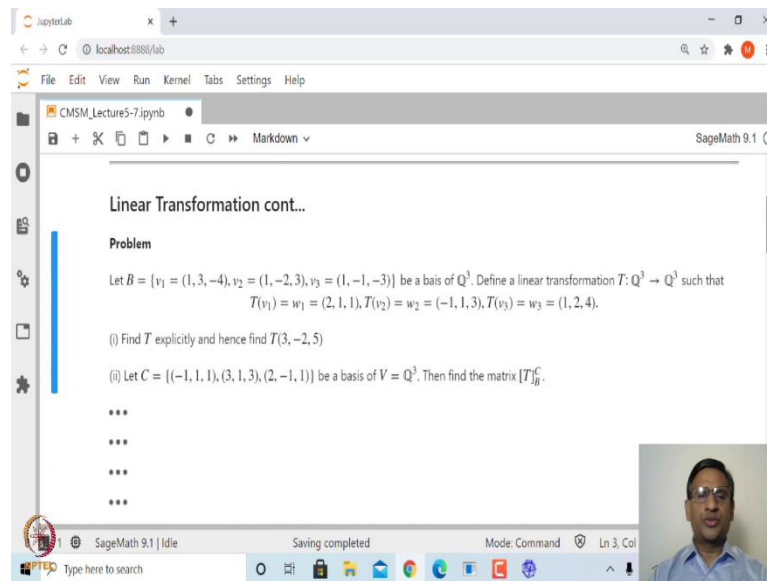


**Computational Mathematics with SageMath**  
**Prof. Ajit Kumar**  
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**Institute of Chemical Technology, Mumbai**

**Linear Transformation cont...**  
**Lecture – 34**  
**Linear Transformations Part 2 with SageMath**

(Refer Slide Time: 00:15)



Welcome to this course on Computational Mathematics with SageMath. In this lecture, we will explore some more concepts in Linear Transformations using SageMath. So, let us start with a problem. So, the problem is as follows.

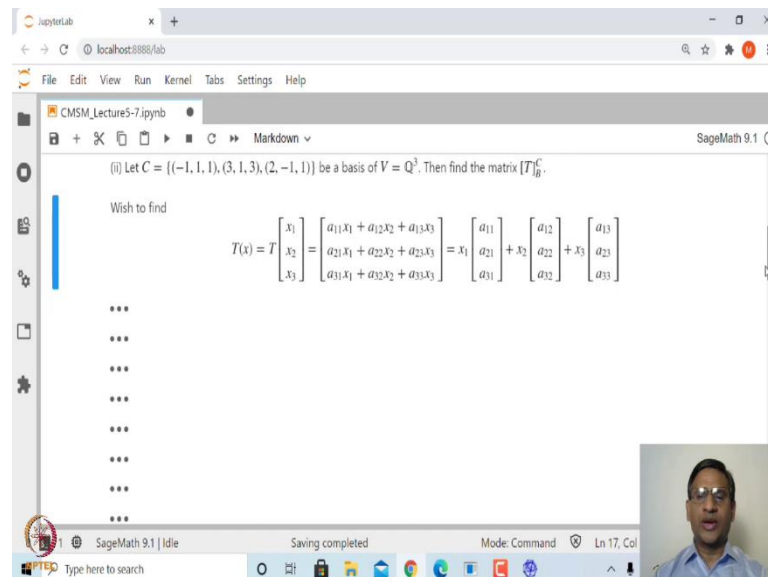
Suppose  $T$  is a linear transformation from  $Q^3$  to  $Q^3$  given by  $T$  of  $v_1$  is equal to  $w_1$ , where  $w_1$  is  $2, 1, 1$ ;  $T$  of  $v_2$  is equal to  $w_2$ , where  $w_2$  is  $-1, 1, 3$ , and  $T$  of  $v_3$  is equal to  $w_3$ , where  $w_3$  is  $1, 2, 4$ , and  $v_1, v_2, v_3$  is a basis, where  $v_1$  is this vector,  $v_2$  is this vector, and  $v_3$  is this vector.

So, you are given, defined a linear transformation  $T$  on basis vectors  $v_1, v_2, v_3$ , and we know that a linear transformation is completely determined once we define its, define it on a basis vector. So, our job is to find  $T$  explicitly. That means, we want to find how  $T$  of

$x_1, x_2, x_3$  should look like, and once you have obtained this, then find, let us say, image of any vector.

And then suppose you have a basis  $C$  which is equal to this, and then you have a basis  $B$  of  $Q^3$ , then find matrix of this linear transformation with respect to bases  $B$  and  $C$ . So, once you have obtained  $T$ , then finding basis will be easy, because these kind of problems we have already come across, how to find matrix of a linear transformation with respect to arbitrary basis.

So, our main task is to find  $T$  explicitly. What do I mean by, what do I mean by that? (Refer Slide Time: 02:15)



Here what we mean is,  $T$  of  $x$ , if  $x$  is  $x_1, x_2, x_3$ , so on, would look like something like this.  $a_{11}x_1$  plus  $a_{12}x_2$  plus  $a_{13}x_3$ , and so on, and this we can write as  $x_1$  times column vector  $a_{11}, a_{21}, a_{31}, \dots$ , plus  $x_2$  times column vector  $a_{12}, a_{22}, a_{32}$ , and  $x_3$  times column vector  $a_{13}, a_{23}, a_{33}$ .

If you want, you can write  $x_1$  times vector  $a$ ,  $x_2$  times vector  $b$ ,  $x_3$  times vector  $c$ , right? So, let us see how do we find this  $a_{11}, a_{21}, a_{31}$ , and so on, ok? (Refer Slide Time: 02:56)

$$T(x) = T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

$$T(x) = T(x_1e_1 + x_2e_2 + x_3e_3) = x_1T(e_1) + x_2T(e_2) + x_3T(e_3)$$

$$T(e_1) = T(a_1v_1 + a_2v_2 + a_3v_3) = a_1T(v_1) + a_2T(v_2) + a_3T(v_3) = a_1w_1 + a_2w_2 + a_3w_3$$

Similarly

$$T(e_2) = b_1w_1 + b_2w_2 + b_3w_3 \text{ and } T(e_3) = c_1w_1 + c_2w_2 + c_3w_3.$$

```

[214]: x = SR.var('x', 4)
[214]: (x0, x1, x2, x3)
***
***
***

```

So, how do we find this? So, let us say  $T$  of  $x$ . Now,  $x$  is, let us say,  $x_1, x_2, x_3$ . Then we can write  $x$  as  $x_1$  times  $e_1$  plus  $x_2$  times  $e_2$  plus  $x_3$  times  $e_3$ , where  $e_1, e_2, e_3$  is a standard basis of  $Q^3$ , right?

So, since  $T$  is linear,  $T$  of  $x$  will be  $x_1$  times  $T$  of  $e_1$  plus  $x_2$  times  $T$  of  $e_2$  plus  $x_3$  times  $T$  of  $e_3$ . Now, do we know  $T$  of  $e_1$ ? We do not know, but what we know is  $T$  of  $v_1$ ,  $T$  of  $v_2$ ,  $T$  of  $v_3$ . However, this  $v_1, v_2, v_3$  is a basis of  $Q^3$ , therefore, any vector can be written as linear combination of  $v_1, v_2, v_3$ . So, in particular, we can write  $e_1$  as scalar, scalar linear combination of  $v_1, v_2, v_3$ .

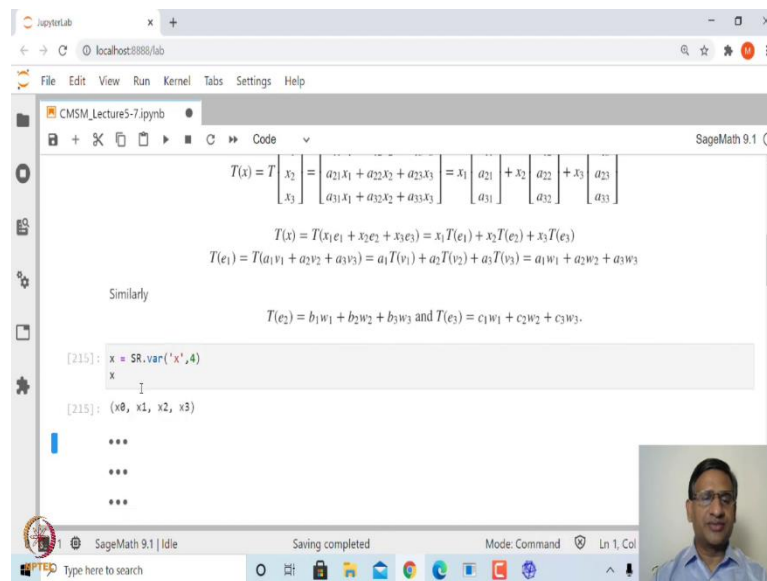
So, suppose, let us say  $e_1$  is equal to  $a_1$  times  $v_1$  plus  $a_2$  times  $v_2$  plus  $a_3$  times  $v_3$ , and then  $T$  of  $e_1$  will be  $a_1$  times  $T$  of  $v_1$  plus  $a_2$  times  $T$  of  $v_2$  plus  $a_3$  times  $T$  of  $v_3$ , which is  $a_1$  times  $w_1$  plus  $a_2$  times  $w_2$  plus  $a_3$  times  $w_3$ . Here what, what are  $a_1, a_2, a_3$ ?  $a_1, a_2, a_3$  are the coordinates of  $e_1$  with respect to basis  $v_1, v_2, v_3$ , right?

Similarly, we can find  $T$  of  $e_2$ , and  $T$  of  $e_2$  will be  $b_1w_1$  plus  $b_2w_2$  plus  $b_3w_3$ , where  $b_1, b_2, b_3$  are coordinates of  $e_2$  with respect to basis  $B$ . Similarly, we can find  $T$  of  $e_3$ . So,

our main task here is to find coordinates of  $e_1, e_2, e_3$ , with respect to basis  $b_1, b_2, b_3$ , and that is it, and after that, we can find out  $T$  of  $e_1$  as this,  $T$  of  $e_2$ ,  $T$  of  $e_3$ , right, ok?

So, let us see how we can do this in Sage. So, first let us declare  $x_1, x_2, x_3$ , as variables. One way to declare this variable, we can declare individually, or you can declare  $x$  is equal to  $SR$  dot  $var$ , and  $x$  is a variable.

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The screenshot shows a JupyterLab interface with a SageMath 9.1 kernel. The code cell contains the following text:

```

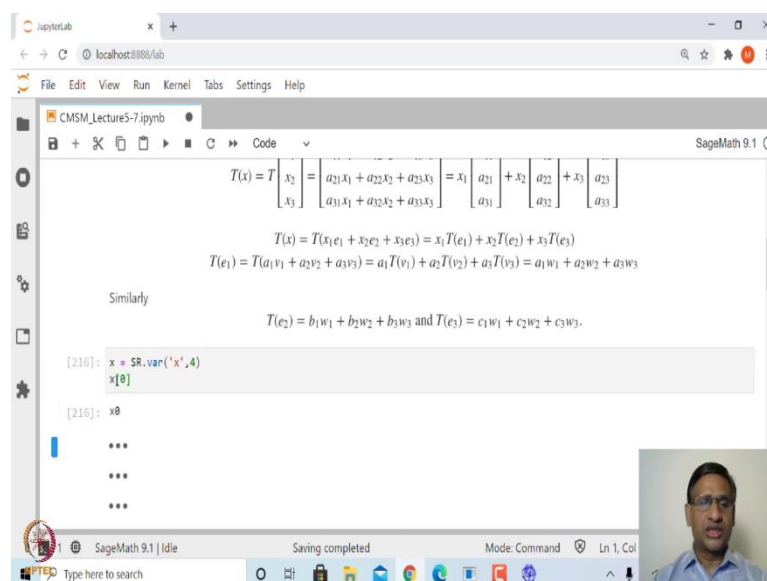
[215]: x = SR.var('x',4)
x
[215]: (x0, x1, x2, x3)
...
...
...

```

The output of the code is displayed below the input, showing the variable  $x$  as a tuple of four symbolic variables:  $(x_0, x_1, x_2, x_3)$ .

So, inside single quote, and 4 denotes here, this  $x$  will be  $x_0, x_1, x_2, x_3$ .

(Refer Slide Time: 05:21)



The screenshot shows the same JupyterLab interface as before, but with the code cell updated to access the first generator of the variable  $x$ :

```

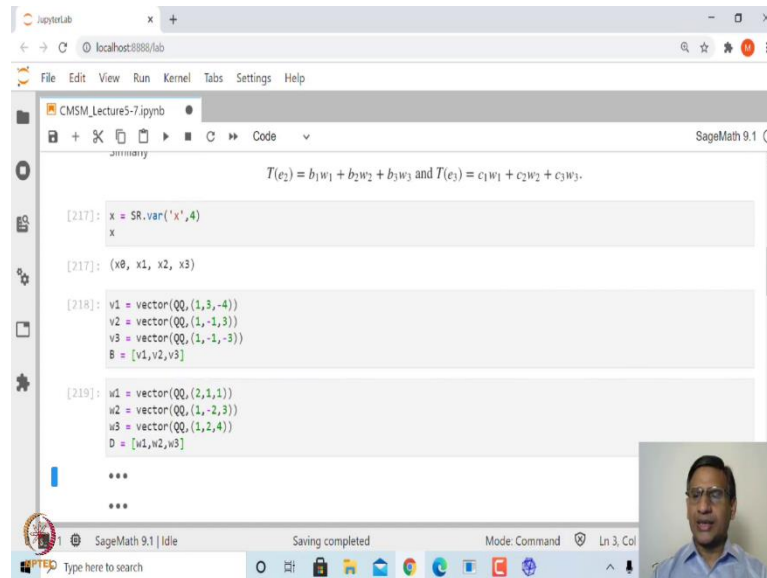
[216]: x = SR.var('x',4)
x[0]
[216]: x0
...
...
...

```

The output of the code is displayed below the input, showing the first generator  $x_0$ .

So,  $x_0$  is actually the first component of this tuple. So, if I say  $x$  of 0, this will be  $x_0$ , and so on, right? So, that is your, that is your, the variables  $x_0$  to  $x_4$ .

(Refer Slide Time: 05:35)



```

[217]: x = SR.var('x',4)
x

[217]: (x0, x1, x2, x3)

[218]: v1 = vector(QQ,(1,3,-4))
v2 = vector(QQ,(1,-1,3))
v3 = vector(QQ,(1,-1,-3))
B = [v1,v2,v3]

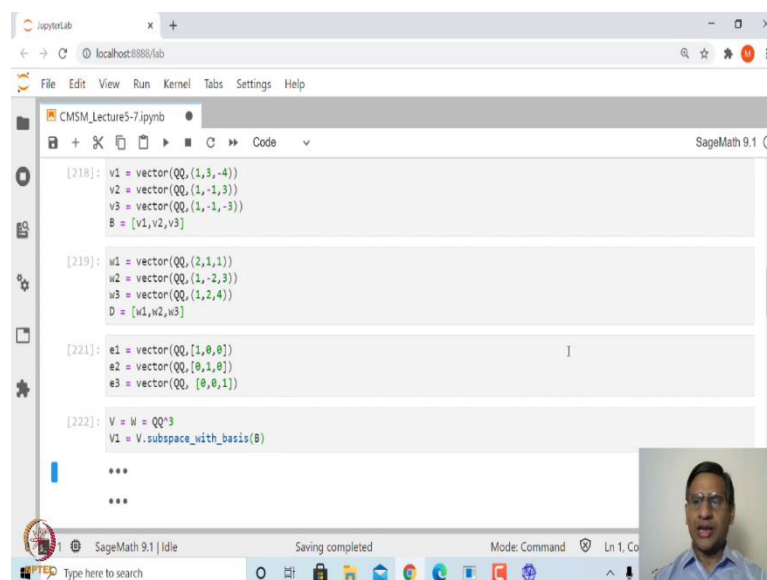
[219]: w1 = vector(QQ,(2,1,1))
w2 = vector(QQ,(1,-2,3))
w3 = vector(QQ,(1,2,4))
D = [w1,w2,w3]

***
***

```

Now, let us look at, let us declare  $v_1, v_2, v_3$ , the basis of the domain, and write  $B$  as list of  $v_1, v_2, v_3$ , ok? Similarly, the images  $w_1, w_2, w_3$ , let us store in list capital  $D$ .

(Refer Slide Time: 05:59)



```

[218]: v1 = vector(QQ,(1,3,-4))
v2 = vector(QQ,(1,-1,3))
v3 = vector(QQ,(1,-1,-3))
B = [v1,v2,v3]

[219]: w1 = vector(QQ,(2,1,1))
w2 = vector(QQ,(1,-2,3))
w3 = vector(QQ,(1,2,4))
D = [w1,w2,w3]

[221]: e1 = vector(QQ,[1,0,0])
e2 = vector(QQ,[0,1,0])
e3 = vector(QQ,[0,0,1])

[222]: V = W = QQ^3
V1 = V.subspace_with_basis(B)

***
***

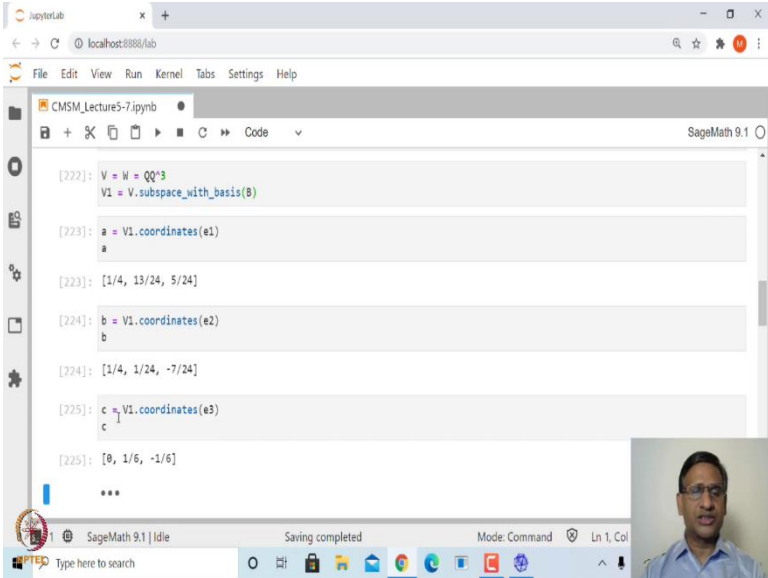
```

Next, let us define  $e_1, e_2, e_3$  as the standard basis of  $Q^3$ . So, this, this is over  $QQ$  we can write, it is not necessary. This  $e_1, e_2, e_3$  are the unit vectors, ok? Next, let us find the

coordinates of  $e_1$  with respect to basis  $B$ . How do we do that? So, first, let us declare  $V$  and  $W$  as domain and co-domain of  $T$ . In this case,  $V$  and  $W$  are the same.

So, that, and it is equal to  $Q^3$ , and let us say  $V_1$ , capital  $V_1$  is equal to the subspace of capital  $V$  with  $B$  as basis. So, that is a capital  $V_1$ . Now, how do I find coordinates of  $e_1$  with respect to basis  $B$ ?

(Refer Slide Time: 06:51)



The screenshot shows a JupyterLab window with a SageMath 9.1 kernel. The code and output are as follows:

```
[222]: V = W = QQ^3
      V1 = V.subspace_with_basis(B)

[223]: a = V1.coordinates(e1)
      a
[223]: [1/4, 13/24, 5/24]

[224]: b = V1.coordinates(e2)
      b
[224]: [1/4, 1/24, -7/24]

[225]: c = V1.coordinates(e3)
      c
[225]: [0, 1/6, -1/6]
```

At the bottom of the JupyterLab window, there is a status bar showing 'SageMath 9.1 | Idle', 'Saving completed', 'Mode: Command', and 'Ln 1, Col'. A small video feed of a man is visible in the bottom right corner of the JupyterLab window.

So, all we need to do is, we simply say capital  $V_1$  dot coordinates, and in the bracket write  $e_1$ , and let us store this in  $a$ .

So, first component will be  $a_1$ , second component  $a_2$ ,  $a_3$ , and so on, right? So, this is your  $a_1$ , this is your  $a_2$ , this is your  $a_3$ . So, we have found out  $a_1$ ,  $a_2$ ,  $a_3$ . Once we have obtained  $a_1$ ,  $a_2$ ,  $a_3$ , we know what is  $T$  of  $e_1$ . Similarly, let us find out  $b$ , the coordinate of  $e_2$  with respect to  $b$ . Similarly, let us find coordinates of  $e_3$  with respect to  $b$ .

(Refer Slide Time: 07:37)

```

[225]: c = V1.coordinates(e3)
c
[225]: [0, 1/6, -1/6]

[226]: Te1 = sum([a[i]*D[i] for i in range(3)]);Te1
[226]: (5/4, -5/12, 65/24)

[227]: Te2 = sum([b[i]*D[i] for i in range(3)]);Te2
[227]: (1/4, -5/12, -19/24)

[228]: Te3 = sum([c[i]*D[i] for i in range(3)]);Te3
[228]: (0, -2/3, -1/6)

***

```

So, once we have found the coordinates of  $e_1, e_2, e_3$  with respect to the basis  $v_1, v_2, v_3$ , let us define  $T$  of  $e_1$ . What will be the  $T$  of  $e_1$ ? The first component of the coordinate of  $e_1$  with respect to  $b$  times  $w_1$ , which is the first component of  $D$ , and so on. So, and all these. This will get you  $T$  of  $e_1$ , that is the  $T$  of  $e_1$ . Similarly, let us find  $T$  of  $e_2$ , and  $T$  of  $e_3$ .

(Refer Slide Time: 08:07)

```

[228]: (0, -2/3, -1/6)

[229]: f = x[1]*Te1+x[2]*Te2+x[3]*Te3
list(f)
[229]: [5/4*x1 + 1/4*x2, -5/12*x1 - 5/12*x2 - 2/3*x3, 65/24*x1 - 19/24*x2 - 1/6*x3]

[230]: show(column_matrix(f))

$$\begin{pmatrix} \frac{5}{4}x_1 + \frac{1}{4}x_2 \\ -\frac{5}{12}x_1 - \frac{5}{12}x_2 - \frac{2}{3}x_3 \\ \frac{65}{24}x_1 - \frac{19}{24}x_2 - \frac{1}{6}x_3 \end{pmatrix}$$


[231]: g(x1,x2,x3) = list(f)
T = linear_transformation(V,W,g)

***

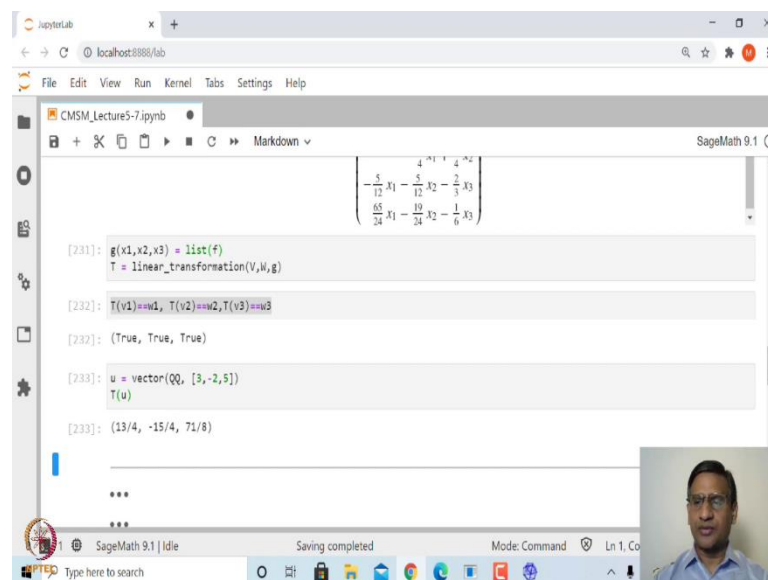
```

So, once we have obtained  $T$  of  $e_1, T$  of  $e_2, T$  of  $e_3$ , then we can find out the  $f$  which is  $x_1$  times  $T$  of  $e_1$  plus  $x_2$  times  $T$  of  $e_2$  plus  $x_3$  times  $T$  of  $e_3$ , and that is what we had here,  $x_1$

times  $T$  of  $e_1$ ,  $x_2$  times  $T$  of  $e_2$ , and  $x_3$  times  $T$  of  $e_3$ . So, that is the components of the linear transformation. Let me show you this as a column matrix.

So, this is your first component, second component, third component. So, we have found  $T$  explicitly. Once we have found  $T$  explicitly, let us, let us find out whether  $T$  of  $e_1$ ,  $T$  of  $v_1$  is  $w_1$ ,  $T$  of  $v_2$  is  $w_2$ . So, how do we do that? First now let me declare a linear transformation with, with these as images. So, let us say  $g$  of  $x_1$ ,  $g$  of  $x_2$ ,  $g$  of  $x_3$  is list of  $f$ ; this is the the list, and then define a linear transformation  $T$  from  $V$  to  $W$  with this image as this  $g$ , and let us, ok?

So, that is the linear transformation. (Refer Slide Time: 09:27)



```

[231]: g(x1,x2,x3) = list(f)
       T = linear_transformation(V,W,g)

[232]: T(v1)==w1, T(v2)==w2,T(v3)==w3

[232]: (True, True, True)

[233]: u = vector(QQ, [3,-2,5])
       T(u)

[233]: (13/4, -15/4, 71/8)
  
```

Now, let us check whether  $T$  of  $v_1$ , is it equal to  $w_2$ ,  $T$  of  $v_2$  is equal to  $w_2$ , and  $T$  of  $v_3$  should be equal to  $w_3$ , and the answer is true. Then we also wanted to find the image of  $T$  of 3 comma minus 2 comma 5, 3 comma minus 2 comma 5. So, let us find that quickly. This is quite easy. I am sure you all of you by now know how to do that.

So, we will declare vector, let me say  $u$  is equal to vector over  $QQ$ , and in the square bracket, write its coordinates 3, minus 2, 5. So, that is the  $u$ , and then let us say  $T$  of  $u$ ,  $T$  in the bracket  $u$ , that is the image, ok? So, we have solved this first part of this problem. The second part is to find the matrix of this  $T$  with respect to basis  $\beta$  on domain, and  $C$  on co-domain.

So, that is easy. I leave that as an exercise for you to do, ok?



(Refer Slide Time: 10:34)

**Composition of Linear Transformations**

Consider the linear transformations  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$  defined by

$$T(x, y, z, w) = (x + z + w, x + y + 2z - w, 2x + y + 3z - 2w)$$

and  $S: \mathbb{R}^3 \rightarrow \mathbb{R}^4$  defined by

$$S(x, y, z) = (x + z, x + 3y + 2z, 2x - y + 3z, y - z).$$

(i) Find the composition  $S \circ T$ .

(ii) Find the matrix  $C$  associated with  $S \circ T$  with respect to the standard bases.

(iii) Hence show that the matrix of the composition is product of the matrices associated with  $T$  and  $S$  with respect to standard bases.

Now, let us look at next problem. If you have two linear transformations, you can take its composition. Of course, it should be compatible. Similarly, you can define sum of two linear transformations.

So, composition, how do we define? So, suppose let us say  $T$  is a linear transformation from  $\mathbb{R}^4$  to  $\mathbb{R}^3$ , and  $S$  is a linear transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^4$ , then the composition  $S \circ T$  makes sense, and it will be a linear transformation from  $\mathbb{R}^4$  to  $\mathbb{R}^4$ .

It is easy to check that if you have two linear transformations  $T$  and  $S$ , then  $S \circ T$  is also a linear transformation. That is a very simple exercise. Similarly, you can check that  $T + S$  will also be a linear transformation. In this case,  $T + S$  will not make sense, but if  $T$  and  $S$  are both linear transformation from, let us say,  $V$  to  $W$ ,  $T + S$  will make sense, and  $T + S$  will also be a linear transformation.

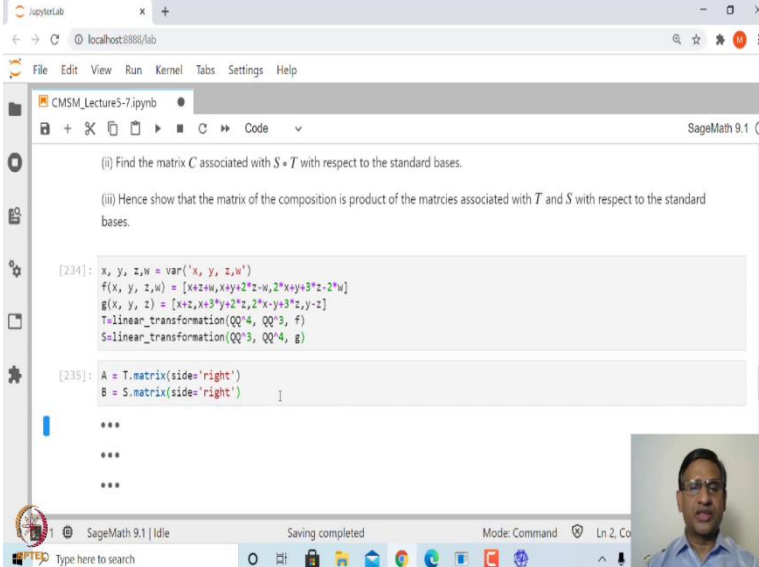
So, once we have a linear transformation  $S \circ T$ , then let us find matrix of this composition, let us say, with respect to standard basis, and then we want to check how is

this matrix of this composition of linear transformation related to matrix of  $T$ , and matrix of  $S$ .

So, that is, the result says that if, if  $C$  is the matrix of linear transformation  $S$  composite  $T$ , and if  $A$  is a matrix of  $T$ , and  $B$  is matrix of  $S$ , then  $C$  will be  $B$  times  $A$ . That is the result, that is the, the third part. So, this is what we want to check.

Now, here we are doing everything with respect to the standard basis. However, the same result will be true with, if you take any basis on domain and co-domain of  $T$ , similarly on the co-domain of  $S$ , right?

(Refer Slide Time: 12:35)



The screenshot shows a JupyterLab window with a SageMath 9.1 kernel. The code in the cell is as follows:

```
(ii) Find the matrix  $C$  associated with  $S \circ T$  with respect to the standard bases.

(iii) Hence show that the matrix of the composition is product of the matrices associated with  $T$  and  $S$  with respect to the standard bases.

[234]: x, y, z, w = var('x, y, z, w')
f(x, y, z, w) = [x+z+w, x+y+2*z-w, 2*x+y+3*z-2*w]
g(x, y, z) = [x+z, x+3*y+2*z, 2*x-y+3*z, y-z]
T=linear_transformation(QQ^4, QQ^3, f)
S=linear_transformation(QQ^3, QQ^4, g)

[235]: A = T.matrix(side='right')
B = S.matrix(side='right')

***
***
***
```

So, let us look at how we can solve this problem. This is quite easy. So, first let us declare  $S$  and  $T$  as linear transformations. So, first let us declare  $x, y, z, w$ , as variables.  $f$  of  $x, y, z$  is the first linear transformation  $T$ .  $g$  of  $x, y$  is equal to second linear transformation, that is the  $S$ , and let us define  $T$  to be linear transformation from  $Q^4$  to  $Q^3$ , and with image as  $f$ ; similarly,  $S$  as linear transformation from  $Q^3$  to  $Q^4$  with image as  $g$ . So, once we have declared  $S$  and  $T$ , now let us declare, let us find matrices of  $T$  and  $S$  with respect to a standard basis.

(Refer Slide Time: 13:29)

```

[236]: U = S*T;
C = U.matrix(side='right')
C

[236]: [ 3  1  4 -1]
[ 8  5 13 -6]
[ 7  2  9 -3]
[-1  0 -1  1]

[237]: B*A==C

[237]: True

Exercise Show that the matrix the composition  $S \circ T$  is product of the matrices of  $T$  and  $S$  with respect to arbitrary bases.

***

```

So, we can simply say, capital A is equal to T dot matrix, side is equal to right. Similarly, B is equal to S dot matrix, side equal to right. So, that will give you the matrices of T and S with respect to standard basis. Now, let us declare the composition. The composition of S and T is nothing but S star T. Similarly, the sum will be S plus T. Now, let us find matrix of this U, which is S composite T with respect to a standard basis. So, this matrix is this. Here you can also print what are these matrices A, and B. Now, let us check whether B into A is equal to C, that is what we wanted to prove, and the answer is yes, it is true. The matrix C is nothing but product of A and B. And of course, you should explore the same thing with respect to arbitrary basis on domain and co-domains of S and T. So that, I will leave it as an exercise, this is quite simple, ok?(Refer Slide Time: 14:22)

```

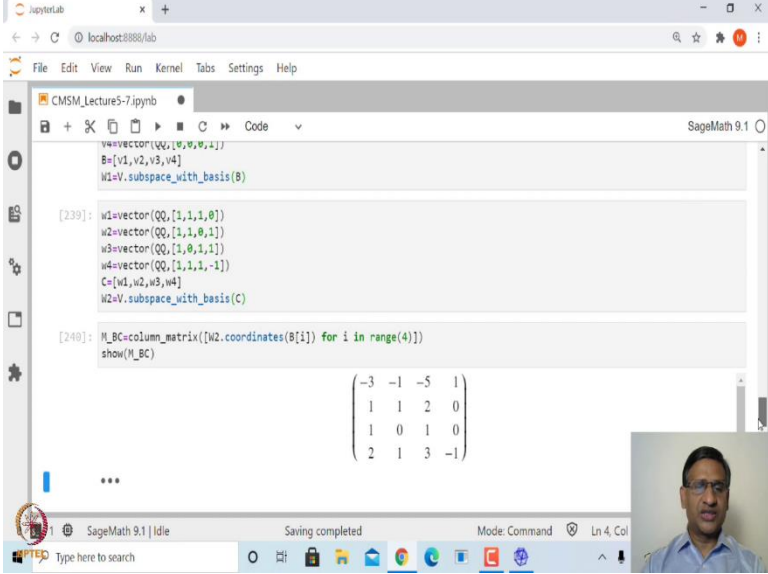
[238]: V=QQ^4
v1=vector(QQ,[1,0,0,0])
v2=vector(QQ,[1,1,0,0])
v3=vector(QQ,[1,0,-1,0])
v4=vector(QQ,[0,0,0,1])
B=[v1,v2,v3,v4]
w1=V.subspace_with_basis(B)

```

Now, let us look at one more thing. Suppose we want to, we have already seen how to find matrix of change of basis.

If you have two basis, then, of a vector space  $V$ , and if you take any vector, then we can write its coordinate with respect to the two basis, and we have seen that the, the two coordinates are related by a matrix. There is a relationship, and that relationship comes from what we call a change of basis matrix, ok? So, this we have already seen.

However, this change of basis matrix can also be obtained as follows. So, you look at a linear transformation which is identity linear transformation from  $V$  to  $V$ , and write matrix of  $I_V$  with respect to basis  $B$  and  $C$ . That matrix is nothing but matrix of change of basis from  $B$  to  $C$ . So, let us look at these. So, suppose we take the space  $V$  is equal to  $\mathbb{Q}\mathbb{Q}$  to the power 4, and  $v_1, v_2, v_3$  as a basis of this, and let us define  $w_1$  to be a subspace of capital  $V$  with  $B$  as basis. (Refer Slide Time: 15:39)



```

v=vector(QQ,[w,w,w,w])
B=[v1,v2,v3,v4]
W1=V.subspace_with_basis(B)

[239]: w1=vector(QQ,[1,1,1,0])
       w2=vector(QQ,[1,1,0,1])
       w3=vector(QQ,[1,0,1,1])
       w4=vector(QQ,[1,1,1,-1])
       C=[w1,w2,w3,w4]
       W2=V.subspace_with_basis(C)

[240]: M_BC=column_matrix([W2.coordinates(B[i]) for i in range(4)])
       show(M_BC)

```

$$\begin{pmatrix} -3 & -1 & -5 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & -1 \end{pmatrix}$$

Similarly, let us define a basis  $w_1, w_2, w_3, w_4$  of  $V$ , and define capital  $W_2$  to be the subspace of capital  $V$  with respect to with  $C$  as a basis, and then we know that, how, we know how to find matrix of change of basis from  $B$  to  $C$ . We have already done this.

So, let us just recall that, how we did. So, the matrix of change of basis from  $B$  to  $C$  can be obtained as the column matrix of the coordinates of  $v_1, v_2, v_3, v_4$  with respect to basis  $w_1, w_2, w_3, w_4$ , and this is this matrix. (Refer Slide Time: 16:24)

```

v=vector(QQ,[w,w,w,w])
B=[v1,v2,v3,v4]
W1=V.subspace_with_basis(B)

[239]: w1=vector(QQ,[1,1,1,0])
w2=vector(QQ,[1,1,0,1])
w3=vector(QQ,[1,0,1,1])
w4=vector(QQ,[1,1,1,-1])
C=[w1,w2,w3,w4]
W2=V.subspace_with_basis(C)

[241]: I_BC=column_matrix([W2.coordinates(B[i]) for i in range(4)])
show(I_BC)

```

$$\begin{pmatrix} -3 & -1 & -5 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & -1 \end{pmatrix}$$

This, actually in the, in the problem I had denoted this by  $I_{BC}$ .

So, let me declare this as  $I_{BC}$ , right? Once we, this is the matrix of change of basis. Now, what is that we want to show? We want to show that this matrix is nothing but matrix of the linear transformation, the identity linear transformation from  $V$  to  $V$  with respect to basis  $B$  on domain, and  $C$  on capital domain, on co-domain, right? So, let us do that. Let us look at. (Refer Slide Time: 16:50)

```

show(I_BC)

[242]: var('x1,x2,x3,x4')
Id(x1,x2,x3,x4)=[x1,x2,x3,x4]
S = linear_transformation(W1, W2, Id)
$.matrix(side='right')

[242]: [-3 -1 -5 1]
[ 1  1  2  0]
[ 1  0  1  0]
[ 2  1  3 -1]

```

So, the, declare variables  $v_1, x_1, x_2, x_3, x_4$ , and identity map  $Id$   $x_1, x_2, x_3, x_4$  is,  $x_1, x_2, x_3, x_4$  as a list, and declare  $S$  as a linear transformation which takes from  $W_1$  to  $W_2$ .

Remember this is with respect to basis  $B$  on domain, and  $C$  on co-domain, that is why here  $v_1$   $W_1$ , and  $W_2$ .

You could have declared this from  $V$  to  $V$ , and then you could have restricted  $S$  to the basis  $B$  and  $C$  on domain, and co-domain, right, ok?

Now, let us find the matrix of  $S$  with respect to these two basis, and this is this, and you can see that these two  $IBC$  is same as matrix of  $IV$ , with respect to basis  $B$  and  $C$ . So, that is what we have proved. Now, let us use this to look at what happens to matrix of a linear transformation when you change the basis, ok?

So, you have a linear transformation, you, you find its matrix with respect to a given basis. Now, you have another set of basis on domain and co-domain, and then find matrix of that same linear transformation with respect to the two new bases, and then we want to check how are they related to each other, ok? So, let us see how are they related.

(Refer Slide Time: 18:22)

**Linear transformation and change of basis**

Consider the linear transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined as

$$T(x_1, x_2, x_3) = (3x_1 + 7x_2 + x_3, 2x_1 + 5x_2 + x_3, -x_1 - 2x_2 + x_3)$$

Let us consider a basis  $B_1 = \{(1, -3, 2), (2, -1, -1), (2, 4, -3)\}$  of domain of  $T$  and a basis  $B_2 = \{(0, 1, -3), (-1, 2, -1), (2, -4, 3)\}$  of the codomain of  $T$ .

Let  $\rho_1$  be the linear transformation that takes the standard basis of  $\mathbb{R}^3$  to the basis  $B_1$  of  $\mathbb{R}^3$ . Let  $\rho_2$  be the linear transformation that takes standard basis of  $\mathbb{R}^3$  to the basis  $B_2$  of  $\mathbb{R}^3$ .

Let  $A$  be the matrix of  $T$  with respect to the standard bases. Let  $B$  be the matrix of  $T$  with respect to bases  $B_1$  and  $B_2$ . Then show that

$$\rho_2 A = B \rho_1 \text{ that is, } \rho_2^{-1} B \rho_1 = A$$

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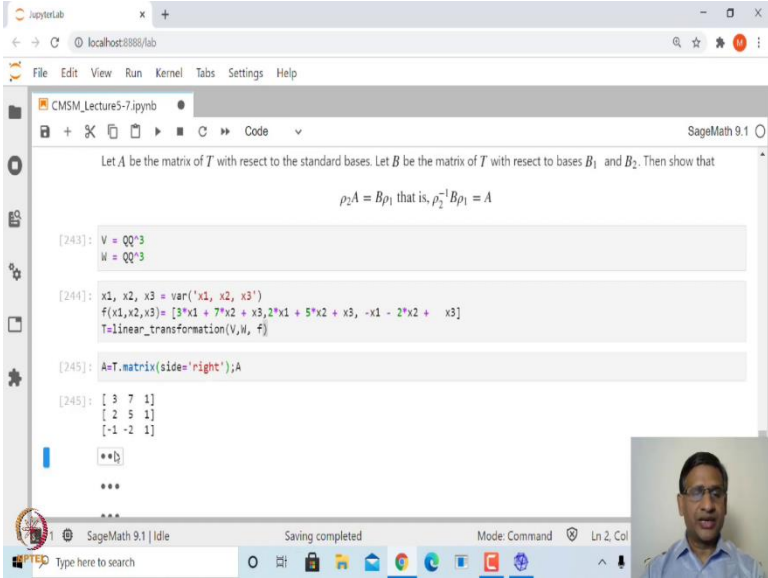
So, let us start with a problem. We have a linear transformation  $T$  which is from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , given by this components, and let us fix a basis  $b_1$  of  $T$ , and  $b_2$ ,  $b_1$  of domain of  $T$ , and  $b_2$  of co-domain of, of  $T$ , right, and let us assume that  $\rho_1$  is the linear transformation

that takes standard basis to  $b_1$ , and  $\rho_2$  a linear transformation that takes standard basis to  $b_2$ .

Now, suppose let us say, capital  $A$  is the matrix of  $T$  with respect to standard basis. Now, we have two basis  $\beta_1$ ,  $b_1$  and  $b_2$  on domain and co-domain. So, let us say capital  $B$  is the matrix of  $T$  with respect to basis  $\beta_1$ ,  $b_1$  and  $b_2$ , and then we want to check how are this  $A$  and  $B$  related.

So, this  $A$  and  $B$  are related by this relation  $\rho_2$  times  $A$ ,  $\rho_2$  is the matrix of the change of basis from standard basis to  $b_1$ , and  $\rho_2$  that is the  $\rho_1$ .  $\rho_1$  is the matrix of change of basis from standard basis to  $b_1$ , and  $\rho_2$  is the matrix of change of basis from standard basis to  $b_2$ . So, this is how they are related,  $A$  and  $B$  are related. In, in particular, you can write this as  $\rho_2$  inverse times  $B$   $\rho_1$  is equal to  $A$ .

So, in this case, actually one can, if you have  $A$ , and two matrices  $A$  and  $B$  which are related by this relation, we say that  $A$  and  $B$  are similar matrices. So, the matrix of  $T$  with respect to different basis are similar matrices, that is what we are proving, ok? So, let us work out this. (Refer Slide Time: 20:21)



The screenshot shows a JupyterLab window with a SageMath 9.1 kernel. The code in the cell is as follows:

```
[243]: V = QQ^3
W = QQ^3

[244]: x1, x2, x3 = var('x1, x2, x3')
f(x1,x2,x3) = [3*x1 + 7*x2 + x3, 2*x1 + 5*x2 + x3, -x1 - 2*x2 + x3]
T = linear_transformation(V,W, f)

[245]: A = T.matrix(sides='right'); A

[245]: [ 3 7 1]
[ 2 5 1]
[-1 -2 1]
```

The output shows the matrix  $A$  as a 3x3 matrix. The status bar at the bottom indicates 'SageMath 9.1 | Idle' and 'Mode: Command'.

So, you start with  $V$  and  $W$  which in this, case both are. both are  $\mathbb{Q}^3$ , and let us define the linear transformation  $T$  with given expressions as a list in  $f$ . So, that is the matrix, that is the  $T$ , and let us define the matrix of  $T$  with respect to standard basis, let me call that as  $A$ . (Refer Slide Time: 20:57)

```

[246]: u1 = vector(QQ, [ 1, -3, 2])
      u2 = vector(QQ, [ 2, -1, -1])
      u3 = vector(QQ, [ 2, 4, 3])
      B1 = [u1, u2, u3]

[247]: v1 = vector(QQ, [0, 1, -3])
      v2 = vector(QQ, [-1, 2, -1])
      v3 = vector(QQ, [ 2, -4, 3])
      B2 = [v1, v2, v3]

[248]: V1 = V.subspace_with_basis(B1)
      W1 = W.subspace_with_basis(B2)

[249]: T2 = T.restrict_domain(V1).restrict_codomain(W1)
      B = T2.matrix(sides='right'); B

[249]: [-43  -6 181]
      [-196 -32 481]
      [-106 -17 259]
      ***

```

Similarly, let us find, let us look at the basis  $b_1$  which is  $u_1, u_2, u_3$ . You can check that this is a basis. This is a basis. Similarly, take  $b_2$  a basis which is  $v_1, v_2, v_3$ . Again you can check that this forms a basis. We need to check that  $b_1$  and  $b_2$  are linearly independent. I have already checked that these are basis.

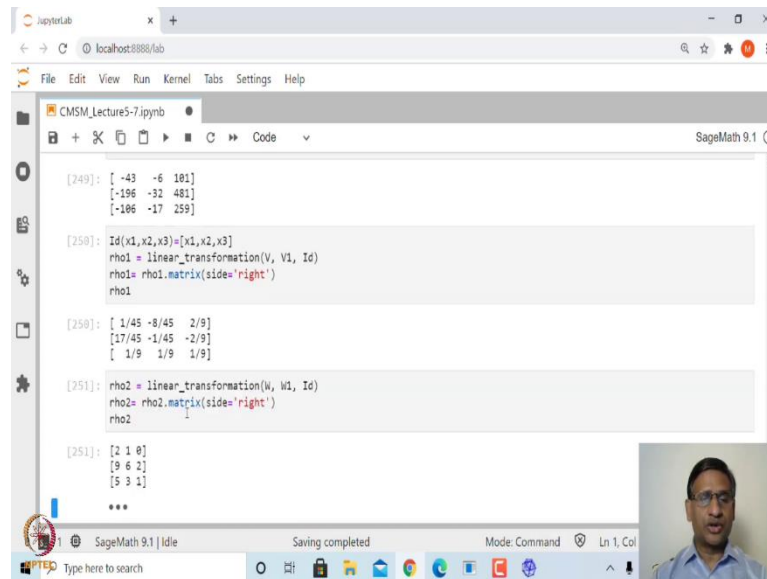
So, I do not need to do that, but when you take arbitrary vectors, and you want to form a basis, you must check that it does form a basis. Now, let us say  $V$  capital  $V_1$  as a subspace of  $V$  with respect to basis  $b_1$ , and  $W_1$  as subspace of  $V$  with basis  $b_2$ , ok? So, this let us run this, and once you have run this, then let us now restrict  $T$  to this basis  $v_1$ , and to subspace  $V_1$  on domain, and  $W_1$  on co-domain.

So that means, we are finding now, linear transformation with respect to the new basis. So, and that matrix of  $T_2$  which is  $T$  restricted on domain and co-domain will, let us call that as  $B$ . So, that is the matrix  $B$ . We have found the matrix  $A$  over here, that is the matrix, somewhere we calculated. Yes, this is the matrix  $A$  which is with respect to standard basis.

Now, we have calculated matrix  $B$  with respect to the basis  $b_1$  and  $b_2$ .

(Refer Slide Time: 22:36)





```

[249]: [ -43  -6 181]
       [-196 -32 481]
       [-186 -17 259]

[250]: Id(x1,x2,x3)=[x1,x2,x3]
rho1 = linear_transformation(V, V1, Id)
rho1 = rho1.matrix(side='right')
rho1

[250]: [ 1/45 -8/45  2/9]
       [17/45 -1/45 -2/9]
       [ 1/9  1/9  1/9]

[251]: rho2 = linear_transformation(W, W1, Id)
rho2 = rho2.matrix(side='right')
rho2

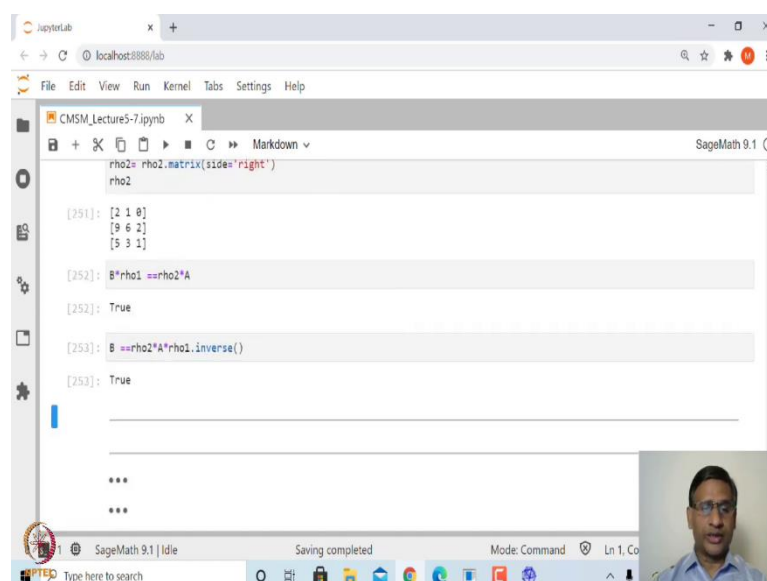
[251]: [2 1 0]
       [9 6 2]
       [5 3 1]
***

```

Now, let us look at, let us find rho 1 and rho 2. What is rho 1? Rho 1 is the matrix of change of basis from standard basis to b1. So, how do we obtain? Just now we saw that we can obtain that as a matrix of identity linear transformation from V to V, and with respect to basis standard basis on domain, and basis b1 on co-domain.

So, let us define a linear transformation with identity as image from V to V1. V1 is subspace of V with basis b1. So, let us find rho 1, and matrix of rho 1, that is rho 1, that is the change of, matrix of change of basis from standard basis to b1. Similarly, let us calculate matrix of change of, matrix of change of basis from standard basis to b2, this is rho 2.

(Refer Slide Time: 23:36)



```

rho2 = rho2.matrix(side='right')
rho2

[251]: [2 1 0]
       [9 6 2]
       [5 3 1]

[252]: B*rho1 == rho2*A
[252]: True

[253]: B == rho2*A*rho1.inverse()
[253]: True

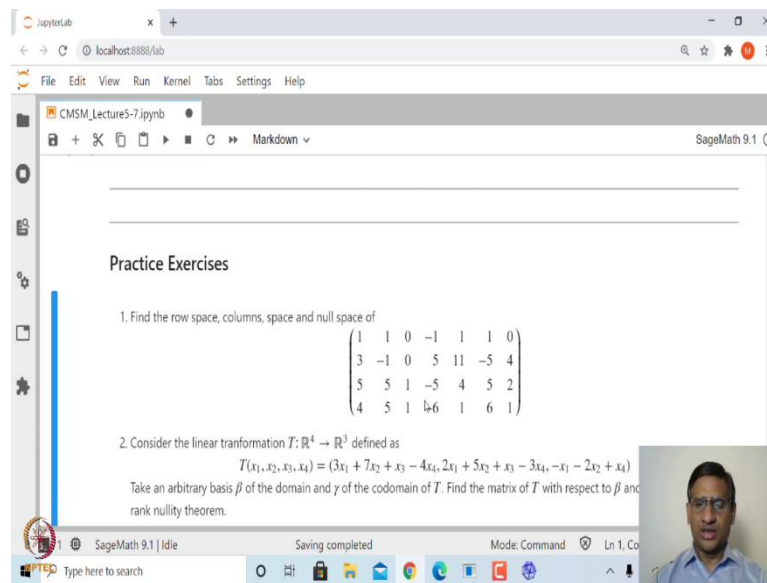
***

```

Now, let us check what we wanted. We can check that  $b_1 B$  times  $\rho_1$  is equal to  $\rho_2$  times  $A$ , and the answer is true, or we can check that  $B$  is equal to  $\rho_2$  times  $A$  times  $\rho_1$  inverse. So, we have verified what we wanted, ok? Now, here we have taken base, standard basis for  $T$ , and then we changed two basis  $b_1$  and  $b_2$ .

You could start with  $b_1, b_2$  on  $T$ , and take another basis let us say  $d_1$  and  $d_2$  on domain and co-domain, and then also you can explore this relation. So that, I will leave it as an exercise, ok?

(Refer Slide Time: 24:20)



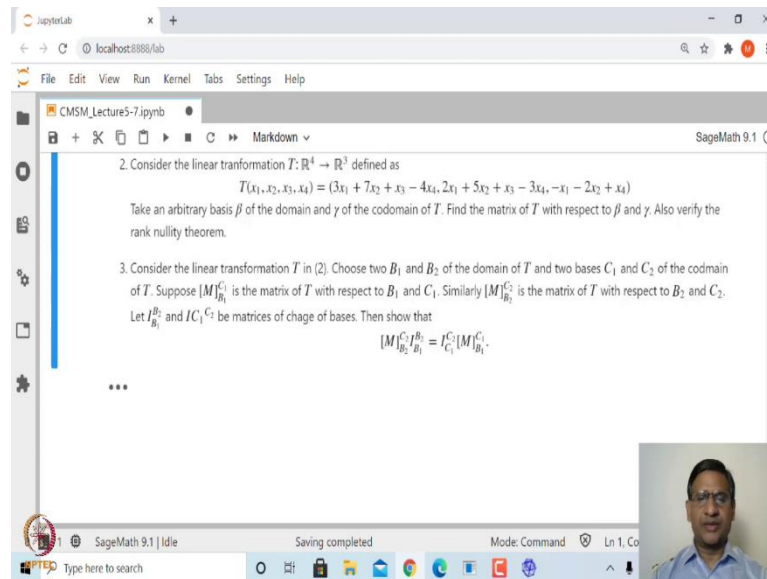
The screenshot shows a JupyterLab window with a SageMath 9.1 notebook titled 'CMSM\_Lecture5-7.ipynb'. The notebook content includes a section titled 'Practice Exercises' with two problems:

- Find the row space, columns, space and null space of
 
$$\begin{pmatrix} 1 & 1 & 0 & -1 & 1 & 1 & 0 \\ 3 & -1 & 0 & 5 & 11 & -5 & 4 \\ 5 & 5 & 1 & -5 & 4 & 5 & 2 \\ 4 & 5 & 1 & -6 & 1 & 6 & 1 \end{pmatrix}$$
- Consider the linear transformation  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$  defined as
 
$$T(x_1, x_2, x_3, x_4) = (3x_1 + 7x_2 + x_3 - 4x_4, 2x_1 + 5x_2 + x_3 - 3x_4, -x_1 - 2x_2 + x_4)$$
 Take an arbitrary basis  $\beta$  of the domain and  $\gamma$  of the codomain of  $T$ . Find the matrix of  $T$  with respect to  $\beta$  and rank nullity theorem.

The interface also shows a video feed of the presenter in the bottom right corner.

Now, let me leave you with three simple exercises for you to practice. The first exercise is, find row space, column space, and null space of this 4 cross, this is 7, matrix, 4 cross 7 matrix.

(Refer Slide Time: 24:36)



And second problem is to consider a linear transformation  $T$  from  $\mathbb{R}^4$  to  $\mathbb{R}^3$  with these components, and then take arbitrary basis on domain and co-domain, and find its matrix with respect to arbitrary basis, and also verify rank-nullity theorem for this, and similarly look at the third problem.

This is, actually I already said that as an exercise. The linear transformation you are given, a linear transformation from  $V$  to  $W$ , find its matrix with respect to two basis  $\beta_1$ , and  $\beta_2$ . Similarly, find its matrix with respect to another two basis  $\gamma_1$ , and  $\gamma_2$ , and then find the relationship between the two matrices which you have obtained, ok?

So, these are the three simple exercises. Let me stop here. In the next class we will look at how to deal with eigenvalues and eigen, eigenvectors, and as you know, these eigenvalues, eigenvectors have numerous applications in linear algebra. So, we will see you in the next class.

Thank you.