis it is diagonalizable and the converse is also true. If the matrix is diagonalizable then it is diagonalized using this invertible matrix P. Then columns of P will be eigenvectors.

Another way of proving that a matrix is diagonalizable or not, is to look at the geometric multiplicity and algebraic multiplicity. In case geometric multiplicity and algebraic multiplicity of every eigenvalue is same then the matrix is diagonalizable. In fact, it is if and only if result.

A matrix A is diagonalizable if and only if for every eigenvalue its geometric multiplicity is same as its geometric multiplicity. And in case you have got n distinct eigenvalues, one can show that eigenvectors corresponding to distinct eigenvalues are linearly independent. If it has n distinct eigenvalues then you will get n eigenvectors corresponding to each eigenvalue. And therefore, they are linearly independent. So, it will form a basis of R n and hence matrix A is diagonalizable. So, these are some of the simple results, which tell you how and when a matrix is diagonalizable and not only that how you can define or how you can diagonalize a matrix.

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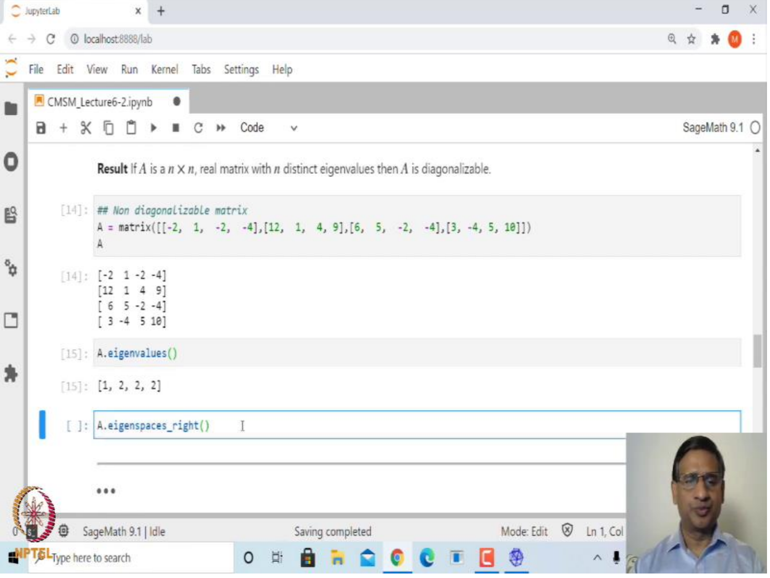

Now, let us take an example. So, let us consider this matrix A, which is again a 4 cross 4 matrix. Now if you find out eigenvalues of A, this has only 2 distinct eigenvalues namely 1 and 2. 2 has algebraic multiplicity 3 and 1 has algebraic multiplicity 1.

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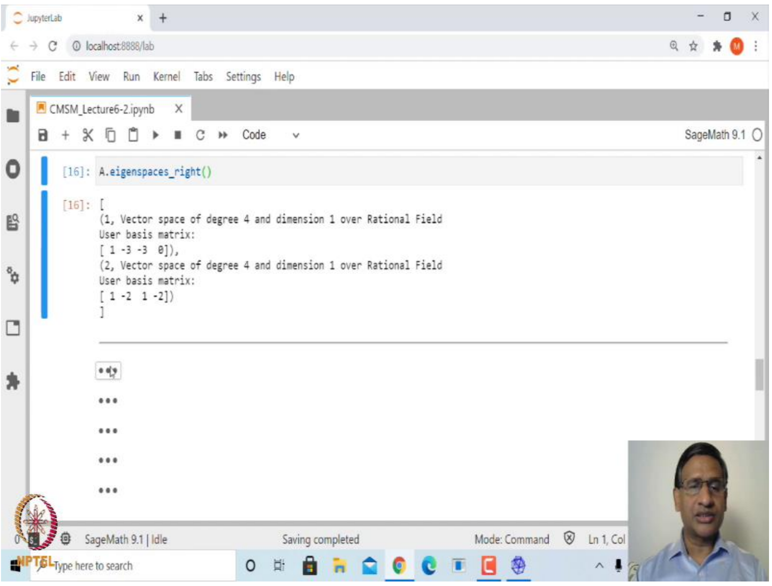

Now, if you look at the eigenmatrix of A, then look at eigenspace of A. So the eigenspace with respect to eigenvalue 2 is 1 dimensional because it has only 1 eigenvector and eigenspace with respect to eigenvalue 2 is also 1 dimensional. That means, algebraic multiplicity of 1 is same as geometric multiplicity of eigenvalue 1, but algebraic multiplicity of eigenvalue 2 is 3 whereas, its geometric multiplicity is 1. Therefore, this matrix is not diagonalizable..

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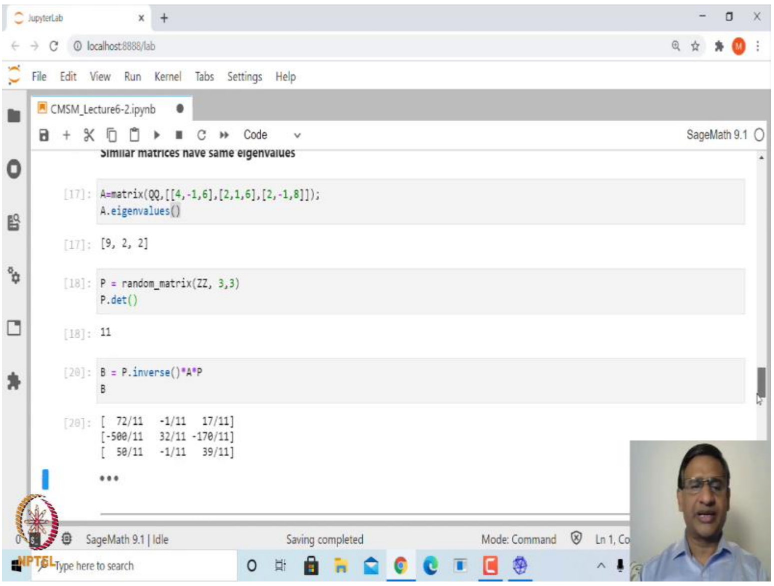

Now, let us look at another result. If you have similar matrices, how did we define similar matrices? If you have two square matrices A and B. We say that A is similar to B. If there exist an invertible matrix, let us say S such that S inverse times A times S is equal to B. And if A and B are similar matrices, they have the same eigenvalues.

Let us just demonstrate this using SageMath. Let us take a matrix A. This matrix has eigenvalue 9 and 2, only 2 distinct eigenvalues. Now, take any random matrix let us say P, a 3 cross 3 random matrix and check if it is invertible and once it is invertible, let us define a matrix B which is P inverse A P. So, that means, A and B are similar. So, I will just print what is B; A and B are similar matrices.

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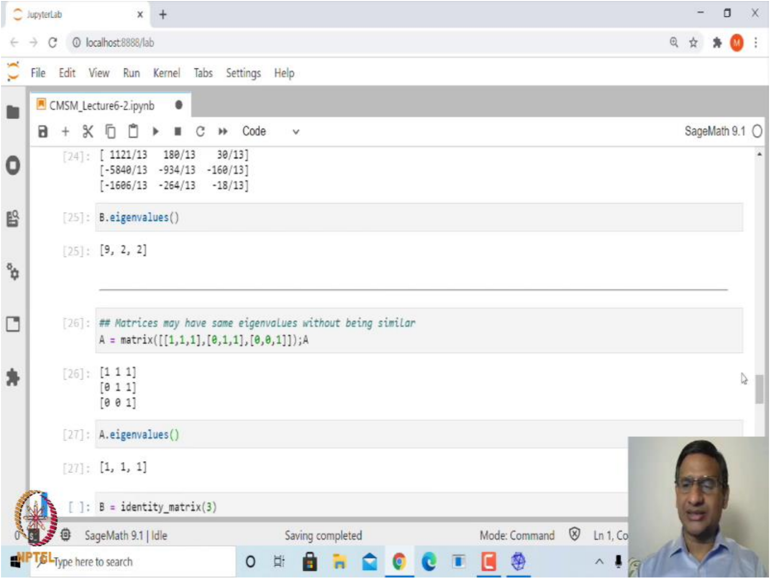

Let us find out what should be eigenvalues of B? Again this is 9, 2 and 2. If I generate once more this random matrix, now this time it is determinant is 0. So, P inverse A P will not not make sense. So, let us regenerate this, now this is invertible and then you can look at B is equal to P inverse A P and find the eigenvalues of B. Again it is 9, 2 and 2. So, it is actually quite easy to show that eigenvalues of similar matrices are the same, right.

Next let us look at an example, where you can have same eigenvalues without matrices, being being similar, right. Similar matrices have the same eigenvalues, but is the converse true? The answer to this is no. So, let us take an example. A is this matrix 1 1 1, 0 1 1, 0 0 1, this is an upper triangular matrix. So, for upper triangular matrix, you can check that the eigenvalues of an upper triangular matrix are diagonal elements. Similarly eigenvalues of a diagonal matrix will be the diagonal entries, similarly lower triangular matrices.

So, this A has 3 eigenvalues 1 1 1, only 1 eigenvalue with multiplicity algebraic multiplicity 3. Now, do we know any other matrix having 1 1 1 as eigenvalues, of course, we can take identity matrix. So, let us take B as identity matrix, 3 cross 3 identity matrix. And then the eigenvalues of B are also 1 1 1.

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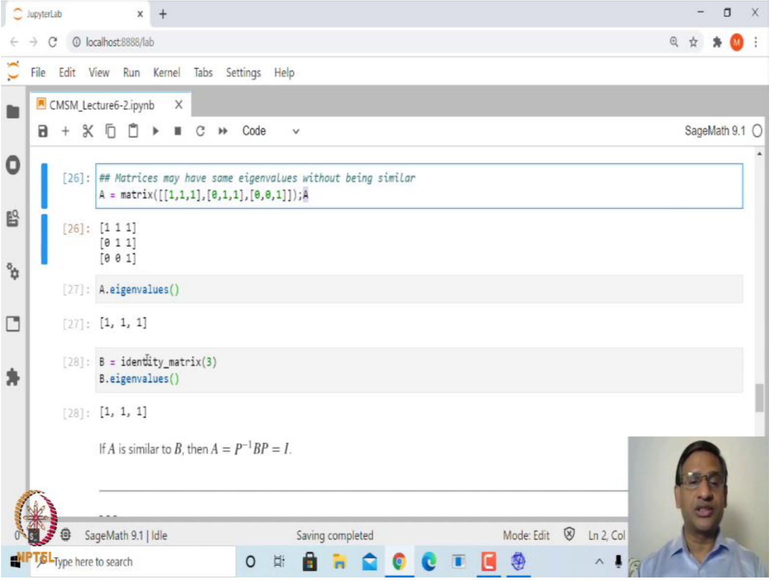

Now, suppose if B is similar to A then what would happen? There exists some matrix S which is invertible matrix , such that S inverse B times s should be A. Now what is S? Let us say there exists invertible matrix P such that P inverse B P should be A. But what is P inverse B P? P is identity matrix. So, P inverse B P is nothing, but P inverse P which is identity. And therefore, A is identity which is not true, A is not identity. This is an example of a matrix A and B, both of them having the same eigenvalues, but they are not similar, right.

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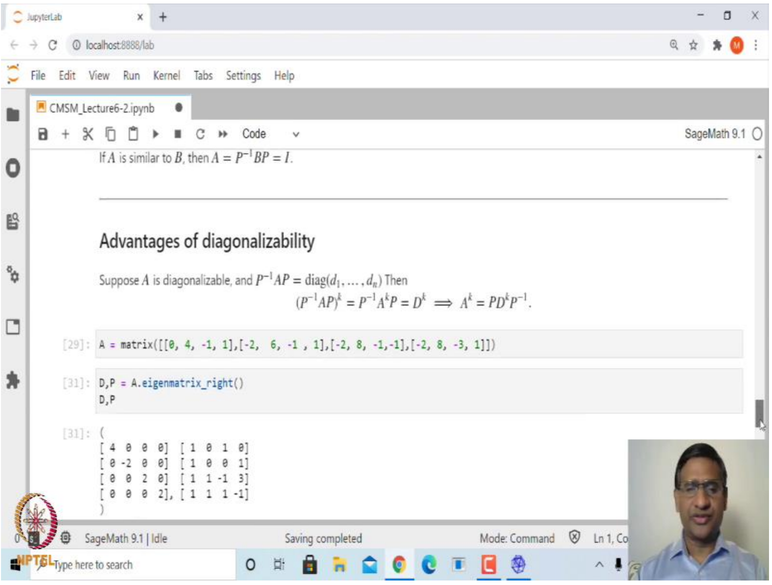

Let us look at what are advantages of diagonalizing a matrix. See diagonal matrix will be the simplest matrix to deal with right. So, in case you are able to diagonalize a matrix many concepts will be very easy to explore. Let us look at one such.

Suppose A is a matrix which is diagonalizable matrix such that P inverse A P is a diagonal matrix whose diagonal entries are d1, d2, dn. Now let us take any power of P inverse A P. You can show that P inverse A P, when you take, let us say square of P inverse A P, what will it be? It will be P inverse A P into P inverse A P. So, P and P inverse you can combine make it identity. What you will be left with, P inverse A square P. So, in general P inverse A P to the power k is nothing, but P inverse times A to the power k times P. And since this is P inverse A P is diagonal matrix D, and power of a diagonal matrix is nothing but take power of each of these diagonal entries. So, that is why di to the power k. From this part you can find out what is A to the power k by multiplying this by P and then on the right hand side multiplying by P inverse. So, A to the power k is P times D to the power k times P inverse.

So, in order to find A to the power k, all you need to do is, once you have obtained this matrix P, by which A is diagonalizable, you have to find D to the power k, which is simple.

Finding power of any diagonal matrix is very simple task. So, that is an advantage of finding power of a matrix and this has lots of applications.

Let us look at this matrix A and check whether this is diagonalizable. So, let us print D and P, this is the same matrix which we started with. So, this is diagonalizable matrix and P is the diagonalzing matrix.

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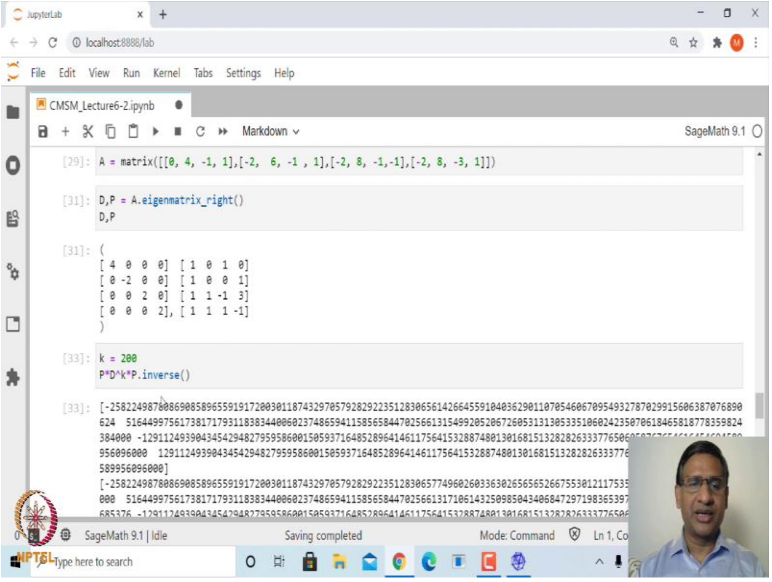

And let us find out, let us say, A to the power 20. What will be A to the power 20? By this result it is nothing but P into D to the power 20 into P inverse. This is A to the power 20. Not just 20, I can find out to the power 200. That is very large number, but in any case we can find it very easily. Let us say power 30, that is A to the power 30.

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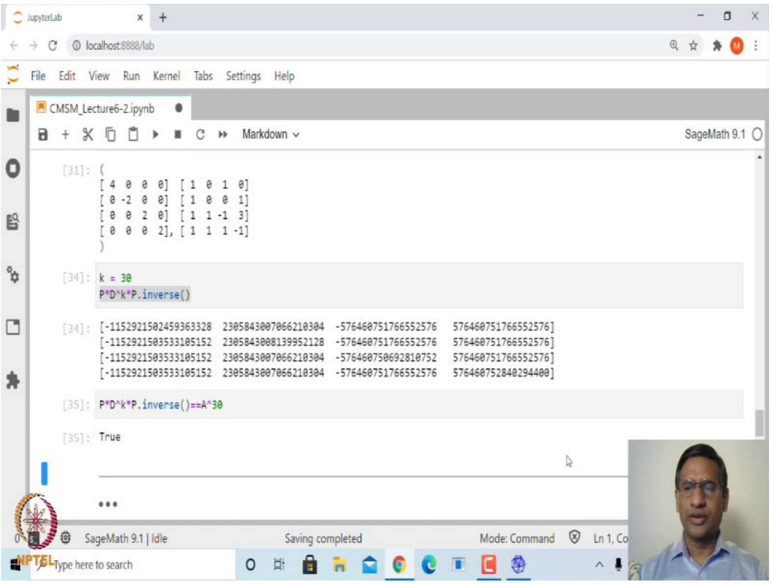

And if you look at is it is this equal to A to the power 30. Let us check if it equal to A to the power 30? The answer should be true right. So, that is one advantage of diagonalizing a matrix, it allows you to find any power of a matrix.

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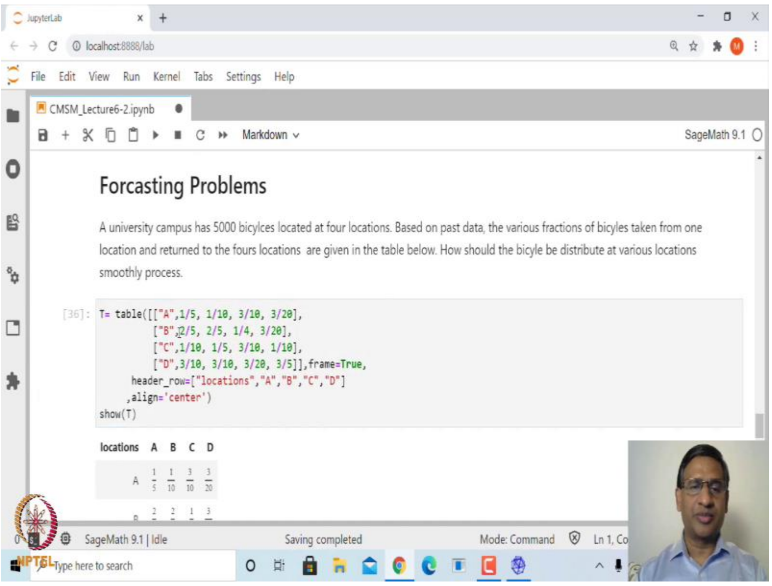

Now, let us look at an application of this diagonalization. Let us look at this problem. Suppose a university campus has, let us say 4 cycle stands and they have 5000 cycles located at 4 locations. And based on some past data the various fraction of cycles that are taken from one place by the students, dropped on another place is given by this table.

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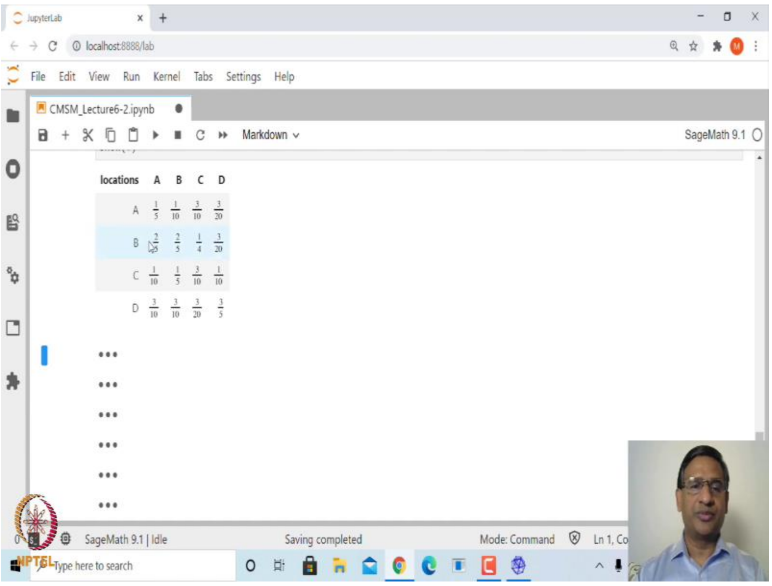

So, let me explain what it means. This table means that one fifth of the cycle hired from location A are dropped at location A. One tenth from B dropped at A. One tenth of cycles hired from or taken from C are dropped at A. One twentieth of cycles that are hired from location D are dropped at location A. This is at the end of the day. So, in the morning you have some cycles and one fifth from A comes at A, one tenth from B comes at A and so on. So, this is a matrix and you can check here. The sum of the first column will be 1, sum of the second column is 1, sum of the third column is 1, sum of the fourth column is 1. So, that means cycles taken from A, all these thing should be back to A, B, C, D. It should not dropped at some other location. This we are not considering right.

So, now let us look at what we can do? Suppose 5000 cycles are there. So, how should one place these cycles at each location, so, that in long run the students or whoever wants to hire or take the cycles should be able to find at least 1 at whichever location he or she goes. So, this process should be smooth right.

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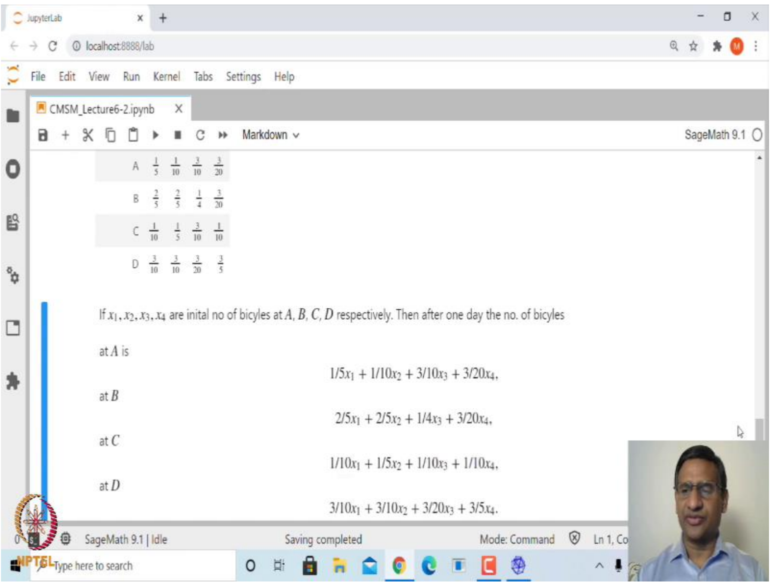

Now, let us look at it. If I look at the fraction of cycles that comes back at A. Suppose you have x1 at A, x2 number of cycles at B, x3 at C and x4 at D. So, x1 times one fifth will come at A, x2 times one tenth will come back at A, three tenth of x3 will come back at A and three twentieth of x4 will come back at A.

So, total number of cycles at A after one day will be this. Similarly you can find out total number of cycles after one day at B is this, and total number of cycles at C at after one day is this, and so on.

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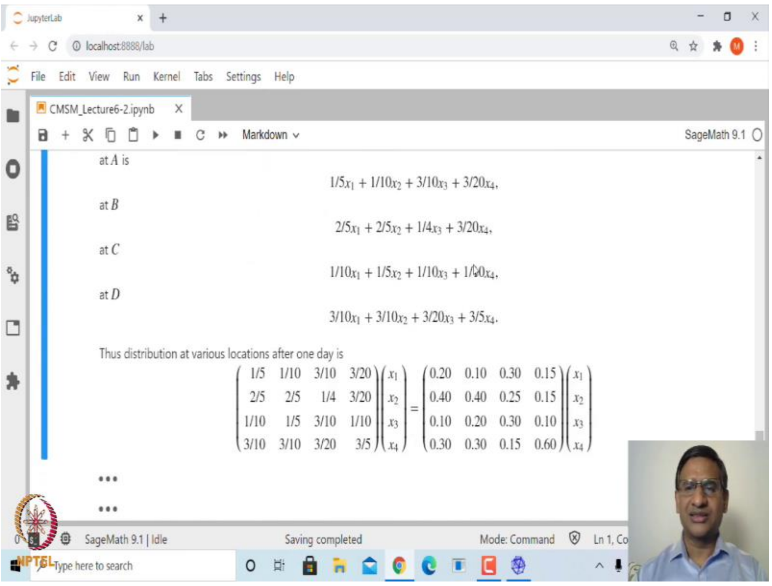

These 4 equations, 4 expressions can be written as a matrix multiplication of this matrix A by x1, x2, x3, x4. Suppose I call this matrix as A times the column vector x1, x2, x3, x4 is the number of cycles that comes back at A.

Now, suppose I call this as x1 dash, x2 dash, x3 dash, x4 dash, that is after one day, then next day will be A times x1 dash, x2 dash, x3 dash, x4 dash, which is nothing but A square times x1, x2, x3, x4. So, that is how you can find after 2 days. After n days, number of cycles that will be at various locations should be equal to A to the power n times x1, x2, x3, x4. So, you see that in long run if I want to find this, I need to find the large power of A.

Now, you just imagine instead of this distributing cycle in a university campus, suppose you want to do this in a big city. I mean big city, you may need few 1000 stands right. And then this is the matrix that you will be dealing with. If there are 1000 stands, 1000 cross 1000. Finding power of a 1000 cross 1000 matrix, even to a computer is quite expensive, computer will also not be able to do it efficiently.

So, that is where this diagonalizability will help. If you can diagonalize this matrix, finding any power of this matrix will be very easy. Now, of course, when diagonalize you have to diagonalize using eigenvalues. Now for finding eigenvalues of 1000 cross 1000 matrix, you need to deal with polynomial of 1000 degree which again is a difficult task. So, that is where numerical method comes into picture. So, you have to find these eigenvalues numerically. So, numerical analysis or numerical methods will be useful in this case.

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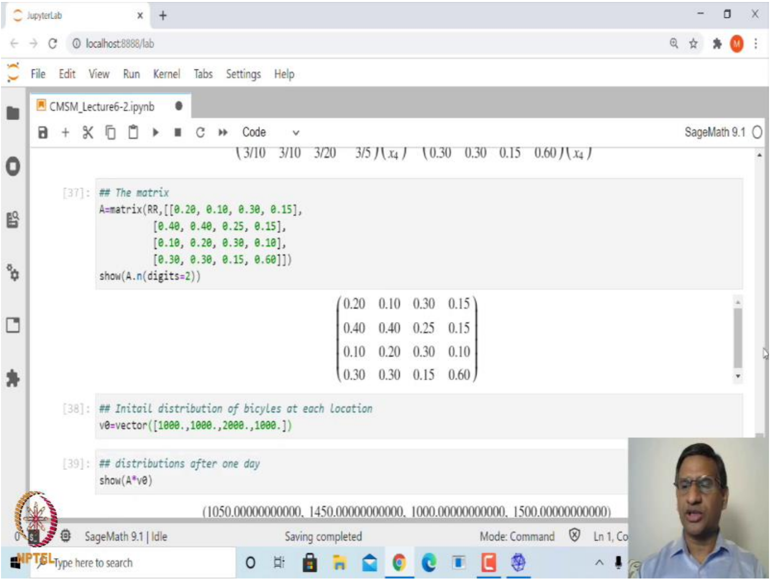

Let us see how we can solve this problem. Let us declare the matrix A as this matrix whose first row is 0.2 0.1, 0.3, 0.15 and so on. So, this way you can check that sum of each column is equal to 1. In fact, in this case you can also show that in case you have sum of each column of a matrix as 1 then 1 will be an eigenvalue. Let us say the initially the distribution of cycle is 1000, 1000, 2000, 1000. So, at location A you put 1000 cycles, at location B another 1000, at location C 2000 and at location D 1000. After one day how many cycles will be there at each location? That can be obtained by A times this initial vector v0.

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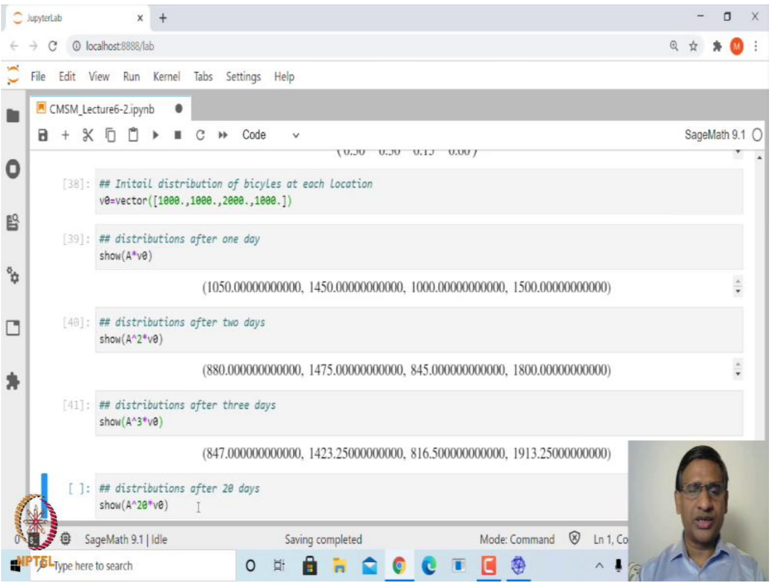

So, after one day the distribution of cycles is something like this. 1050 at the first location, 1450 at the second location, 1000 at the third location and 1500 at fourth location. After second day, that means, you need to find A times the distribution at each location after 1 day which is same as saying A square times v0.

So, after two days the distribution of cycles have become like this. Of course, it is possible that you might get these fraction value, in that case you have to round it off. And after three days, A to the power 3 times v0, this is the distribution, after let us say 20 days what will be the distribution of cycles?

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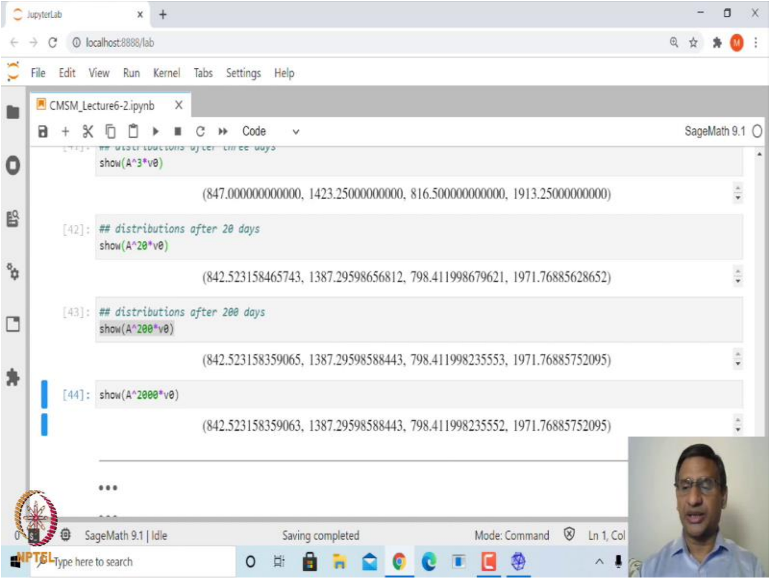

And it says that it is going to be 842.52 at A, 1387.29 at B, 798.41 at C and 1971 at D. Now, instead of 20 days, suppose we find out at 200 days, after 200 days it is almost very similar to what you got after 20 days right. So, again you can see here now this is going to be very much stable, this is what is called stationary vector. So, and let us see what happens after let us say not just 200

days, but let us say after 2000 days, 2000 days again you can see here this is almost very similar. So, it it is actually converging to some fixed vector.

So, that means, if I have these many cycles at these locations the next day will also be almost similar. So, that means, this process will run smoothly. Now this is what is called forecasting problem and there are various types of forecasting problems.

For example, you can look at migration problem or let us say, even Google search is very similar to this. So, you can see here this is actually a real life problem and in which you you may have to deal with matrix of very large dimension. So, that means, 1000 by 1000 or if you are dealing with let us say, Google search the it will deal with all the websites that are connected to each other. That is a millions of websites. So, you may have to deal with million by million matrix and in that case you cannot find power of a matrix. The only way to do is using diagonalizability concept. And, that means, you need to find eigenvalues eigenvectors. But finding eigenvalues eigenvectors of a very large matrix it is not an easy task it is quite expensive. So, you may have to appeal to numerical methods.

We will actually look at some more applications of diagonalizability, also in solving system of ordinary differential equations sometimes later. This has got numerous applications.

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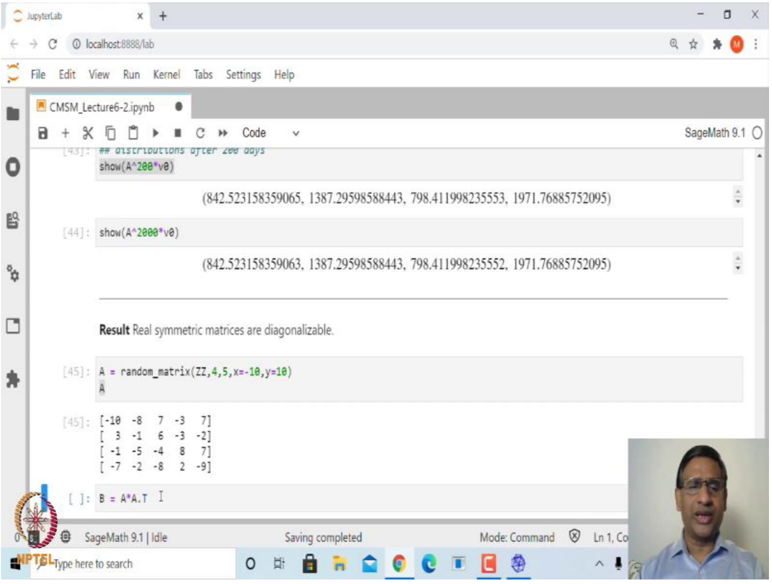

Let me at the end, also look at this, if you have a real matrix then it is always diagonalizable. So, for example, if I take any matrix A let us say this is a 4 cross 5 matrix and suppose from this I want to generate a symmetric matrix, how do I generate? I can I can take A into A transpose.

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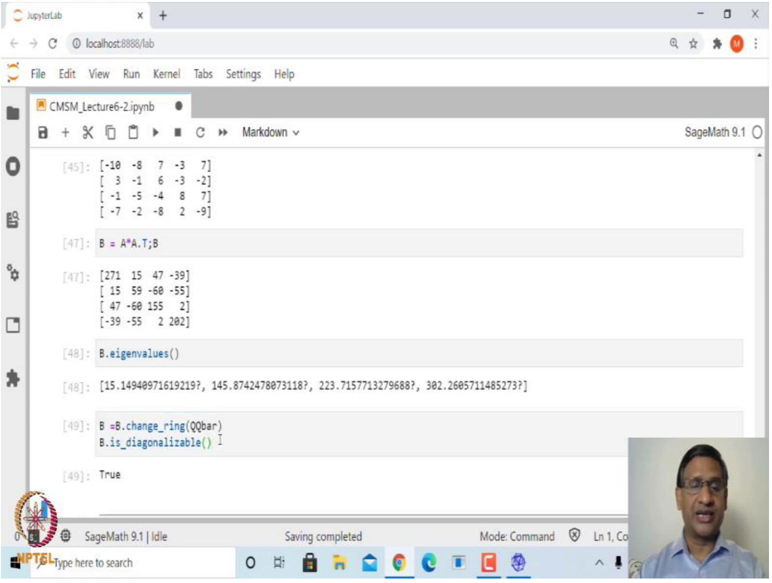

Let us say this B is going to be a 4 cross 4 matrix. Now this is a symmetric matrix or you could have taken B as A transpose A that would have given you 5 cross 5 matrix. And then let us find out eigenvalues of B. These are the all real eigenvalues. Actually you can see here, this is question mark means this is approximate eigenvalue.

So, it has computed eigenvalues numerically, that is what I said that, if you want to deal with arbitrary matrix finding eigenvalues explicitly in closed form will be difficult task. But then you need to appeal to numerical analysis to find eigenvalues numerically.

So, now if you look at whether B is diagonalizable or not. So, let us change the ring of B convert this ring the underlying in field and probably the entries are taken to QQ bar, extended rational field. And then check whether B is diagonalizable, the answer is true and this you can do it with any matrix. Let me generate another random matrix B and let us find out what are eigenvalues.

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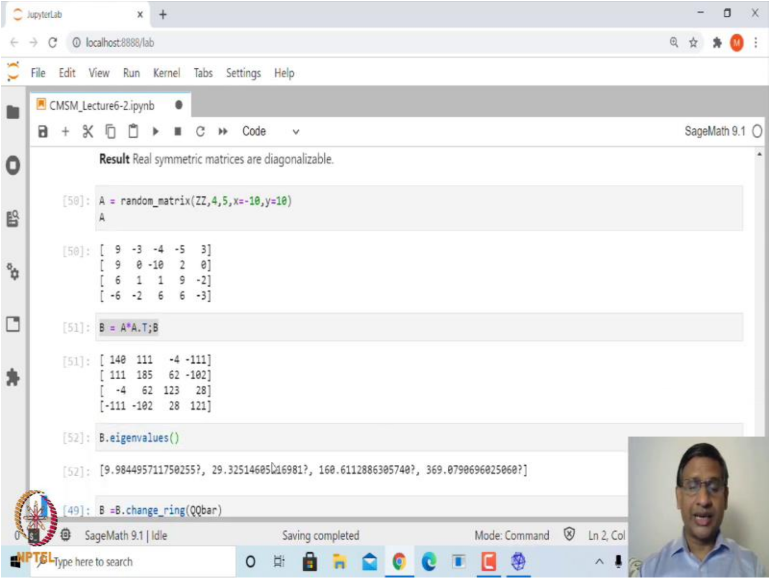

One thing you should notice here that all eigenvalues of this B are non negative actually. For example, instead of B, A transpose A.

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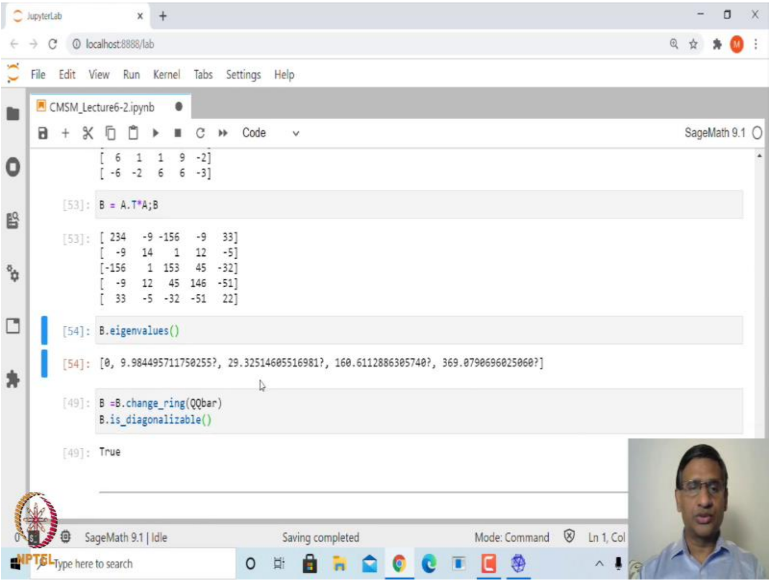

If I look at A transpose times A, this will be 5 cross 5 matrix. Now, if you look at eigenvalues of B. Then one of the eigenvalue should be very close to 0, this is one of the eigenvalue is 0. And you can check whether this matrix is diagonalizable or not. So, this is diagonalizable.

It is actually easy to show that any real symmetric matrix is diagonalizable. You can define the matrix P which diagonalizes and you can show that the columns of P will be orthogonal. The eigenvectors with respect to distinct eigenvalues of a symmetric matrix will be orthogonal.

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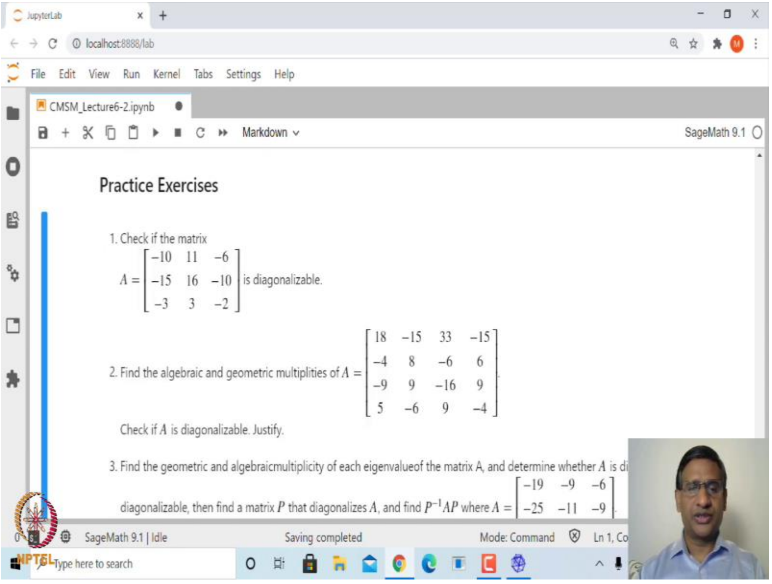

Now, let me leave you with few exercises. First look at this matrix A, diagonalize this. Then next matrix is again 4 cross 4 matrix. And find the algebraic and geometric multiplicities of A and check if it is diagonalizable.

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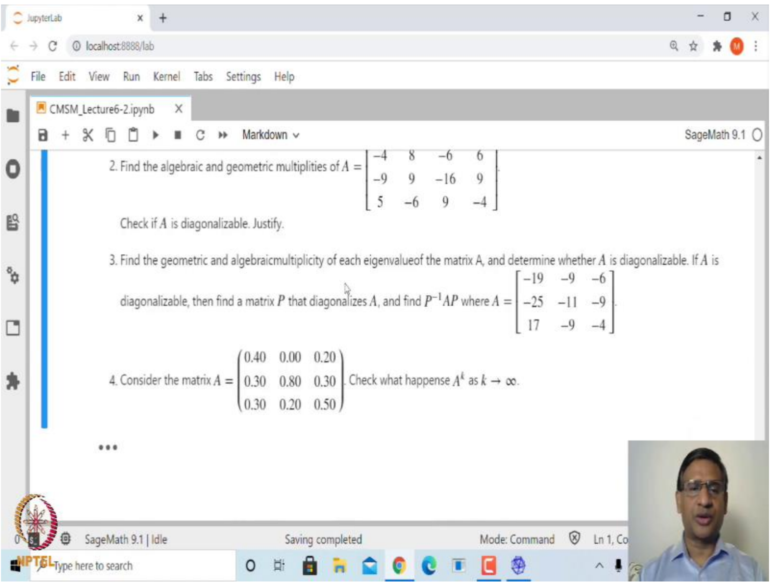

The next problem is to find again algebraic and geometric multiplicity of each of this eigenvalue of A and check whether it is diagonalizable. So, this is again simple exercise. And the last one is look at this matrix A. So, this matrix again the sum of the each column is 1 and each entry is nonnegative such matrices are also known as stochastic matrix.

So, you have stochastic matrix and in which case you can take any power, look at the large power of A. You will see that it converges to a constant matrix. In fact, each column will turn out to be eigenvector with respect to eigenvalue 1.

As I already said if you have a stochastic matrix it always has 1 as eigenvalue. You will get an eigenvector that is what we saw in case of this cycle distribution problem. So, try to solve these problems, it is quite easy. Of course, we will be posting solution of these problems as well.

Thank you very much.