Computational Mathematics with SageMath
Prof. Ajit Kumar
Department of Mathematics
Institute of Chemical Technology, Mumbai

Linear Transformation cont…
Lecture – 34
Linear Transformations Part 2 with SageMath

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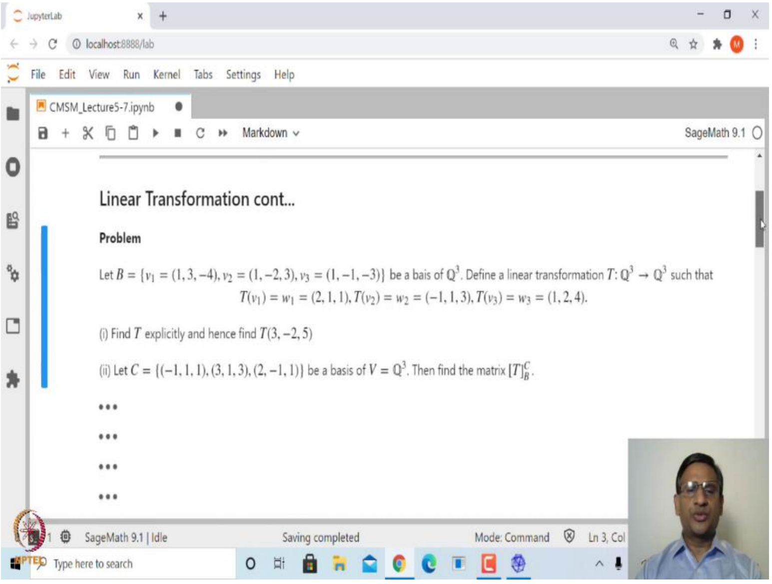


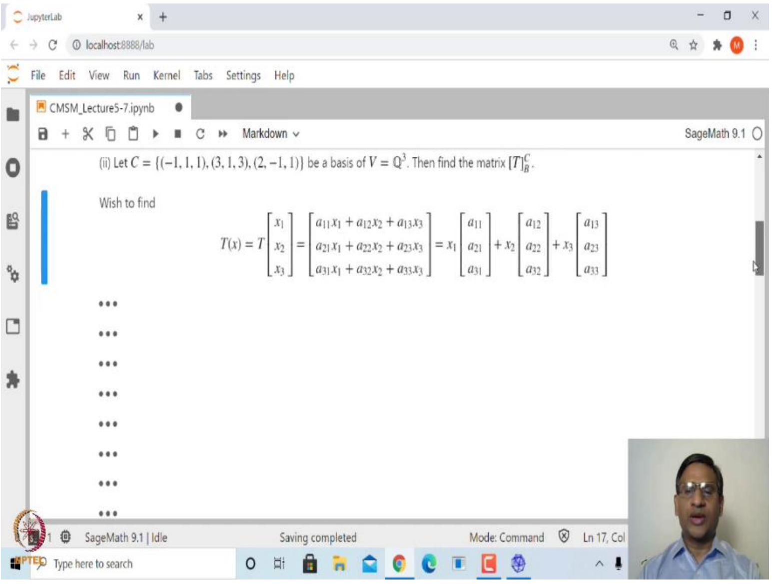
Welcome to this course on Computational Mathematics with SageMath. In this lecture, we will explore some more concepts in Linear Transformations using SageMath. So, let us start with a problem. So, the problem is as follows.

Suppose T is a linear transformation from Q 3 to Q 3 given by T of v1 is equal to w1, where w1 is 2, 1, 1; T of v2 is equal to w 2, where w 2 is minus 1, 1, 3, and T of v3 is equal to w 3, where w 3 is 1, 2, 4, and v1, v2, v3 is a basis, where v1 is this vector, v2 is this vector, and v3 is this vector.

So, you are given, defined a linear transformation T on basis vectors v1, v2, v3, and we know that a linear transformation is completely determined once we define its, define it on a basis vector. So, our job is to find T explicitly. That means, we want to find how T of x1, x2, x3 should look like, and once you have obtained this, then find, let us say, image of any vector.

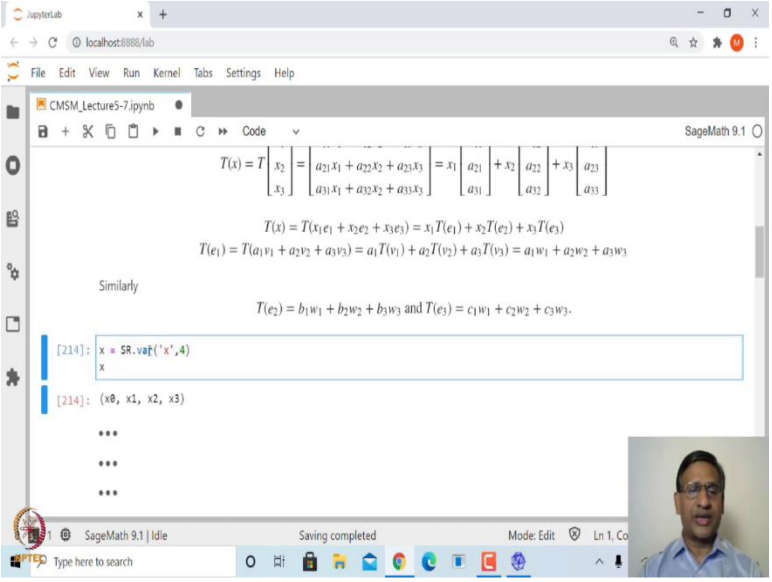
And then suppose you have a basis C which is equal to this, and then you have a basis B of Q 3, then find matrix of this linear transformation with respect to bases B and C. So, once you have obtained T, then finding basis will be easy, because these kind of problems we have already come across, how to find matrix of a linear transformation with respect to arbitrary basis.

So, our main task is to find T explicitly. What do I mean by, what do I mean by that? (Refer Slide Time: 02:15)



Here what we mean is, T of x, if x is x1, x2, x3, so on, would look like something like this. a11*x1 plus a12*x2 plus a13*x3, and so on, and this we can write as x1 times column vector a11, a21, a31,…, plus x2 times column vector a12, a22, a32, and x3 times column vector a13, a23, a33.

If you want, you can write x1 times vector a, x2 times vector b, x3 times vector c, right? So, let us see how do we find this a11, a21, a31, and so on, ok? (Refer Slide Time: 02:56)



So, how do we find this? So, let us say T of x. Now, x is, let us say, x1, x2, x3. Then we can write x as x1 times e1 plus x2 times e2 plus x3 times e3, where e1, e2, e3 is a standard basis of Q 3, right?

So, since T is linear, T of x will be x1 times T of e1 plus x2 times T of e2 plus x3 times T of e3. Now, do we know T of e1? We do not know, but what we know is T of v1, T of v2, T of v3. However, this v1, v2, v3 is a basis of Q 3, therefore, any vector can be written as linear combination of v1, v2, v3. So, in particular, we can write e1 as scalar, scalar linear combination of v1, v2, v3.

So, suppose, let us say e1 is equal to a1 times v1 plus a2 times v2 plus a3 times v3, and then T of e1 will be a1 times T of v1 plus a2 times T of v2 plus a3 times T of v3, which is a1 times w1 plus a2 times w2 plus a3 times w3. Here what, what are a1, a2, a3? a1, a2, a3 are the coordinates of e1 with respect to basis v1, v2, v3, right?

Similarly, we can find T of e2, and T of e2 will be b1*w1 plus b2*w2 plus b3*w3, where b1, b2, b3 are coordinates of e2 with respect to basis B. Similarly, we can find T of e3. So, our main task here is to find coordinates of e1, e2, e3, with respect to basis b1, b2, b3, and that is it, and after that, we can find out T of e1 as this, T of e2, T of e3, right, ok?

So, let us see how we can do this in Sage. So, first let us declare x1, x2, x3, as variables. One way to declare this variable, we can declare individually, or you can declare x is equal to SR dot var, and x is a variable.

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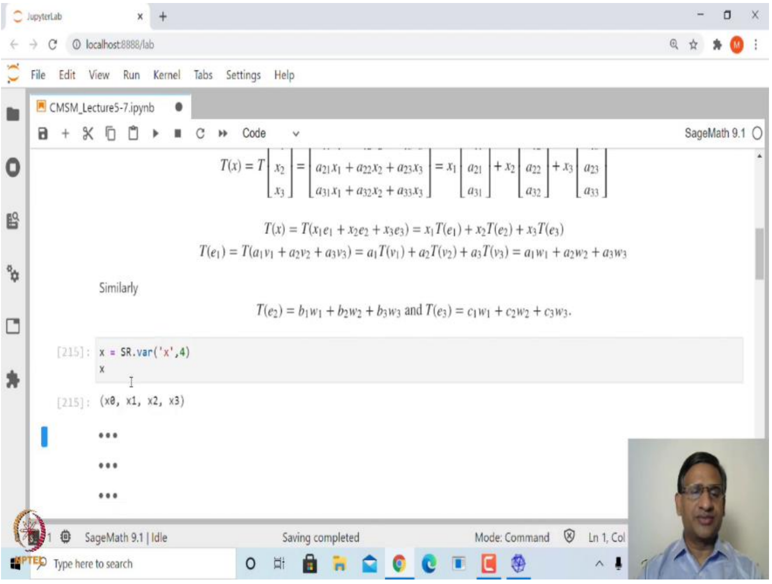

So, inside single quote, and 4 denotes here, this x will be x0, x1, x2, x3.

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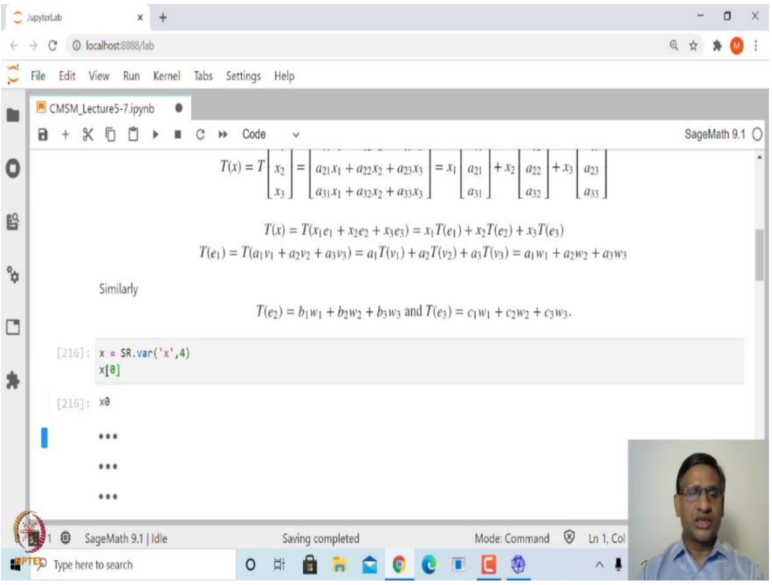

So, x0 is actually the first component of this tuple. So, if I say x of 0, this will be x0, and so on, right? So, that is your, that is your, the variables x0 to x4.

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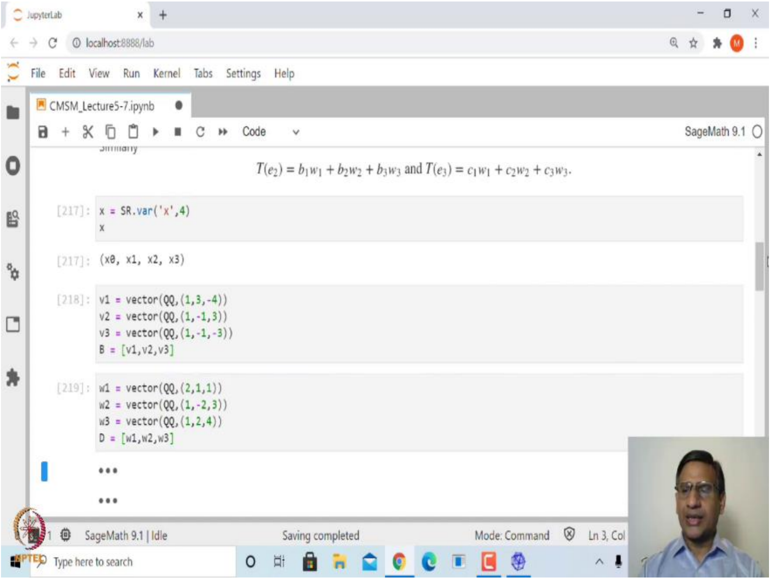

Now, let us look at, let us declare v1, v2, v3, the basis of the domain, and write B as list of v1, v2, v3, ok? Similarly, the images w1, w2, w3, let us store in list capital D.

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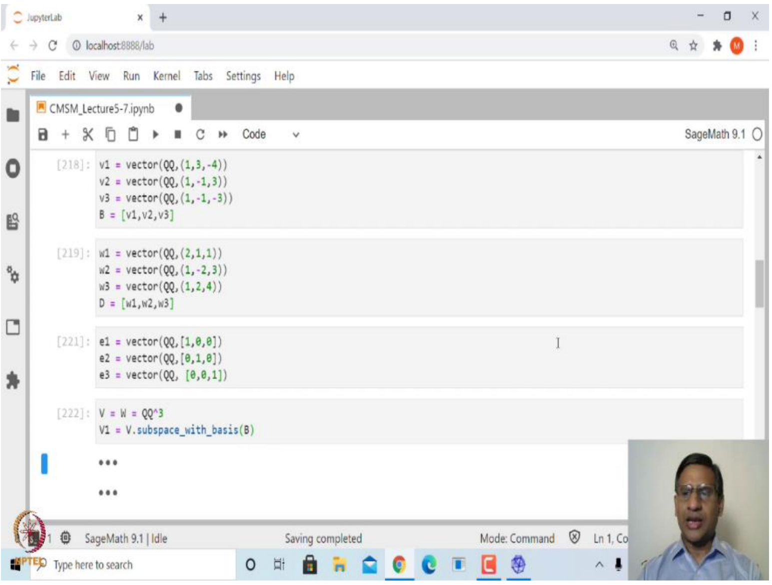

Next, let us define e1, e2, e3 as the standard basis of Q 3. So, this, this is over QQ we can write, it is not necessary. This e1, e2, e3 are the unit vectors, ok? Next, let us find the coordinates of e1 with respect to basis B. How do we do that? So, first, let us declare V and W as domain and co-domain of T. In this case, V and W are the same.

So, that, and it is equal to Q 3, and let us say V1, capital V1 is equal to the subspace of capital V with B as basis. So, that is a capital V1. Now, how do I find coordinates of e1 with respect to basis B?

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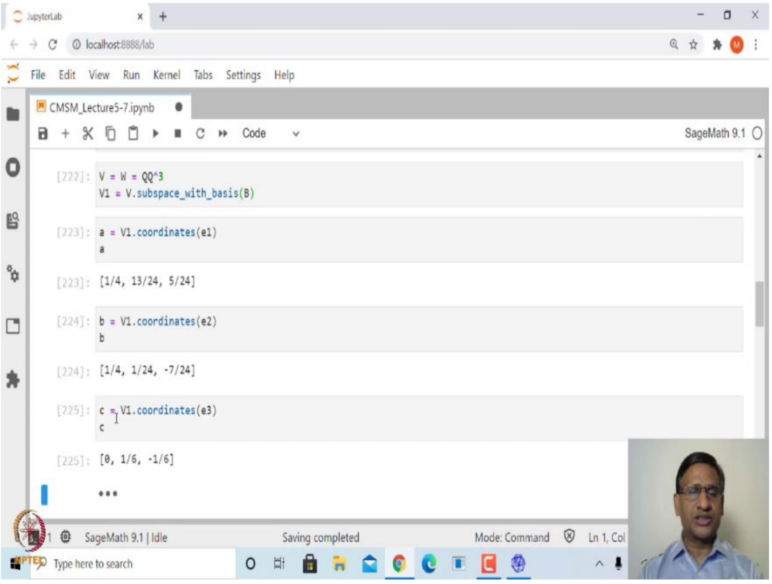

So, all we need to do is, we simply say capital V1 dot coordinates, and in the bracket write e1, and let us store this in a.

So, first component will be a1, second component a2, a3, and so on, right? So, this is your a1, this is your a2, this is your a3. So, we have found out a1, a2, a3. Once we have obtained a1, a2, a3, we know what is T of e1. Similarly, let us find out b, the coordinate of e2 with respect to b. Similarly, let us find coordinates of e3 with respect to b.

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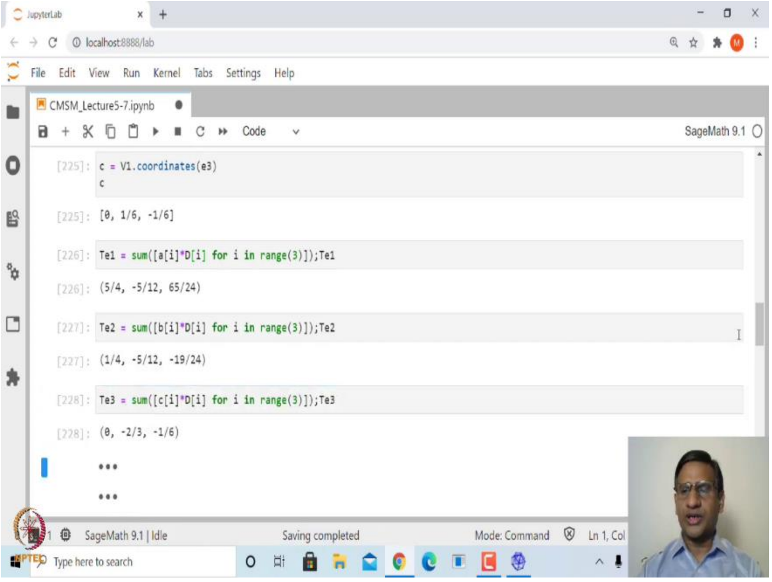

So, once we have found the coordinates of e1, e2, e3 with respect to the basis v1, v2, v3, let us define T of e1. What will be the T of e1? The first component of the coordinate of e1 with respect to b times w1, which is the first component of D, and so on. So, and all these. This will get you T of e1, that is the T of e1. Similarly, let us find T of e2, and T of e3.

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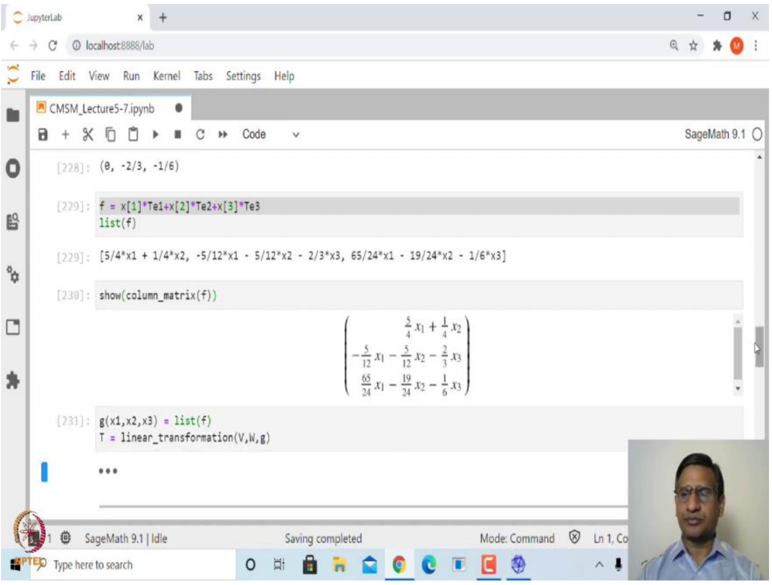
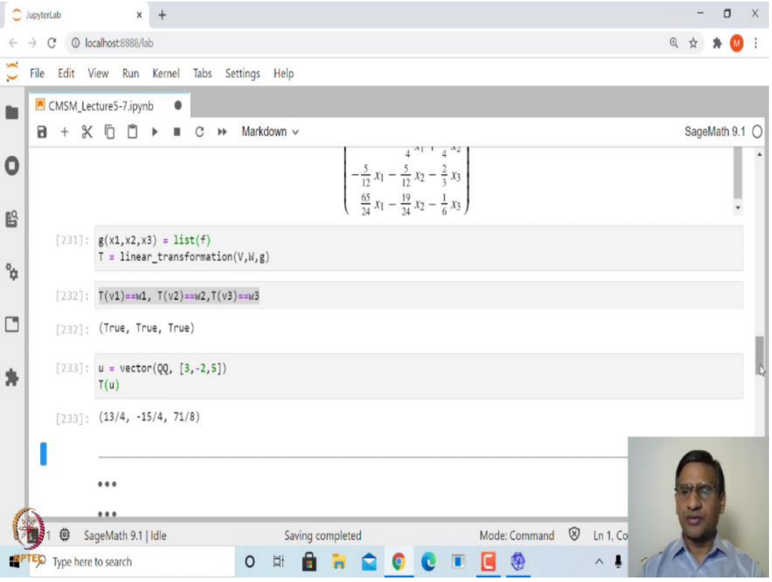


So, once we have obtained T of e1, T of e2, T of e3, then we can find out the f which is x1 times T of e1 plus x2 times T of e2 plus x3 times T of e3, and that is what we had here, x1 times T of e1, x2 times T of e2, and x3 times T of e3. So, that is the components of the linear transformation. Let me show you this as a column matrix.

So, this is your first component, second component, third component. So, we have found T explicitly. Once we have found T explicitly, let us, let us find out whether T of e1, T of v1 is w1, T of v2 is w 2. So, how do we do that? First now let me declare a linear transformation with, with these as images. So, let us say g of x1, g of x2, g of x3 is list of f; this is the the list, and then define a linear transformation T from V to W with this image as this g, and let us, ok?

So, that is the linear transformation. (Refer Slide Time: 09:27)



Now, let us check whether T of v1, is it equal to w2, T of v2 is equal to w2, and T of v3 should be equal to w3, and the answer is true. Then we also wanted to find the image of T of 3 comma minus 2 comma 5, 3 comma minus 2 comma 5. So, let us find that quickly. This is quite easy. I am sure you all of you by now know how to do that.

So, we will declare vector, let me say u is equal to vector over QQ, and in the square bracket, write its coordinates 3, minus 2, 5. So, that is the u, and then let us say T of u, T in the bracket u, that is the image, ok? So, we have solved this first part of this problem. The second part is to find the matrix of this T with respect to basis beta on domain, and C on co-domain.

So, that is easy. I leave that as an exercise for you to do, ok?

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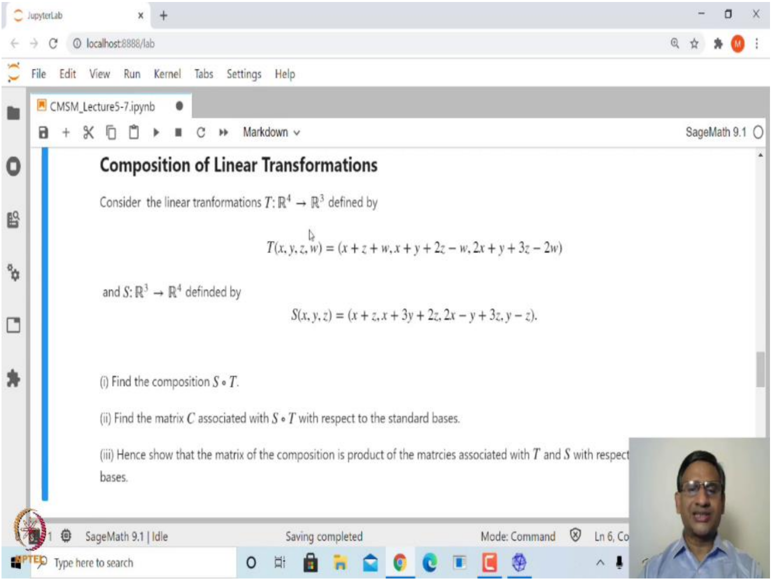

Now, let us look at next problem. If you have two linear transformations, you can take its composition. Of course, it should be compatible. Similarly, you can define sum of two linear transformations.

So, composition, how do we define? So, suppose let us say T is a linear transformation from R 4 to R 3, and S is a linear transformation from R 3 to R 4, then the composition S composite T makes sense, and it will be a linear transformation from R 4 to R 4.

It is easy to check that if you have two linear transformations T and S, then S composite T is also a linear transformation. That is a very simple exercise. Similarly, you can check that T plus S will also be a linear transformation. In this case, T plus S will not make sense, but if T and S are both linear transformation from, let us say, V to W, T plus S will make sense, and T plus S will also be a linear transformation.

So, once we have a linear transformation S composite T, then let us find matrix of this composition, let us say, with respect to standard basis, and then we want to check how is this matrix of this composition of linear transformation related to matrix of T, and matrix of S.

So, that is, the result says that if, if C is the matrix of linear transformation S composite T, and if A is a matrix of T, and B is matrix of S, then C will be B times A. That is the result, that is the, the third part. So, this is what we want to check.

Now, here we are doing everything with respect to the standard basis. However, the same result will be true with, if you take any basis on domain and co-domain of T, similarly on the co-domain of S, right?

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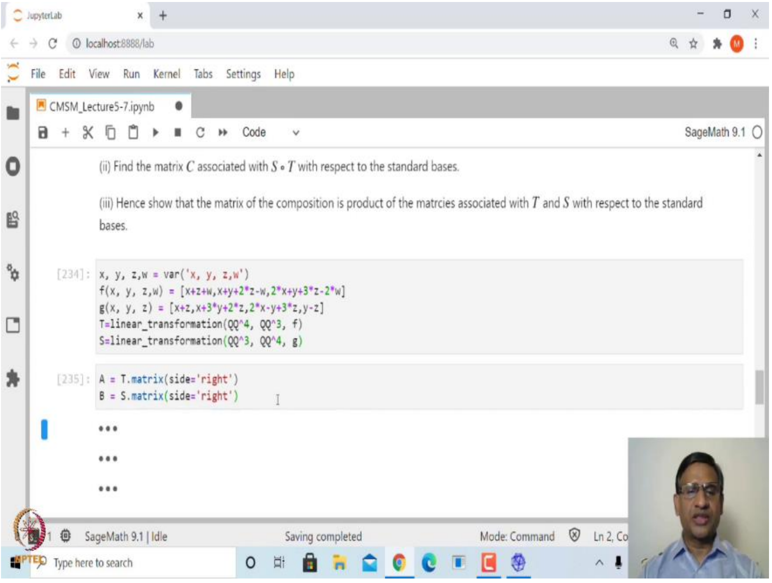

So, let us look at how we can solve this problem. This is quite easy. So, first let us declare S and T as linear transformations. So, first let us declare x, y, z, w, as variables. f of x, y, z is the first linear transformation T. g of x, y is equal to second linear transformation, that is the S, and let us define T to be linear transformation from Q 4 to Q 3, and with image as f; similarly, S as linear transformation from Q 3 to Q 4 with image as g. So, once we have declared S and T, now let us declare, let us find matrices of T and S with respect to a standard basis.

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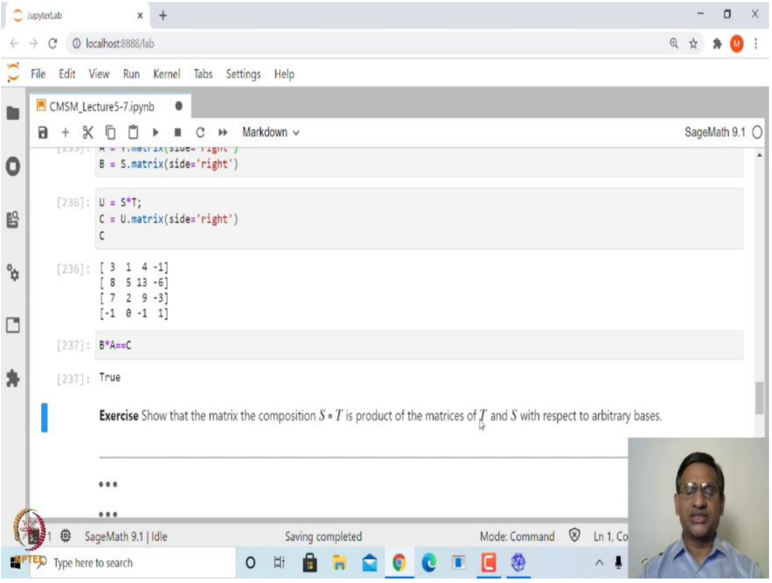
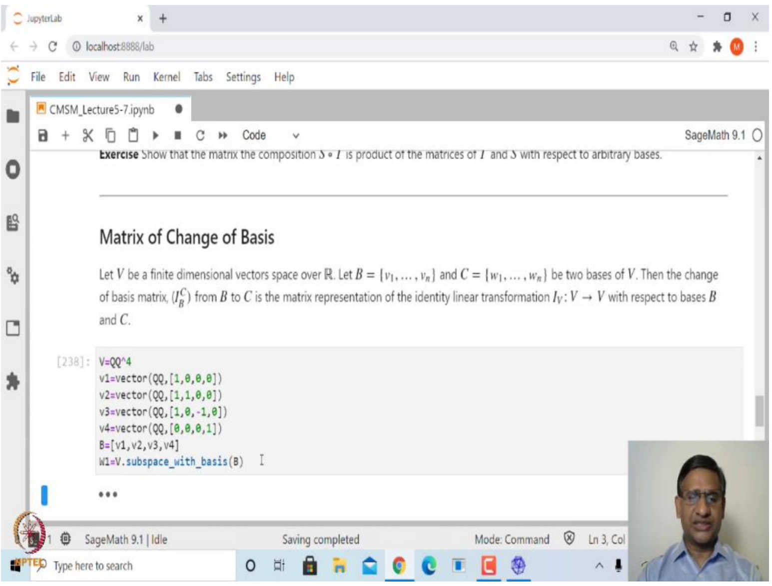


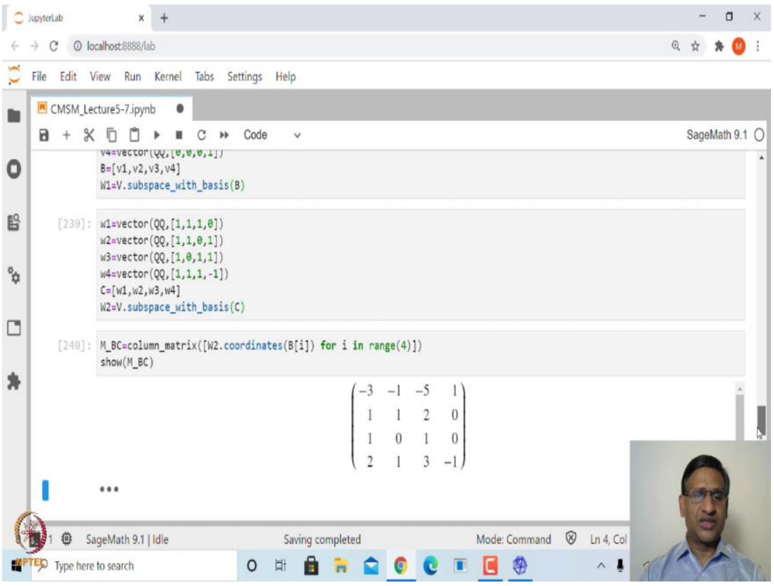
So, we can simply say, capital A is equal to T dot matrix, side is equal to right. Similarly, B is equal to S dot matrix, side equal to right. So, that will give you the matrices of T and S with respect to standard basis. Now, let us declare the composition. The composition of S and T is nothing but S star T. Similarly, the sum will be S plus T. Now, let us find matrix of this U, which is S composite T with respect to a standard basis. So, this matrix is this. Here you can also print what are these matrices A, and B. Now, let us check whether B into A is equal to C, that is what we wanted to prove, and the answer is yes, it is true. The matrix C is nothing but product of A and B. And of course, you should explore the same thing with respect to arbitrary basis on domain and co-domains of S and T. So that, I will leave it as an exercise, this is quite simple, ok?(Refer Slide Time: 14:22)



Now, let us look at one more thing. Suppose we want to, we have already seen how to find matrix of change of basis.

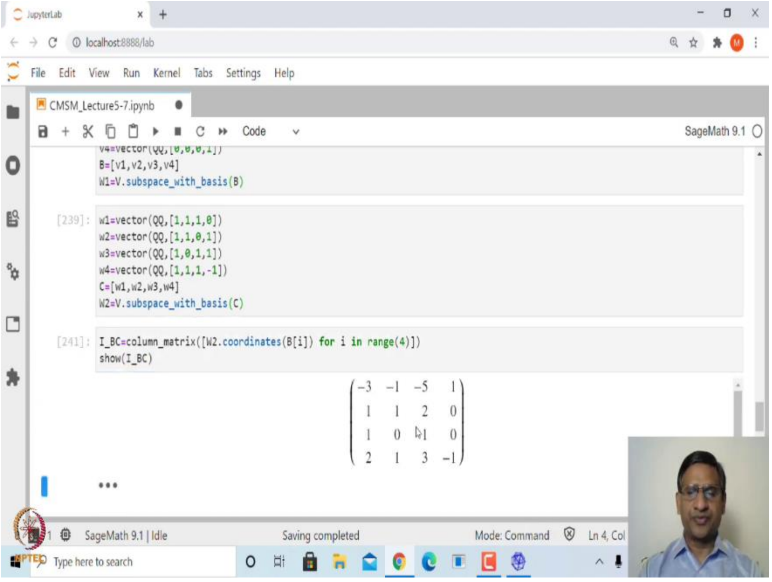
If you have two basis, then, of a vector space V, and if you take any vector, then we can write its coordinate with respect to the two basis, and we have seen that the, the two coordinates are related by a matrix. There is a relationship, and that relationship comes from what we call a change of basis matrix, ok? So, this we have already seen.

However, this change of basis matrix can also be obtained as follows. So, you look at a linear transformation which is identity linear transformation from V to V, and write matrix of I V with respect to basis B and C. That matrix is nothing but matrix of change of basis from B to C. So, let us look at these. So, suppose we take the space V is equal to QQ to the power 4, and v1, v2, v3 as a basis of this, and let us define w1 to be a subspace of capital V with B as basis. (Refer Slide Time: 15:39)



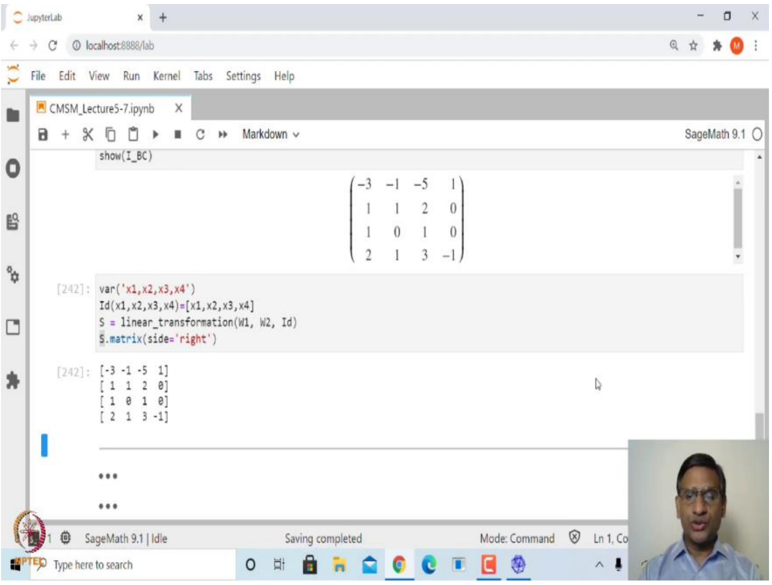
Similarly, let us define a basis w1, w2, w3, w4 of V, and define capital W 2 to be the subspace of capital V with respect to with C as a basis, and then we know that, how, we know how to find matrix of change of basis from B to C. We have already done this.

So, let us just recall that, how we did. So, the matrix of change of basis from B to C can be obtained as the column matrix of the coordinates of of v1, v2, v3, v4 with respect to basis w1, w2, w3, w4, and this is this matrix. (Refer Slide Time: 16:24)



This, actually in the, in the problem I had denoted this by I BC.

So, let me declare this as I BC, right? Once we, this is the matrix of change of basis. Now, what is that we want to show? We want to show that this matrix is nothing but matrix of the linear transformation, the identity linear transformation from V to V with respect to basis B on domain, and C on capital domain, on co-domain, right? So, let us do that. Let us look at. (Refer Slide Time: 16:50)



So, the, declare variables v1, x1, x2, x3 x4, and identity map Id x1, x2, x3, x4 is, x1, x2, x3, x 4 as a list, and declare S as a linear transformation which takes from W1 to W2.

Remember this is with respect to basis b1 B on domain, and C on co-domain, that is why here v1 W1, and W2.

You could have declared this from V to V, and then you could have restricted S to the basis B and C on domain, and co-domain, right, ok?

Now, let us find the matrix of S with respect to these two basis, and this is this, and you can see that these two I BC is same as matrix of IV, with respect to basis B and C. So, that is what we have proved. Now, let us use this to look at what happens to matrix of a linear transformation when you change the basis, ok?

So, you have a linear transformation, you, you find its matrix with respect to a given basis. Now, you have another set of basis on domain and co-domain, and then find matrix of that same linear transformation with respect to the two new bases, and then we want to check how are they related to each other, ok? So, let us see how are they related.

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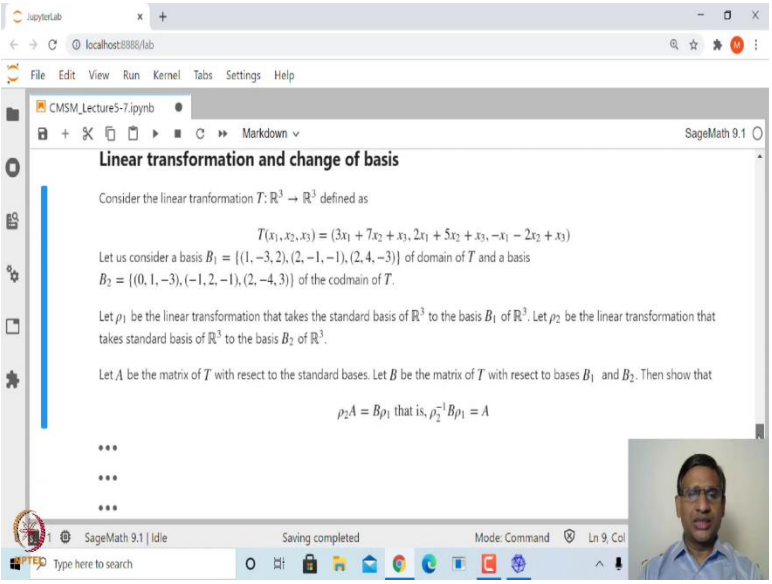
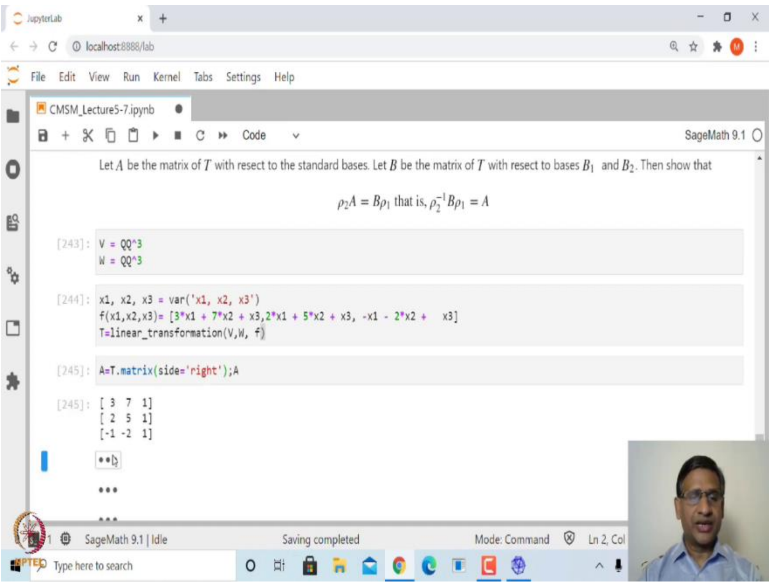


So, let us start with a problem. We have a linear transformation T which is from R 3 to R 3, given by this components, and let us fix a basis b1 of T, and b2, b1 of domain of T, and b2 of co-domain of, of T, right, and let us assume that rho 1 is the linear transformation that takes standard basis to b1, and rho 2 a linear transformation that takes standard basis to b2.

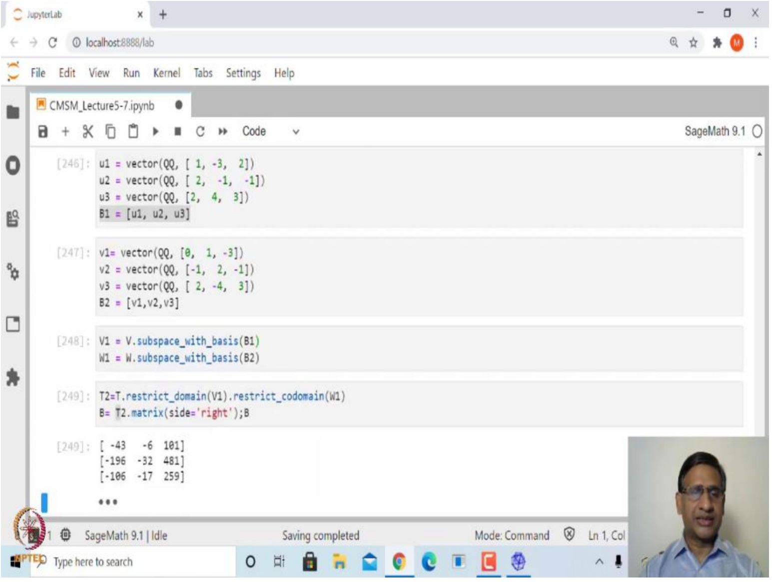
Now, suppose let us say, capital A is the matrix of T with respect to standard basis. Now, we have two basis beta 1, b1 and b2 on domain and co-domain. So, let us say capital B is the matrix of T with respect to basis beta 1 b1 and b2, and then we want to check how are this A and B related.

So, this A and B are related by this relation rho 2 times A, rho 2 is the matrix of the change of basis from standard basis to b1, and rho 2 that is the rho 1. Rho 1 is the matrix of change of basis from standard basis to b1, and rho 2 is the matrix of change of basis from standard basis to b2. So, this is how they are related, A and B are related. In, in particular, you can write this as rho 2 inverse times B rho 1 is equal to A.

So, in this case, actually one can, if you have A, and two matrices A and B which are related by this relation, we say that A and B are similar similar matrices. So, the matrix of T with respect to different basis are similar matrices, that is what we are proving, ok? So, let us work out this. (Refer Slide Time: 20:21)



So, you start with V and W which in this, case both are. both are Q 3, and let us define the linear transformation T with given expressions as a list in f. So, that is the matrix, that is the T, and let us define the matrix of T with respect to standard basis, let me call that as A. (Refer Slide Time: 20:57)



Similarly, let us find, let us look at the basis b1 which is u1, u2, u3. You can check that this is a basis. This is a basis. Similarly, take b2 a basis which is v1, v2, v3. Again you can check that this forms a basis. We need to check that b1 and b2 are linearly independent. I have already checked that these are basis.

So, I do not need to do that, but when you take arbitrary vectors, and you want to form a basis, you must check that it does form a basis. Now, let us say V capital V1 as a subspace of V with respect to basis b1, and W1 as subspace of V with basis b2, ok? So, this let us run this, and once you have run this, then let us now restrict T to this basis v1, and to subspace V1 on domain, and W1 on co-domain.

So that means, we are finding now, linear transformation with respect to the new basis. So, and that matrix of T 2 which is T restricted on domain and co-domain will, let us call that as B. So, that is the matrix B. We have found the matrix A over here, that is the matrix, somewhere we calculated. Yes, this is the matrix A which is with respect to standard basis.

Now, we have calculated matrix B with respect to the basis b1 and b2.

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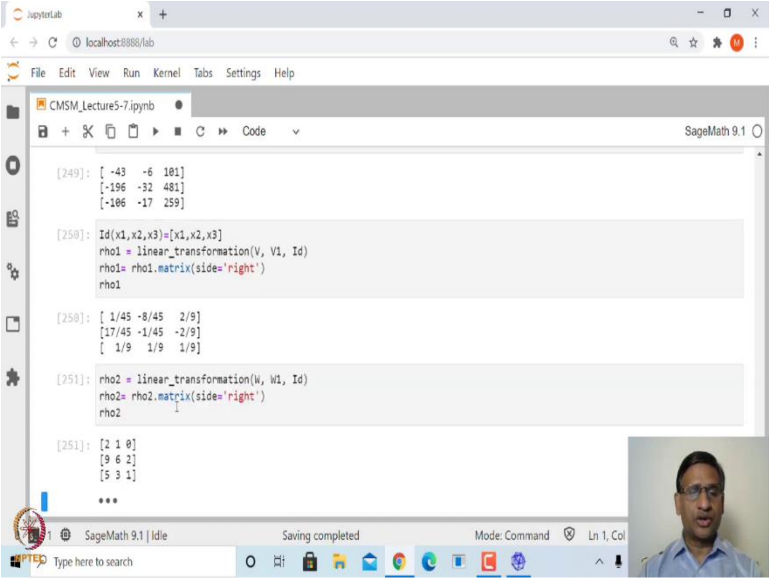

Now, let us look at, let us find rho 1 and rho 2. What is rho 1? Rho 1 is the matrix of change of basis from standard basis to b1. So, how do we obtain? Just now we saw that we can obtain that as a matrix of identity linear transformation from V to V, and with respect to basis standard basis on domain, and basis b1 on co-domain.

So, let us define a linear transformation with identity as image from V to V1. V1 is subspace of V with basis b1. So, let us find rho 1, and matrix of rho 1, that is rho 1, that is the change of, matrix of change of basis from standard basis to b1. Similarly, let us calculate matrix of change of, matrix of change of basis from standard basis to b2, this is rho 2.

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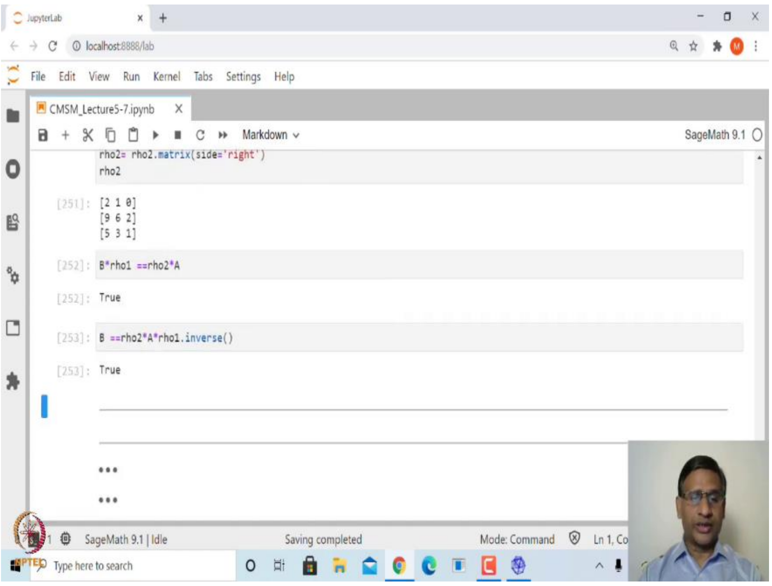

Now, let us check what we wanted. We can check that b1 B times rho 1 is equal to rho 2 times A, and the answer is true, or we can check that B is equal to rho 2 times A times rho 1 inverse. So, we have verified what we wanted, ok? Now, here we have taken base, standard basis for T, and then we changed two basis b1 and b2.

You could start with b1, b2 on T, and take another basis let us say d1 and d2 on domain and co-domain, and then also you can explore this relation. So that, I will leave it as an exercise, ok?

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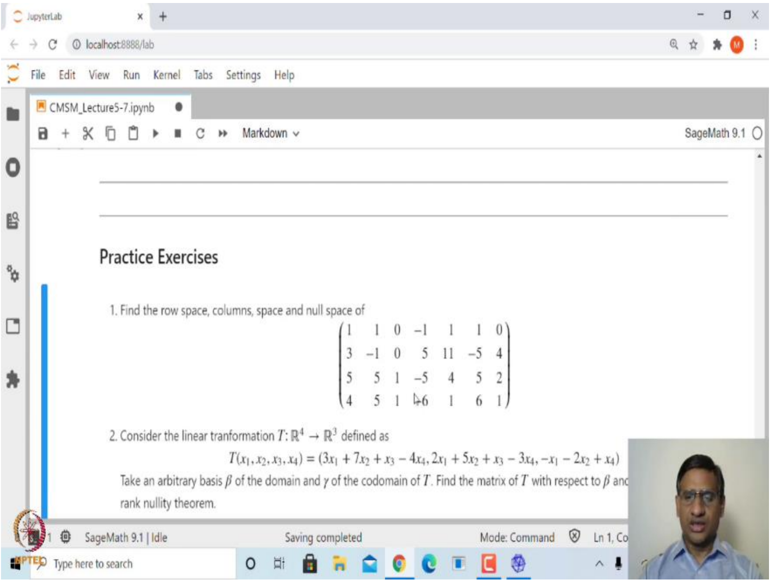

Now, let me leave you with three simple exercises for you to practice. The first exercise is, find row space, column space, and null space of this 4 cross, this is 7, matrix, 4 cross 7 matrix.

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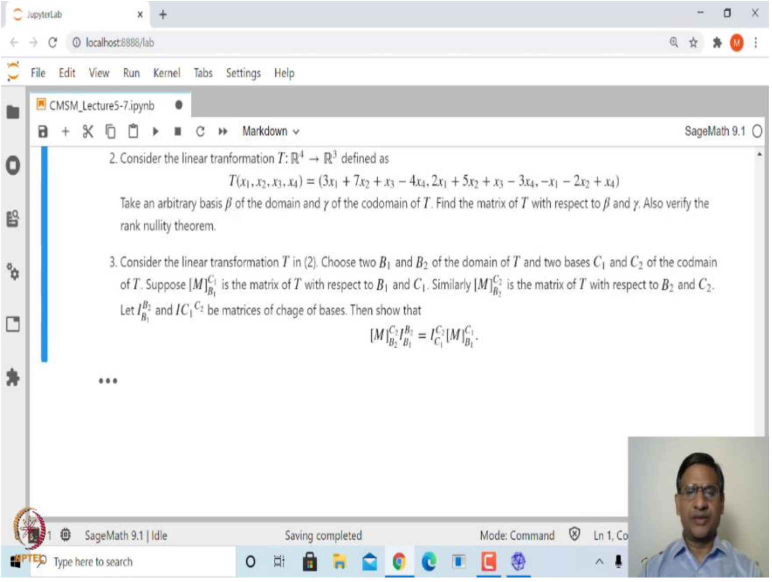

And second problem is to consider a linear transformation T from R 4 to R 3 with these components, and then take arbitrary basis on domain and co-domain, and find its matrix with respect to arbitrary basis, and also verify rank-nullity theorem for this, and similarly look at the third problem.

This is, actually I already said that as an exercise. The linear transformation you are given, a linear transformation from V to W, find its matrix with respect to two basis beta 1, and beta 2. Similarly, find its matrix with respect to another two basis c1, and c2, and then find the relationship between the two matrices which you have obtained, ok?

So, these are the three simple exercises. Let me stop here. In the next class we will look at how to deal with eigenvalues and eigen, eigenvectors, and as you know, these eigenvalues, eigenvectors have numerous applications in linear algebra. So, we will see you in the next class.

Thank you.