Computational Mathematics with SageMath
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Linear Transformations
Lecture – 33
Linear Transformations Part 1 with SageMath

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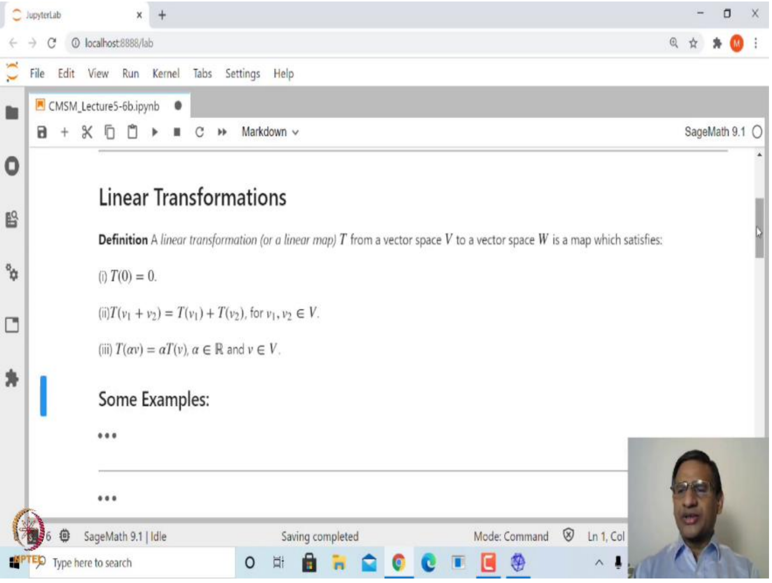


Welcome, to the 33rd lecture on Computational Mathematics with SageMath. In this lecture, we will look at Linear Transformations, and we will see how to define linear transformations in SageMath, and explore various concepts related to linear transformations. So, so far we have defined vector space, subspaces, and various concepts related to subspaces, like vectors which are linearly independent, dependent, span of vectors, and things like that.

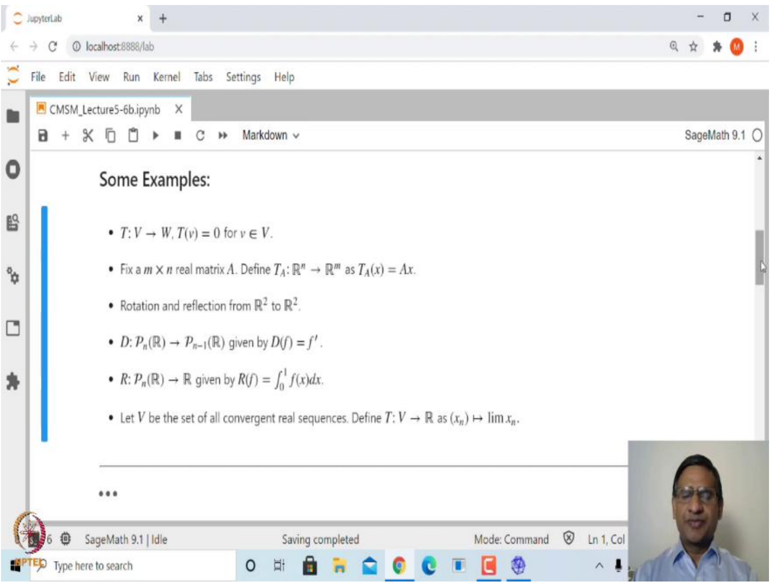
Now, suppose you are given two vector spaces, let us say over same field; let us consider this over R, then you would like to define some map between the two vector spaces. But if you look at vector space; vector space has two defining properties or structures, namely vector addition, and scalar multiplication.

So, you do not want any arbitrary map, but a map which preserves these two defining structures of a vector space, and this is what is called linear transformations. So, let us define what is linear transformation, and then we will see how to explore this in SageMath.

So, linear transformation T from vector space V to a vector space W; here we are assuming V and W vector spaces over the same field this. So, the linear map is, linear map, or linear transformation is a map from V to W, which satisfy these three properties; namely, first T should map 0 to 0, 0 of V to 0 of W. Here we are using both 0 with the same notation; one can denote 0 V here, and 0 W here, just to distinguish, and second property says that T preserves the vector addition.

So, T of v1 plus v2 should be same as T of v1 plus T of v2 for any two v1 v2 in capital V. So, this simply says that, either you first add two vectors, and take its image, is same as saying you first find images of v1 and v2 and then add them in the co-domain space. And similarly, T of alpha times v should be equal to alpha times T of v for any real number alpha, and any vector v in capital V right?

So, what it says? T preserves this scalar multiplication. So, either you multiply it v by scalar alpha, and then take its image, that is same as saying, first take its image, and then multiply by alpha. So, that is any T which satisfies these three properties is known as linear transformation from V to W, or a linear map from V to W, ok? (Refer Slide Time: 03:26)



Now, let us look at some simple examples. I am sure you must have seen most of these examples. So, first from V to W, T from V to W, we can define as T of v is equal to 0; for every vector is mapped to 0, this is a zero linear, you can check that is a linear transformation, and this is what is called zero linear transformation.

Similarly, if you fix any m cross n real matrix, and then define T A from R n to R m as T A of x going to A times x, this is matrix multiplication. Again you can check that this is a linear transformation. For example, A times zero vector will be 0, A times x1 plus x2 will be A x1 plus A x2, similarly A times alpha x is alpha times A times x. You can also look at rotation and reflections in R 2; this can be thought of as a map from R 2 to R 2.

So, take any vector, and rotate it by some fixed angle theta, and similarly reflect about, let us say, x-axis or you can reflect about any axis; but let us think of reflection about x-axis. When you reflect about x-axis, any vector x,y will go to x comma minus y. So, that again is a linear transformation.

Similarly, you can define from set of all polynomials of degree less than equal to n over R, to set of all polynomials of degree less than equal to n minus 1 over R, and this is the derivative. So, take D of any polynomial as derivative of this. So, that again is a linear map; we know that derivative of sum of two functions is nothing but sum of their derivatives, and derivative of alpha, scalar multiple times a polynomial is nothing but scalar multiple times this derivative.

Similarly, you can define some, from set of polynomials of degree less than equal to n to R, as R of f, as integral of f of x, 0 to 1. This we know that the polynomial, set of polynomials that is the integrable function. So, this forms a vector space over R, and you can define similarly, R as a vector space over R, that is a linear transformation, right?

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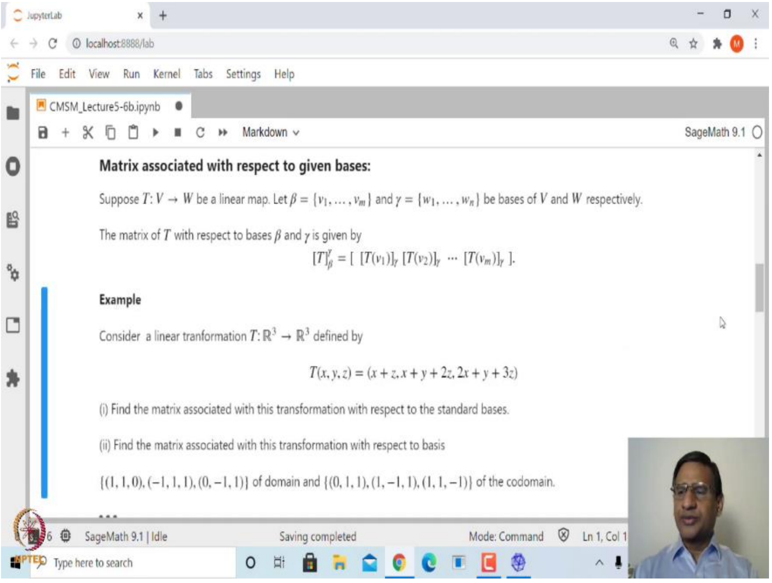

Or you can take V to be set of all convergent real sequences, and define T from V to R as T of any sequence x n as limit of x n, that again is a linear transformation. So, these are very simple examples of linear transformations, and you take any two subspaces, it is not very difficult to obtain some linear transformations, right?

So, now let us look at how we can define matrix associated with a linear transformation with respect to a given basis. So, if you look at this example, T A from R n to R m, where A is m cross n matrix; then this is defined by matrix multiplication.

And in fact, any linear map from R n to R m can be obtained using this. That means, we can find a matrix such that this map is nothing but matrix multiplication. So, that is what we will look at how to find that matrix. So, therefore, if you have to look at linear transformation from R n to R m, all you need to do is define that matrix associated with given linear transformation, and work with that.

So, how do we find matrix of a linear transformation? So, let us assume that V and W are finite dimensional subspaces, finite dimensional vector spaces over, let us say R, and T from V to W, a linear map. Now, fix the bases beta of V, let us say v1, v2,…, vm, and gamma of W which is w1 to wn; V is m dimensional, and W is n dimensional, right?

Next, to find the matrix of this T with respect to bases beta on domain, and gamma on codomain, all you need to do is look at the image of v1 under T, T v 1. So, that is a vector in W, and since gamma is a basis of W, this T of, T of v1 can be written as linear combination of w1 to wn. That means, we can find the coordinate of T of v1 with respect to basis gamma, and that coordinate vector, you put it in the first column of this matrix.

Similarly, take the find the coordinate of T of v2, and put it as a second column, and so on. So, ith column of this, this matrix is going to be the coordinate vector of T of vi with respect to basis gamma, that is what you need to do in order to find this matrix.

So, let us look at an example, and see how we can obtain this matrix, ok? So, look at example T from R 3 to R 3 or Q 3 to Q 3 defined by T of x, y, z is equal to x plus z, x plus y plus 2 z, 2 x plus y plus 3 z, and first let us see how we can define this in SageMath, and then find the matrix associated with this linear transformation with respect to standard basis on domain and co-domain.

Similarly, next, let us change bases on domain and co-domain; the bases let us say beta is this of R 3 on domain, and this is let us say gamma on the co-domain, and let us find matrix of this linear transformation, right, ok?

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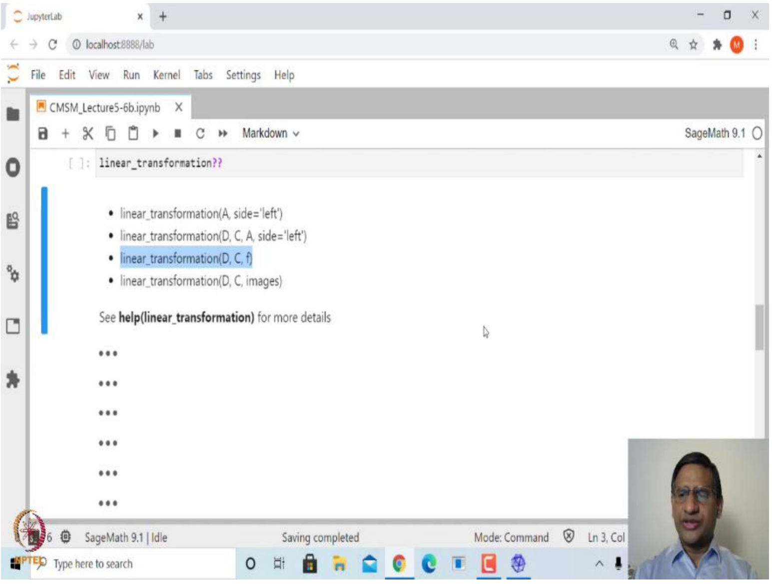

So, first, you can look at in built function linear underscore transformations, and then it will tell you how to define linear transformations in SageMath along with several examples.

So, if you look at this help, there are actually four ways in which one can define linear transformation. One linear underscore transformation, you can mention the matrix, and then you can define side is equal to left or right, by default side is left. So, left means it will be, think of A acting on, let us say a vector x as x transpose A, whereas if you say right, it will think of as A x which is what we saw as an example.

Similarly, the second way of defining linear transformation is to define linear underscore transformation, mention this domain D, C as co-domain, and the matrix, and then mention the side equal to left or right? Or you can define the set of images, let us say f as a function.

So, here we could have declared, let us say, let me say, this is T is equal to f; f (x, y, z) is equal to list this as a list, of course, x, y, z you need to define as a variable, and then declare the linear underscore transformation D comma C, where D is domain, C is co-domain, and f as the image function, or you can simply declare these images as a list of images.

So, these are four different ways in which you can define linear transformations in a, from vector space to another vector space. So, mostly we will look at this way of defining, other two are also very similar, ok?

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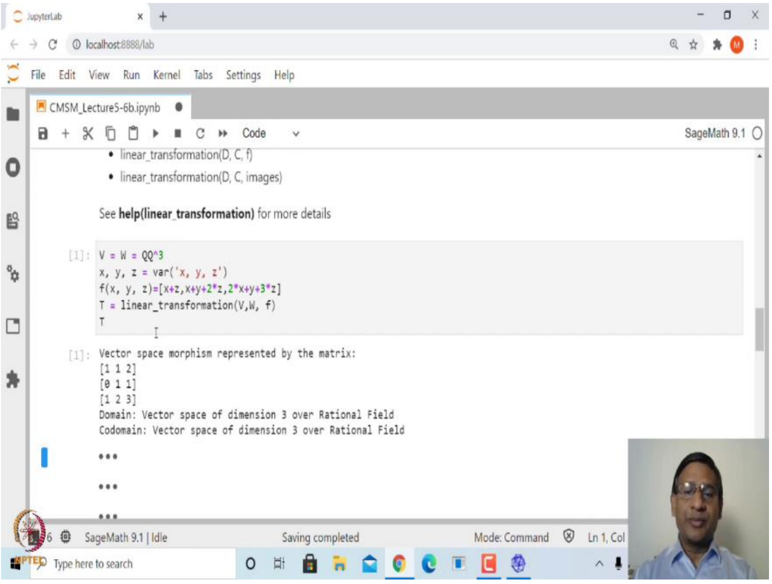

So, let us first declare V and W as vector spaces QQ 3.

So, we will think of this above linear transformation as a linear transformation from R, Q 3 to Q 3, not from R 3 to R 3. You can do from R 3 to R 3 as well. So, all you need to do is just change QQ to RR, and x, y, z as variables.

So, the first, then declare f(x, y, z) as a function whose elements are, whose output is list of the coordinates of this linear transformation, and then declare T is equal to, T is equal to linear underscore transformation, and V is the domain, W is the co-domain, and f is the image function.

So, if I run this, it says that, T is a vector space morphism represented by a matrix this. So, actually already it gives you a matrix representation of this linear transformation, and this matrix is nothing but matrix with respect to standard basis, ok?

And it says that, domain is a space of dimension 3 over rational field; similarly co-domain is this space of dimension 3 over rational field. Now, various, again here T is an object, and all the methods that you want to apply on T, you can obtain by T dot tab.

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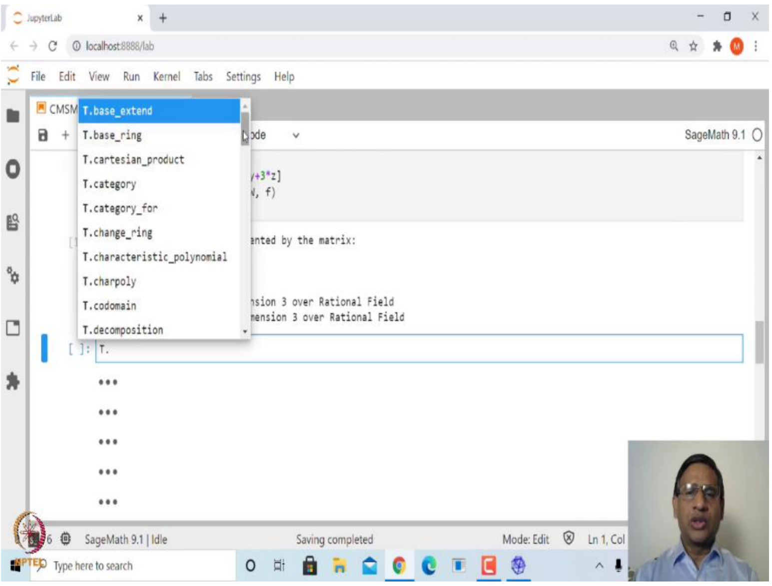

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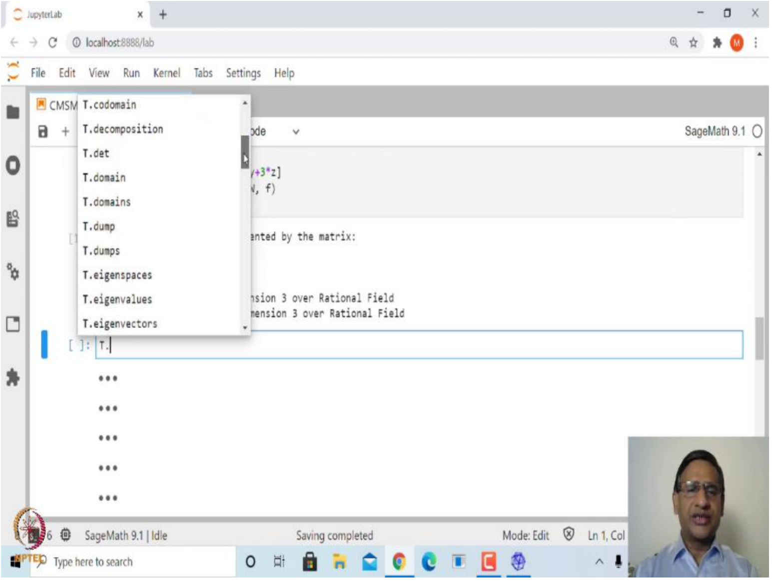

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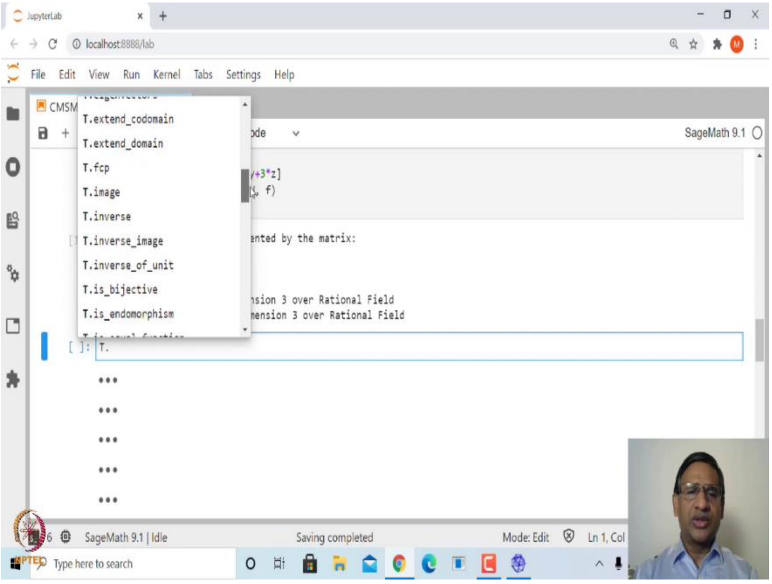

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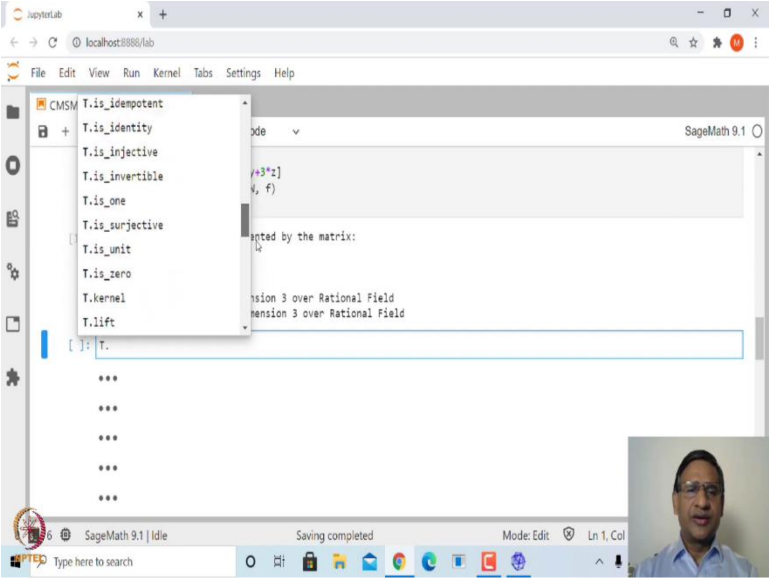

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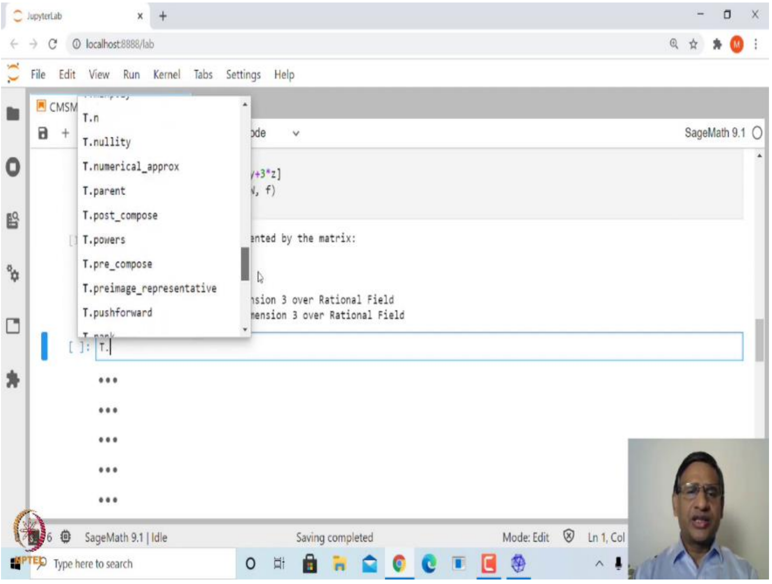

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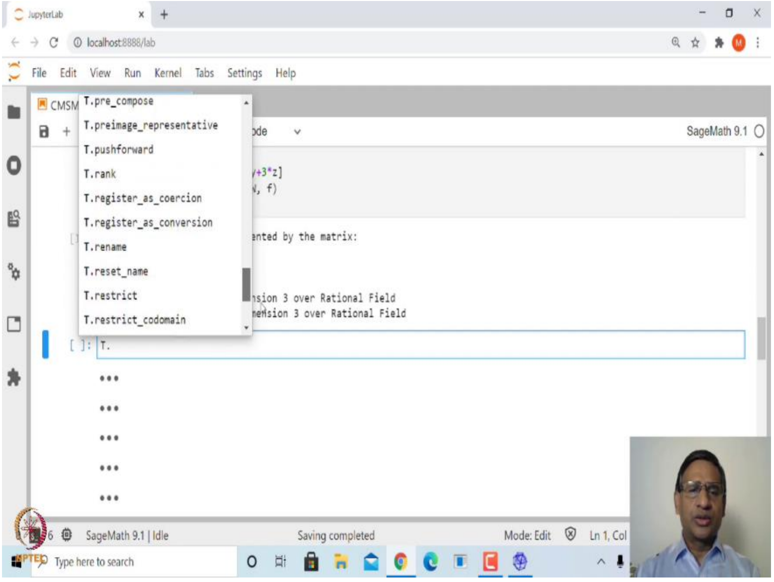

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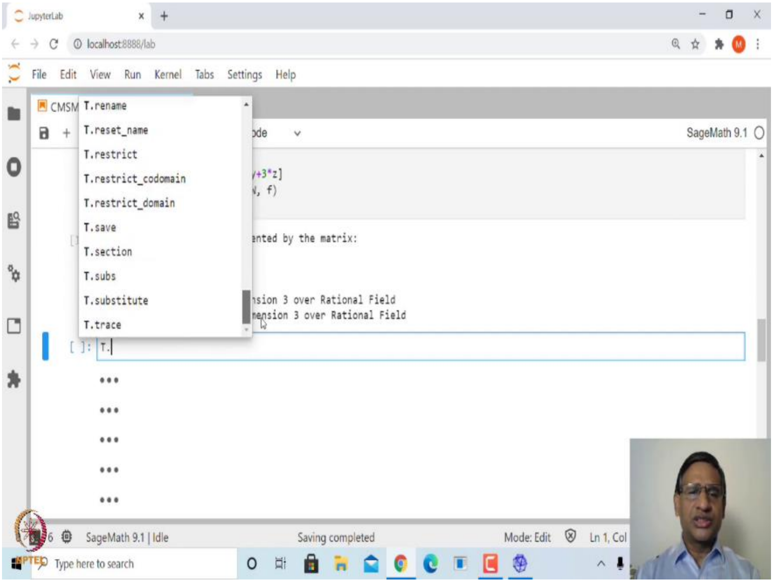

So, you can simply use, you can use T dot, and pres tab; you will see all the methods like domain, co-domain, image, kernel, all these things are there, you can go through this list, and explore some of these things, right?

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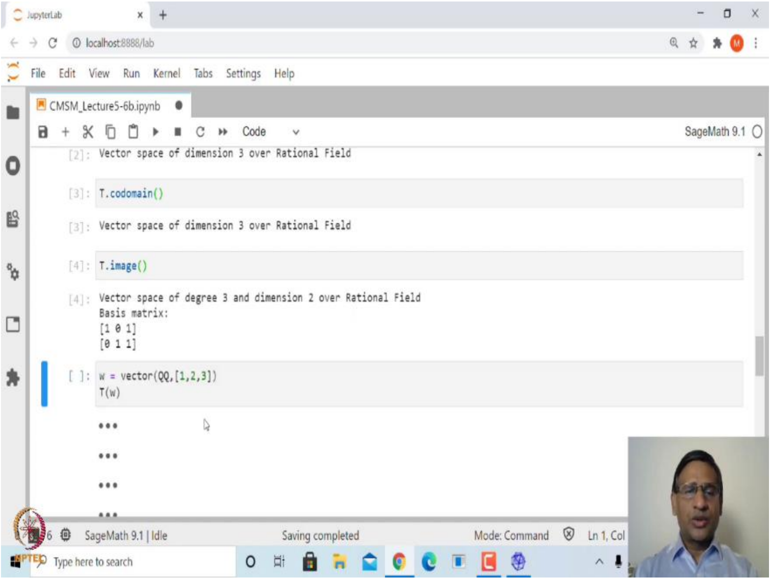

Next let us find, let us explore some of these methods. So, if I say T dot domain, it will give me what is the domain of this, which is Q Q 3. Similarly, I can say T dot co-domain, that will give me again QQ to the power 3, and then you can say T dot image, so that will give you set of all images as a vector in Q 3, and subspace spanned by the, those images.

So, it says that, image is a subspace of dimension 2 over QQ, over Q right? Similarly you can find the image of any vector.

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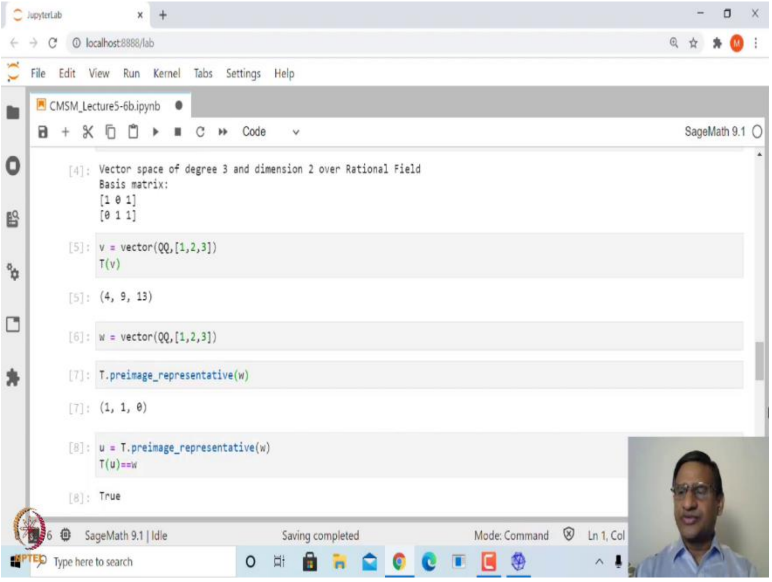

So, if I declare a vector, let us say W, which is vector 1, 2, 3; let me write this as v is equal to vector 1, 2, 3, and the image of v can be obtained as T in the bracket v. Similarly, if I have, let us say, a vector w in the co-domain, that is again 1, 2, 3, and then let us check whether this is pre-image of some vector.

So, you, the sage has inbuilt method called pre image underscore representative; because when you find pre-image of an element in the codomain, that is, the pre-image is actually a subset of the domain set. So, it cannot report all the elements, but it gives you representative from that sub subset, right? So, this is representative, suppose if I call this pre-image representative as u, and check whether T of u we get as w; the answer is true, ok?

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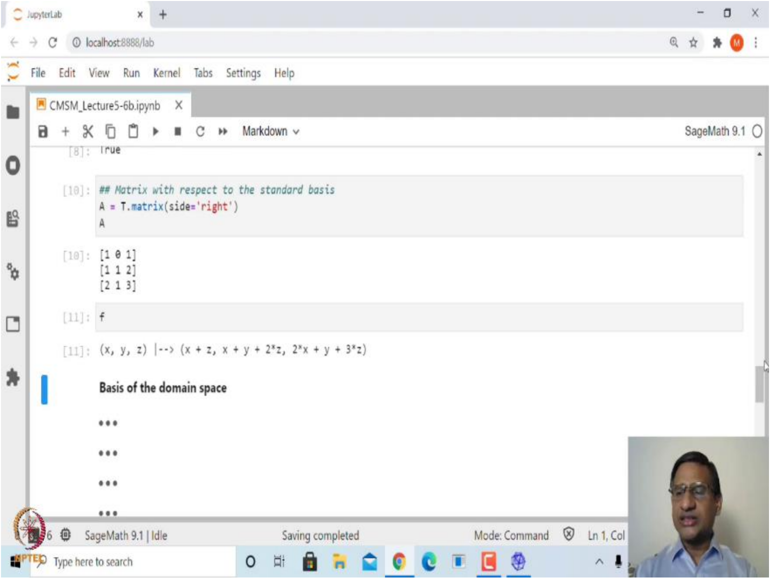

Similarly, you can explore other concepts of T, other methods on T by looking at T dot tab, and select from that list, and then explore. Now, let us see how we can find matrix representation of this T, first with respect to standard basis.

So, first, with respect to a standard basis, all you need to do is, say T dot matrix, and then mention what is the side; side is, by default the side is left. So, if you want the side is right, you have to say side equal to right. So, that is the; let us see what is this, this matrix A? This is this matrix.

So, you can see here now, if I look at A times x, y, z; the column vector x, y, z, what will you get? The first coordinate is x plus z, that was the first coordinate of T. Similarly second coordinate is going to be x plus y plus 2 z, which was the second coordinate of T; if, let me also show you what is this f.

So, f of x, y, z is x plus z; that is the first coordinate, second coordinate x by plus y plus 2 z which you will get as A times x, the second coordinate of A times x, y, z, and third row in this multiplication is going to be 2 x plus y plus 3 z which is the image.

So, this is with respect to standard basis, the matrix with respect to standards basis on both domain, and co-domain. Now, let us change basis on domain and co-domain, and see what will be the matrix associated with this.

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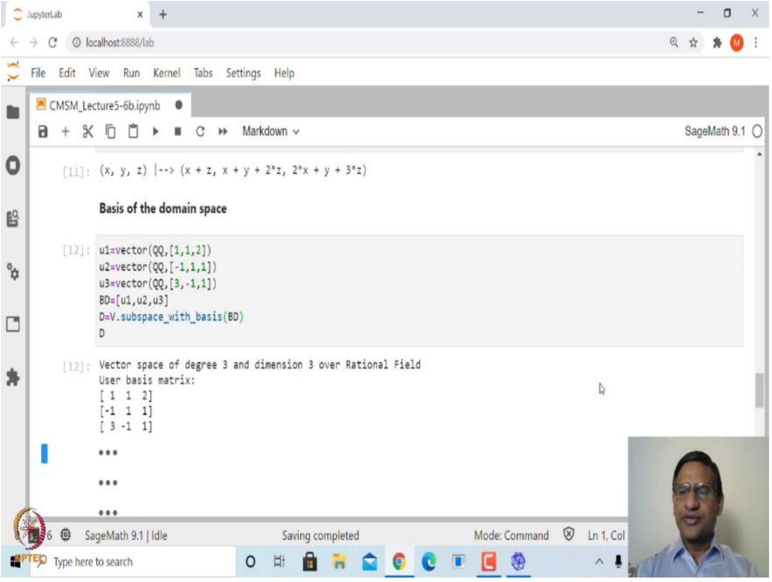

So, let us define vectors u 1, u 2, u 3 in Q 3, and then let us say BD as a list of u 1, u 2, u 3, and define D to be the subspace of V with basis v1, sorry u 1, u 2, u 3 that is the basis of the domain.

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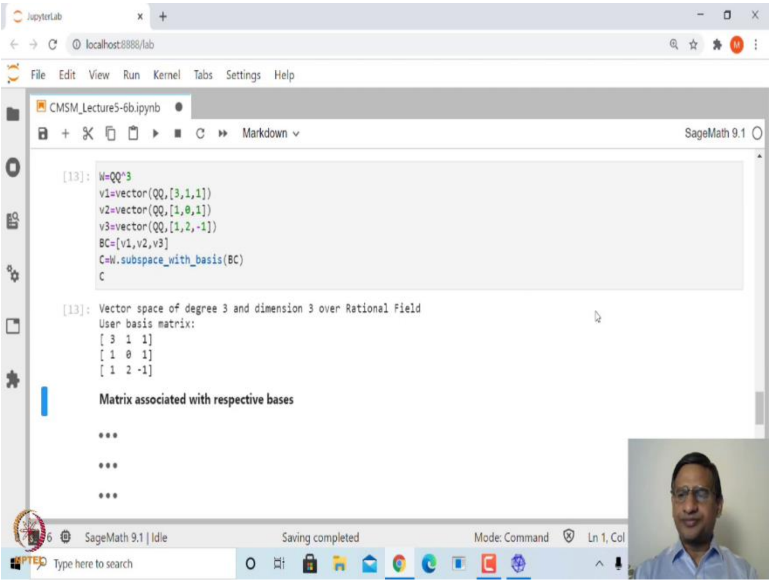

Similarly, let us find, or define a basis of co-domain. So, let us declare that as v 1, v 2, v 3, and BC as a list of v 1, v2, v 3. Again you can check that is v 1, v2, v 3 forms a basis, and then find capital C to be the subspace of W with basis v 1, v2, v 3, so on. So, that is the subspace. Of course, in this is going to be the entire W, but basis will be different.

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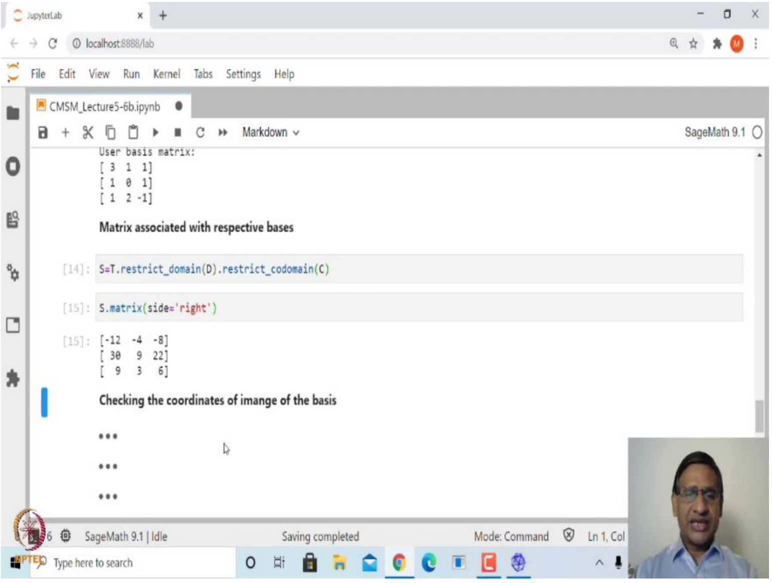

Now, we have to, in order to find matrix of T with respect to bases, say BD on the domain, and BC on co-domain; how do we do it? So, all you need to do is restrict T on this domain D, which is the subspace of v spanned by subspace of v with bases BD.

Similarly, restrict T to a to co-domain C, that is S, and then all you need to do is, find the matrix of S, and with side equal to right option. So, that is the matrix of T with respect to bases, let us say BD on the domain, and BC on the co-domain, that is very simple.

However, as I said, we can, but manually how does one find this matrix? All we need to do is, find the image of each element in the basis of the domain, and the image of the first element, the image of the first element, in this case, u 1, find its coordinate with respect to the basis on the co-domain, and whatever the coordinates you get, that you put it in the first column.

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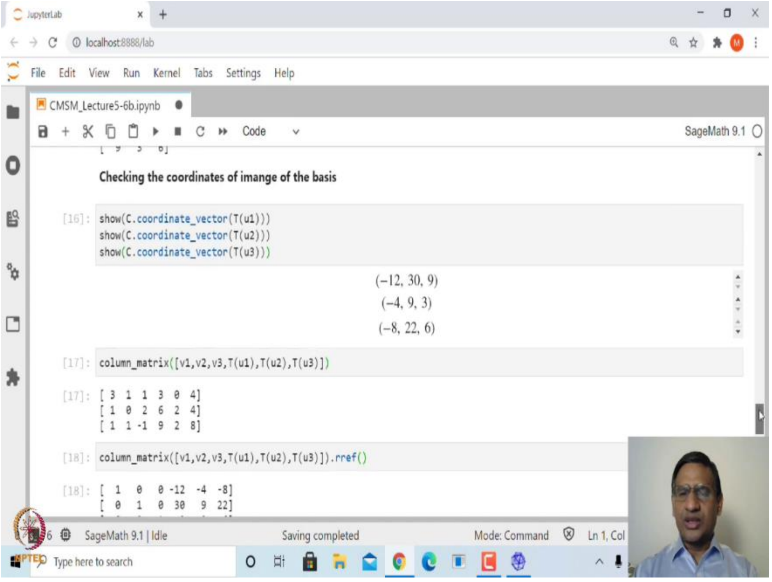


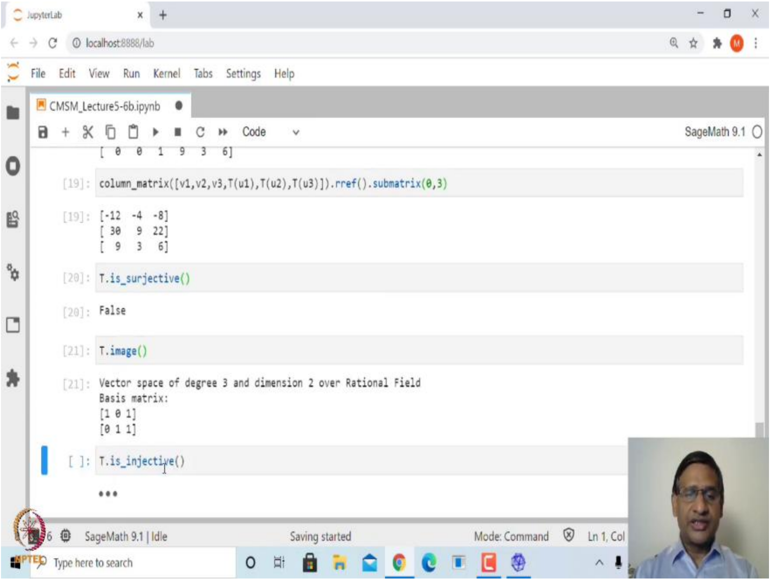
So, let us do that. So, let us first find what are the coordinates of C with respect to coordinates of T of u 1 with respect to the basis C; coordinates of u 2, T u 2 with respect to basis v 1, v2, v 3, and coordinate of u 3 with respect to basis v 1, v2, v 3. So, let us look at these, these are the coordinates.

Now, if you look at the first coordinate of the first T of u 1, and if you put this in column, that is nothing but the first column of the matrix of S; second image the coordinate of the T of u, u 2 that is nothing but the first, second column, and so on. So, that is how you can obtain the, the matrix of a linear transformation.

Let us do this. How do we find these coordinates? So, all we need to do is, we can create a column matrix T of, column matrix with first three columns as v 1, v 2, v 3, the basis vector of the co-domain, and then obtain T of u 1, T of u 2, T of u 3, and then apply RREF.

When you apply RREF, the third column in this RREF will give me the coordinate of T of u 1 with respect to v 1, v 2, v 3; the third, fourth column of RREF is going to give me the coordinate of the vector T of u 2 with respect to matrix, with respect to basis v 1, v 2, v 3, and so on.

So, this is the matrix. Now apply RREF to this matrix, and what you see, the last three rows, and three columns is nothing but the matrix of T with respect to basis BC on the domain, BD on the domain, and BC on the codomain, right? So, this is how you can obtain again matrix of a linear transformation using RREF. So, once again RREF is very useful. (Refer Slide Time: 21:21)



Now, you can extract this matrix of this linear transformation using this submatrix option. So, this sub, this matrix, RREF of this matrix, and apply submatrix method. So, its starting row is 0, and starting column is 3. So, remaining all these things it will take. So, this is what you get. So, that is the matrix.

Next let us look at, check whether this map T, that is linear map T, is surjective or not. So, it is individual function to check, whether T dot is underscore surjective. We already saw what is the image of T, T dot image, T dot image gave me subspace of dimension 2; whereas the co-domain is of dimension 3. So, therefore, that image is not full space, and hence T is not surjective.

Similarly, you can check this, we have already done; we can find out whether T is injective, whether T is one-one.

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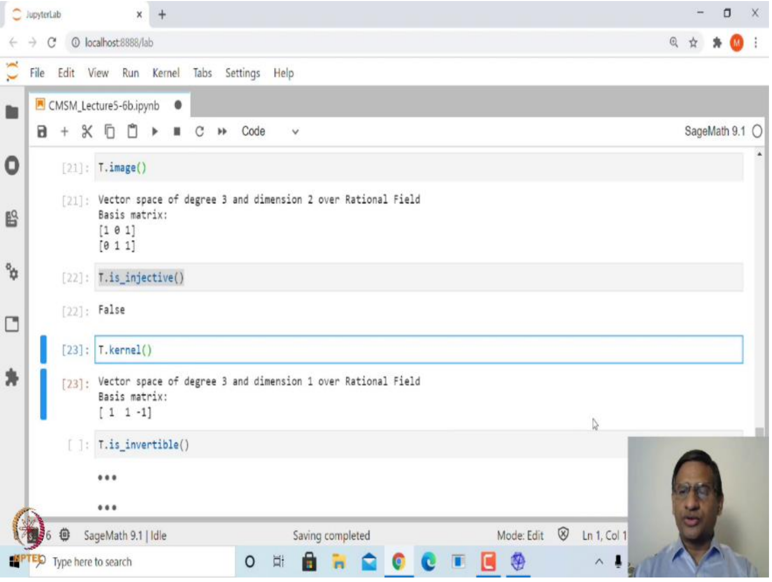

So, if I check that, then it is false, so, that means T is not one-one. T is not one-one is actually same as saying kernel of T is nontrivial. So, there are some vectors which, non zero vectors, which is mapped to zero vector right? Kernel of T.

So, you can check, you can check, find out T dot kernel, and you can see here this, this gives you one-dimensional subspace. So, it is, T is not bijective; T is not bijective, because it is not surjective, it is not even injective, so of course T is not invertible. So, one can show that a linear map T from V to W is injective if, and only if kernel T is a zero subspace; there is no nonzero vector which is in the kernel, right?

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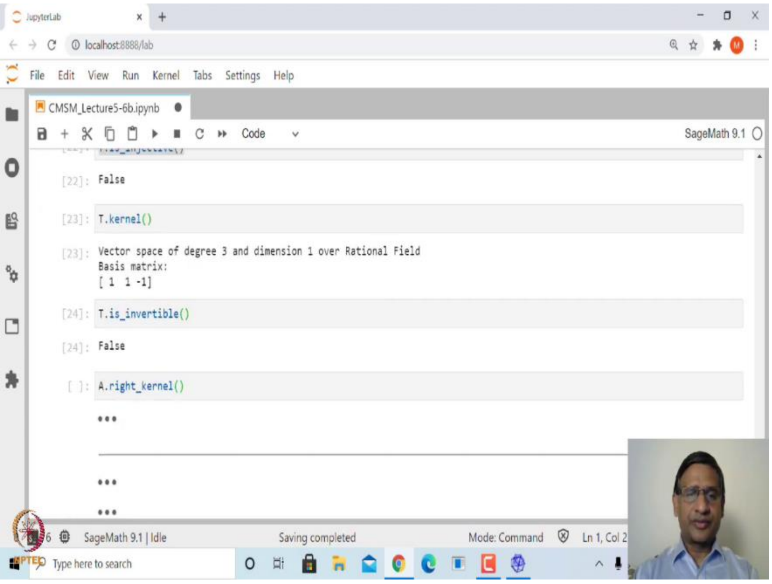

So, this is to check whether T is invertible or not. So, it says false obviously, and we already found the kernel of this, so I do not need to execute this. Similarly, you can also find. So, we have declared A as a matrix of S, ok?

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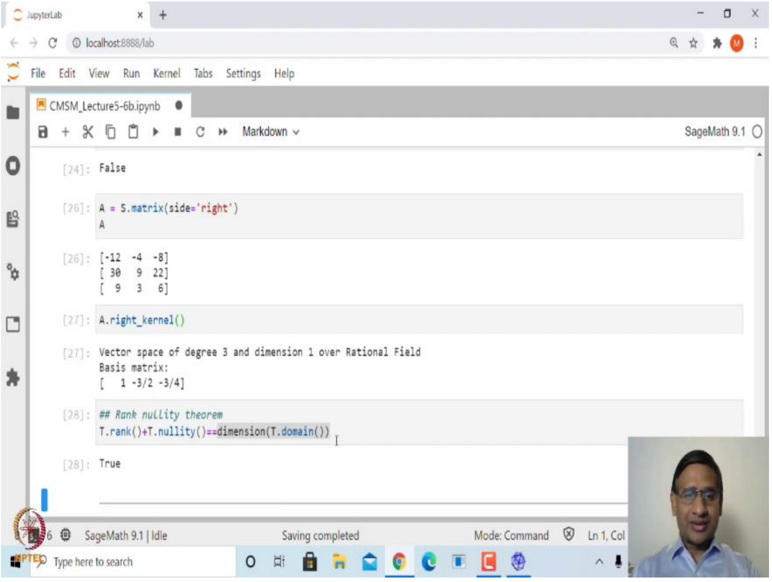

So, let us define the matrix of T with respect to given basis, with respect to given basis in A. So, let us store the matrix of T with respect to the basis BD on domain, and BC on co-domain as A, and so, if you look at A is this matrix.

If you look at A dot right underscore kernel; then this is what you get, this is the kernel of A, right? Similarly, you can verify rank-nullity theorem of a linear transformation. So, the rank of a linear transformation is nothing but rank of the matrix associated to the, associated to T with respect to any basis.

Similarly, nullity is nothing but dimension of the null space of T, or dimension of the kernel of T, and if you add these two, the rank of T plus nullity of T, that is nothing but dimension of the co-domain, ok? So, right?

So, if you look at, in terms of matrix, suppose A is m by n matrix, then A can be thought of as linear map from R n to R m right, namely x going to A x. So, rank, find rank of A;

that is, rank of A is going to be the image of the, the subspace spanned by the column vectors, which will be the image, right?

So, as a map, the image rank of A is nothing but dimension of the image of A, and similarly, the nullity is going to be dimension of the null space of A, and when you add these two, we know that it is going to be number of columns. But number of columns is nothing but the domain of A as a map. So, that is how you can verify rank-nullity theorem for a matrix, and the same thing can be translated to a rank-nullity theorem on any matrix, of any matrix, right?

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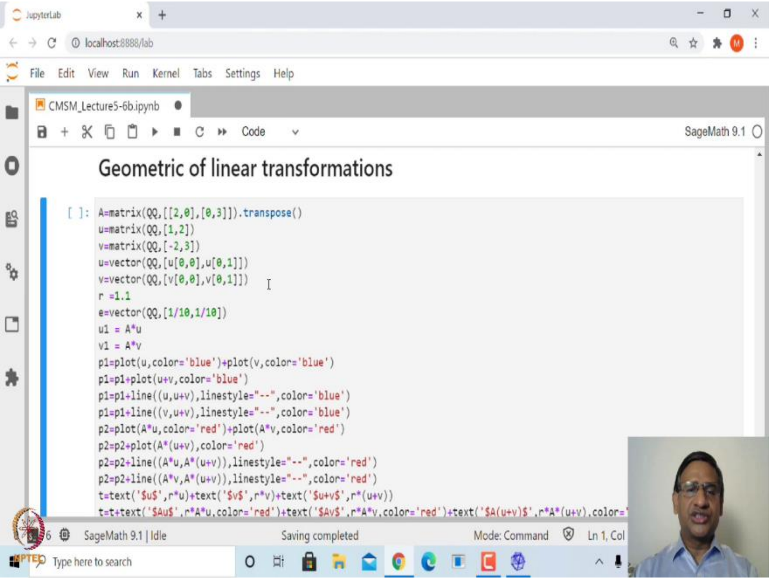

So, as I said, whatever you want to do, all the concepts that you want to do on a linear map from finite dimensional vector space to another finite dimensional vector space, can be translated to the corresponding concepts to a matrix associated with that linear transformation, and many things are exactly the same.

So, for example, you could find out trace of a matrix, trace of a linear transformation, and you can see that, the trace of that is same as the trace of the matrix associated to, with that linear transformation, right? Let us look at geometrically, geometric meaning of linear transformation.

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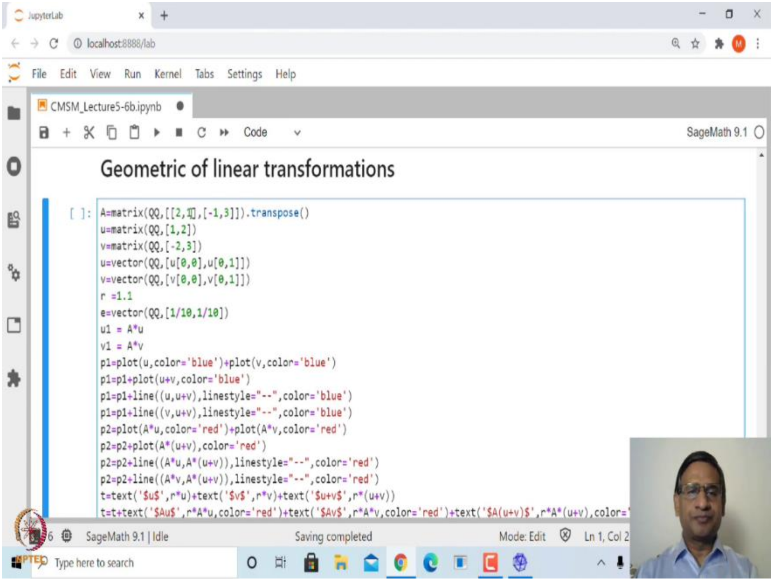

So, suppose we take a linear transformation A, which is given by 2 by 2 matrix; let us say 2, 0; 0, 3 or let me just say 2, 1, and let us say, minus 1, 3, and let us see what will be image of sum of vectors.

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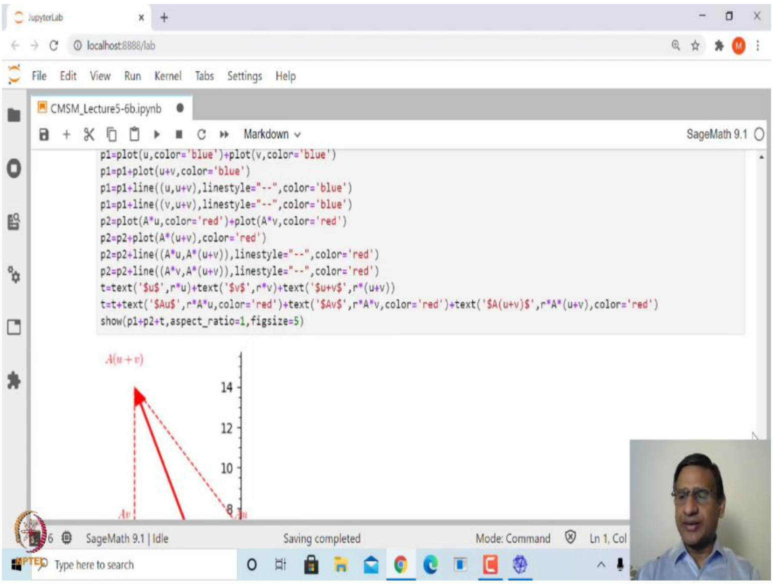

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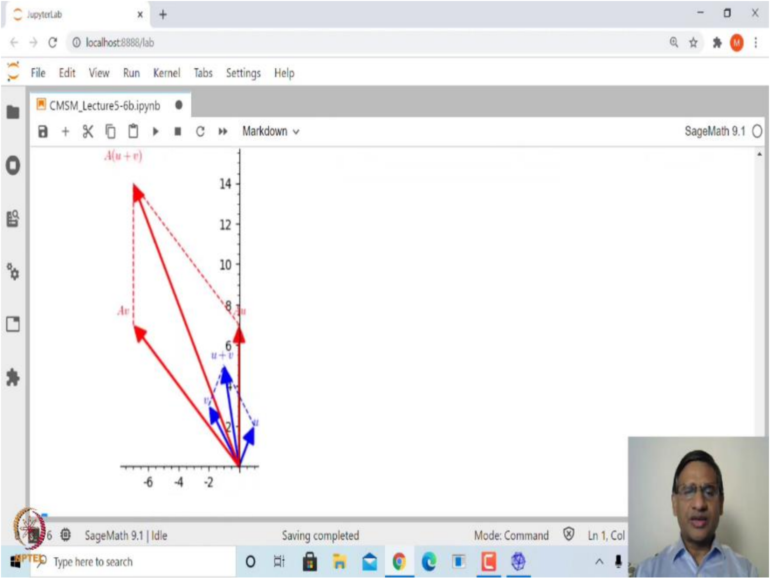

So, you see here blue vectors u, and v, and this is u plus v, and this is A of u, this is A of u, this is A of v, and this is A u plus A v, which is same as A of u plus v. So, this is same A u plus A v; you can see here this is plotted as A u, A times u plus v, but these two are same as by sum of vectors A u and A v, this is equal to this. So, that is what is happening.

So, this is just to demonstrate how this sum of two vectors gets mapped to, mapped by this matrix, and you can change this matrix. For example, let me declare this as diagonal matrix, this is a diagonal matrix, let us say this is 0.

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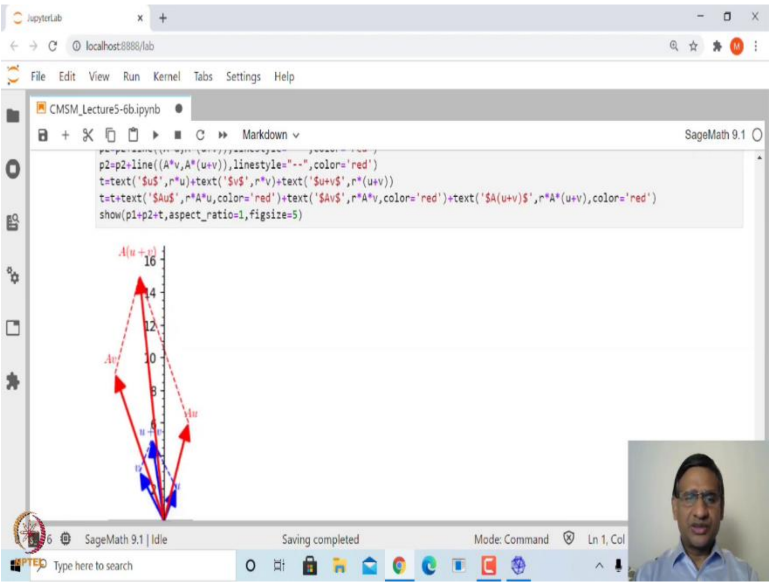
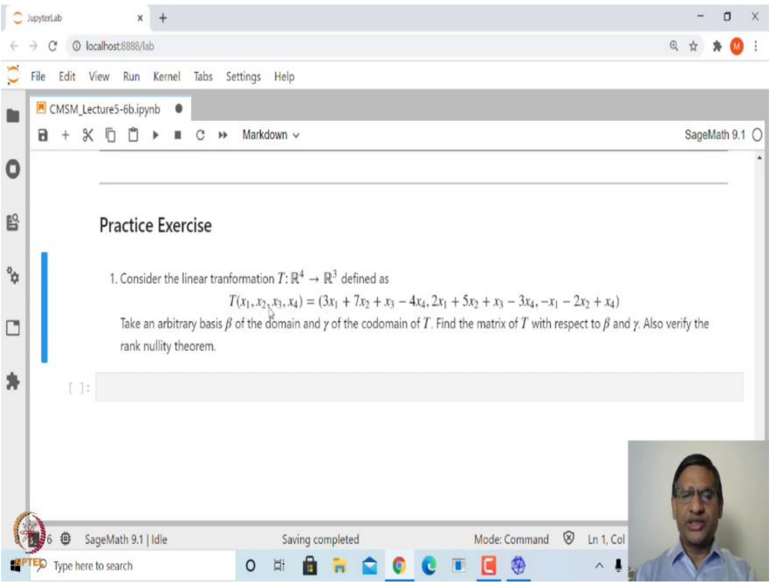


So, now what will happen?

You will see that u and v will actually be scaled in the same direction. It will scaled in the same direction. There is small bit of problem here; I have to just look into it, that must be just something do with the plotting, ok?

So, you can take various matrices, and different vectors, and then explore, explore this geometric meaning of linear transformation. You can even have some figure, let us say figure of a house plotted, and then apply linear transformation to this, and then see how the house gets transformed under this linear transformation. I leave that as an exercise. (Refer Slide Time: 29:35)



So, now let me leave you one simple exercise. So, take a linear transformation T from R 4 to R 3, and define this T as x 1, T of x 1, x 2, x 3, x 4 as this first coordinate is 4, 3 x 1 plus 7 x 2 plus x 3 minus 4 x 4; second coordinate is 2 x 1 plus 5 x 2 plus x 3 minus 3 x 4, and third coordinate is minus x 1 minus 2 x 2 plus x 4. Find matrix of T with respect to some arbitrary basis beta on the domain, and gamma on co-domain, and also find the matrix of T with respect to, ok this is, ok?

So, and then verify the rank-nullity theorem, that is the simple exercise. It is quite easy, all you need to do is define this. You can define this map in a different way as we have declared in, there are four, five ways, in which linear transformation can be defined in SageMath. So, you can explore that.

Thank you very much.