Computational Mathematics with SageMath
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Subspace associated with a matrix
Lecture – 32
Matrix Spaces with SageMath

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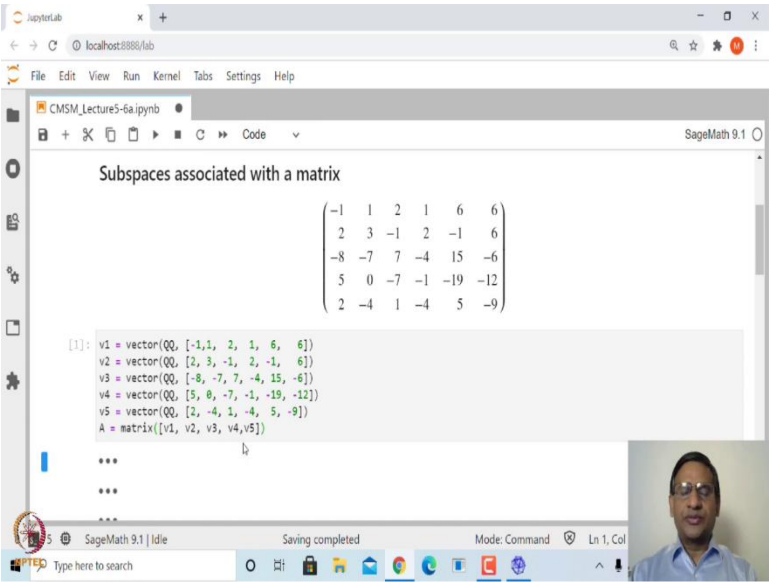

Welcome to the 32nd lecture on Computational Mathematics with Sagemath. In this lecture, we will look at subspaces associated with a matrix. So, let us begin with an example. So, let us consider this matrix A, which is a 5 cross 6 matrix. If we think of each entry coming from a rational number, then it has 5, sorry, it has 6 columns, each column has got 5 entries, therefore, each column can be thought of as a vector in Q 5. Similarly, each row can be thought of as vector from Q 6.

So, we can think of subspace spanned by all the rows of this matrix, and subspace spanned by all the columns of this, this matrix. And similarly, there are other subspaces associated with this matrix. So, let us look at how we can find these subspaces in SageMath.

So, first let us declare v1, v2, v3, v4, as rows, and matrix A as row matrix v1 to v5.

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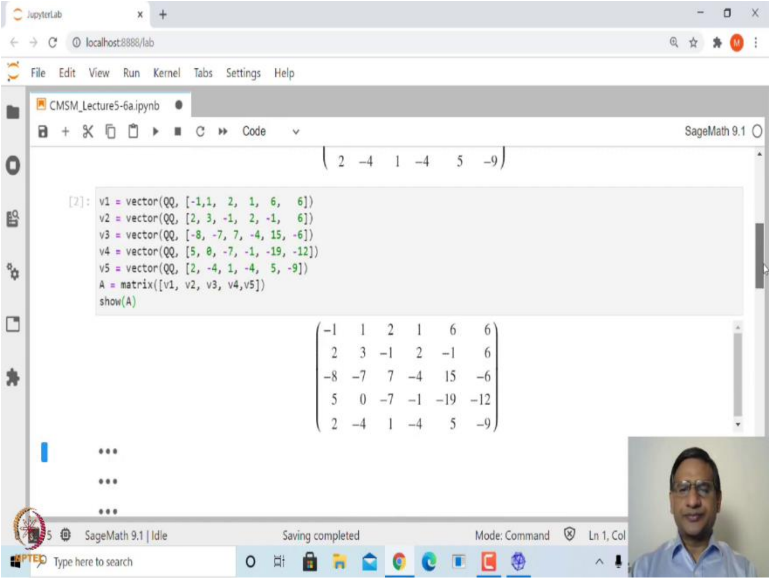
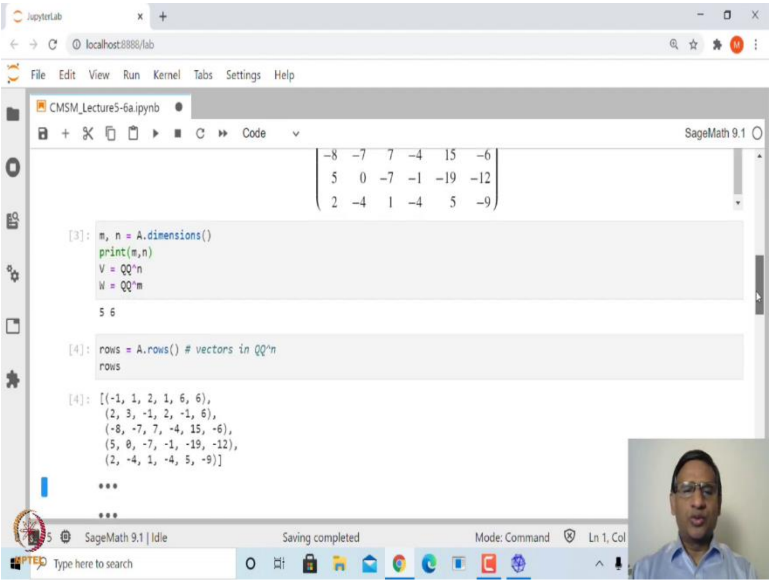


Now, if you look at this A, this is exactly what is this matrix here, right? (Refer Slide Time: 01:58)



So, what we will do? We will look at, first, let us find out dimension of this matrix A, and store these dimensions in m comma n, so m will be number of rows, and n will be number of columns.

And let us define subspace, define V as a vector space over Qn, and subspace, and vector space W as Q to the power m. So, Q to the power, n is what? Here n is number of columns. So, Q n is going to be Q to the power 6, and Q to the power m is Q to the power 5, right?

Next, let us look at rows of A. So, A dot rows will extract all the rows of A, and it will convert into a vector. So, each row is going to a vector in Q to the power n; that means, Q to the power 6.

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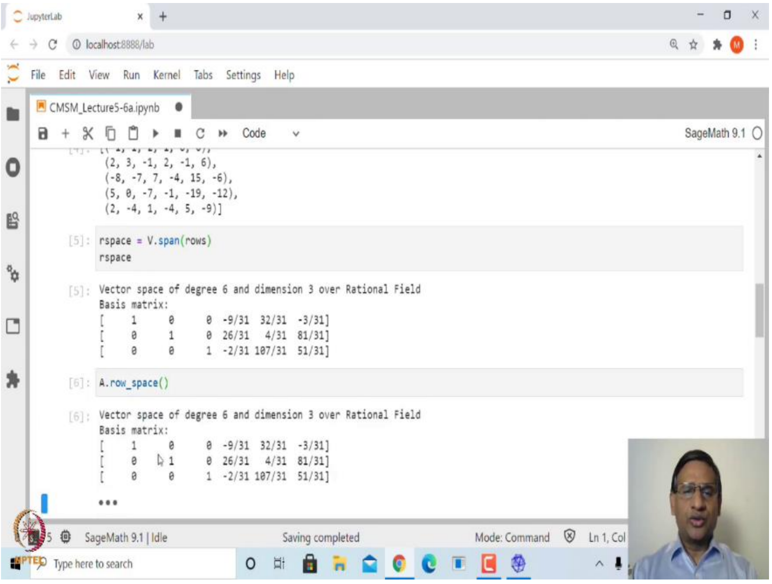

Now, so we have got these 5 rows. So, we can find out subspace spanned by these rows. So, we can find V dot span of rows vector, so that is a subspace of Q to the power n, which is of degree 6. That means, 6 entries, and dimension 3.

But we know how to obtain this subspace using RREF. So, we can apply RREF on rows, and also get subspace. However, sage has an inbuilt function to find row space. You can simply use A dot row underscore space. So, when you do that, you, you will get a subspace of a particular dimension, and with degree, so you can see here, both these subspaces obtained by V dot span, and obtained by inbuilt function row underscore space are the same.

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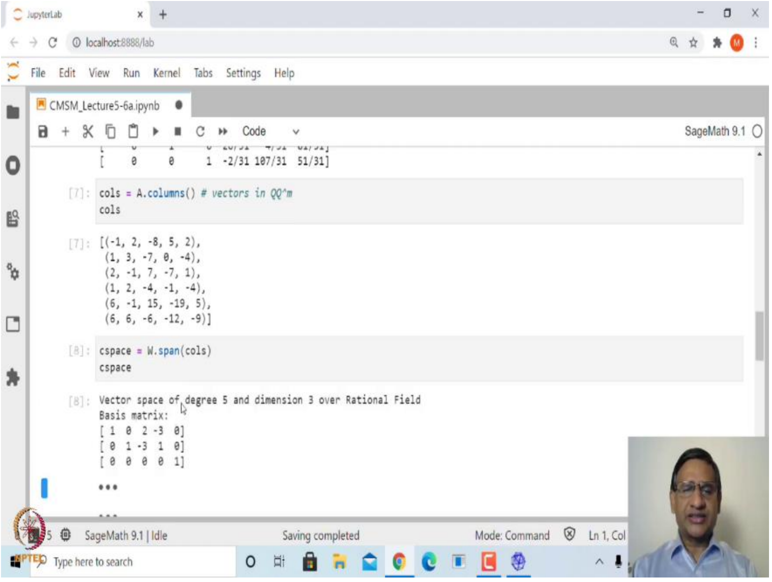

Similarly, you can extract columns of A by A dot columns. So, these are the columns, and here the first column is this, which is a vector in Q to the power of 5, and then we can obtain the subspace spanned by these 6 columns. So, we can say W dot span, so W is a, is a vector space, Q to the power6. So, when we find, it says that vectors, this vector space is of degree 5, that means, each vector has got 5 entries, and this is of dimension 3.

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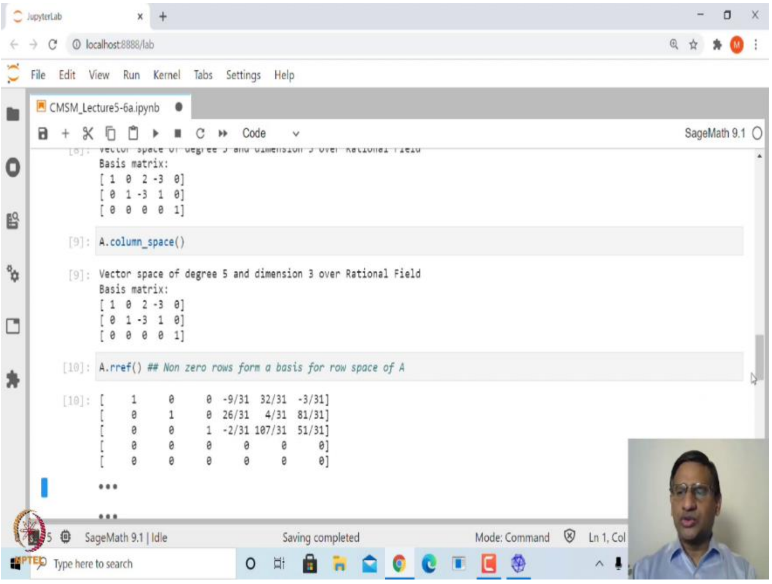

Similarly, you can find these using inbuilt function, column underscore space, and this gives you again a subspace of dimension 3, and degree 5, ok? So, if you see here, the dimension of the row space, and the dimension of column space both are same.

So, in fact, rows, dimension of row space is known as row rank of a matrix, and dimension of column space is known as column rank of a matrix, and for any matrix, these two, that is row space, row rank, and column rank are always same, which is what is called rank of a matrix, ok?

Now, I just said you could have also obtained row space and column space using RREF. So, if I apply RREF, this is reduced row echelon form, to a matrix A, to this matrix A, then you will get basis vectors, which are the non-zero rows of this, it forms a basis vector. So, again you can see here, there are only 3 non-zero rows, and these 3 non-zero rows are same as what you obtained as a basis for row space when we use span of rows.

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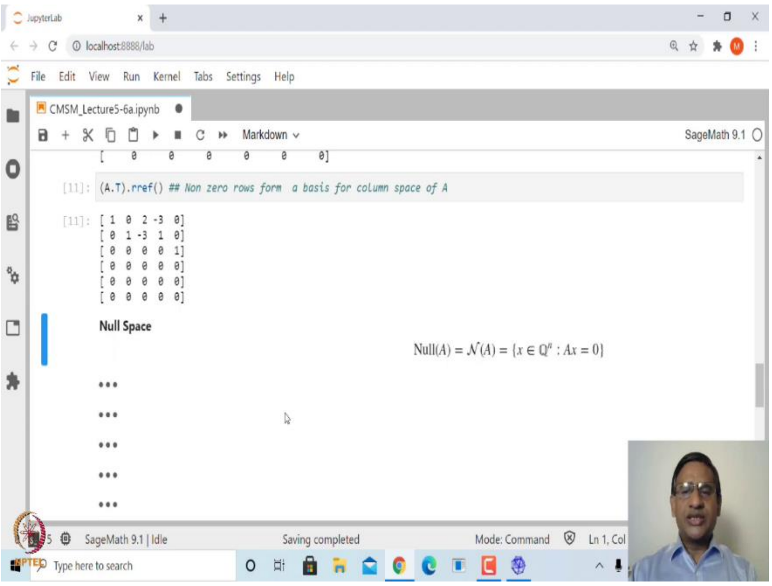

Similarly, you can apply RREF to A transpose, take A transpose, so, and then apply RREF, that will give you basis for column space. So, here if, again here you have 3 non-zero rows, and if I take these 3 non-zero rows, that forms a basis vector for the column space, and that is exactly same as this subspace, column subspace, right?

So, we have seen how to find two subspaces associated with, with matrix namely row space, and column space, and we also know that the dimension of row space, and dimension of column space are the same, and that is called rank of a matrix.

Now, there is another subspace associated with this matrix A, that is called null space of A, or in terms of linear transformation, this is known as kernel of A. So, what is this? This is set of all x in Q n, such that, A times x is equal to 0. So, that is a solution of this homogeneous system of linear equations. So, let us see how we can find this null space of A. It is quite easy to check that this forms a vector subspace of Q n.

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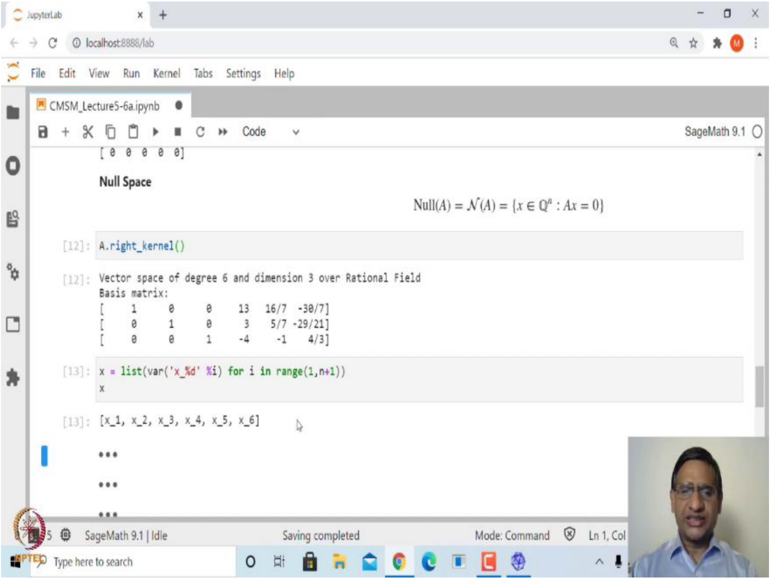

So, sage has inbuilt method to find this null space, and it is A right underscore kernel. So, if you look at this A dot right underscore kernel, this is a subspace of dimension 3, and degree 6. That means, it is a subspace of Q to the power6, and this has got 3 basis vectors, so it is of dimension 3. So, that is how you can find null space of A using an inbuilt method in SageMath, right?

However, let us look at how we can find this manually. So, if I look at this equation A x equal to 0, A is 5 by 6 matrix, and x will be a vector in Q n, Q to the power6. So, let us write Q as x1, x2, x3, up to x6. So, A x will give me 5 equations in 6 variables, x1 to x5. So, let us write these variables, let us declare these variables as x1 to x6.

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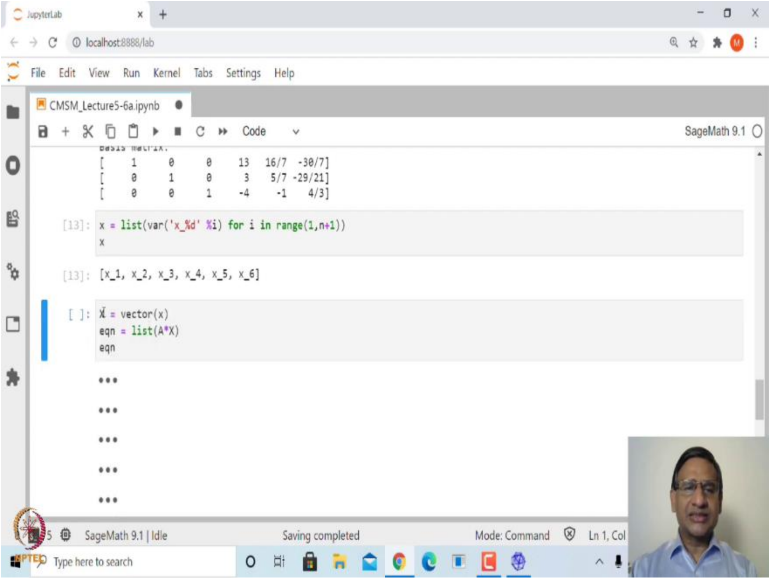

And then let us declare capital X as vector defined by these variables x1 to x6. So, it will be a column matrix, x1 to x6. Let me just print X, and show you what it is. So, this is a column matrix, x1 to x6. So, vector, so it is a column matrix.

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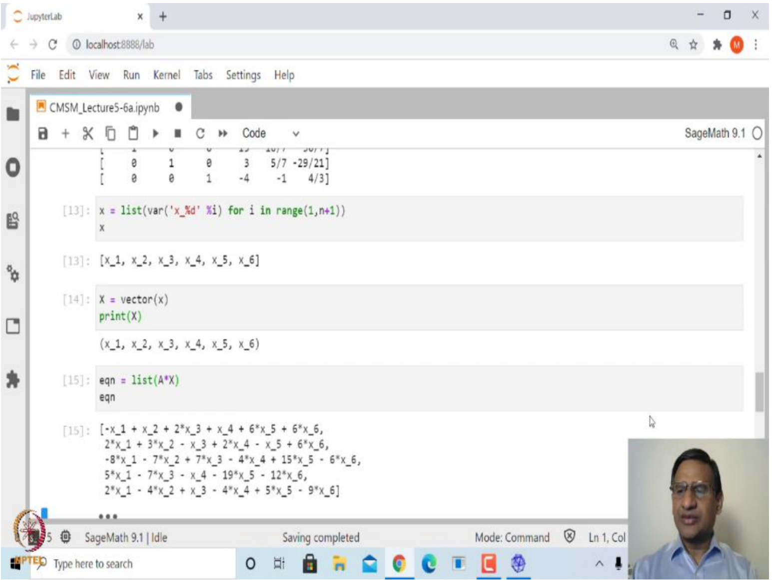

Now, let us find out A times X, and create a list of this. So, when we create A times X, it will give me list of 6 rows, and this, so therefore, if I want to solve A X equals to 0, it amounts to solving first element in this list equal to 0, second element in this list equal to 0, and so on, right?

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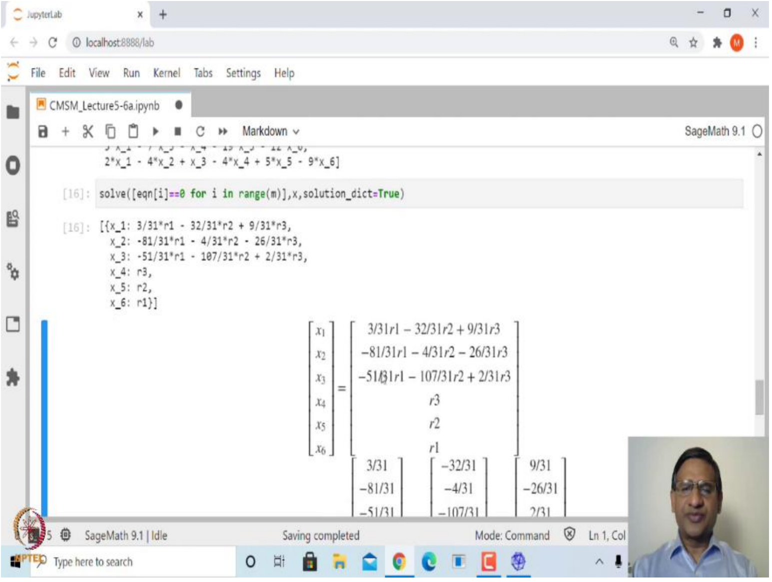

So, let us find it. Let us solve equation i equal to 0, double equal to 0 for i in range m. Since this has m number of rows, we will have to say i is in range m, and then let us declare solution dictionary equal to true, so that we will get solution of this homogeneous system of linear equations, 5 equations in 6 variables.

It is taking little bit of time. So, that is the solution. x1 can be given in terms of r1, r2, r3; x2 also in terms of r1, r2, r3; x3 is also in terms of r1, r2, r3, and x4 is r3, x5 is r2, x6 is r1. That means, this 6 equations, sorry 5 equations in this 6 variables has 3 free variables, namely r1, r2, r3, and therefore, actually it has got only 3 vectors which will be solution of this.

So, let us see how we can obtain. So, how do we obtain this? So, let us write this x1, x2, x3, x4, x5, x6, as it has been reported, this 3 by 31 times r1, and so on. This is column matrix.

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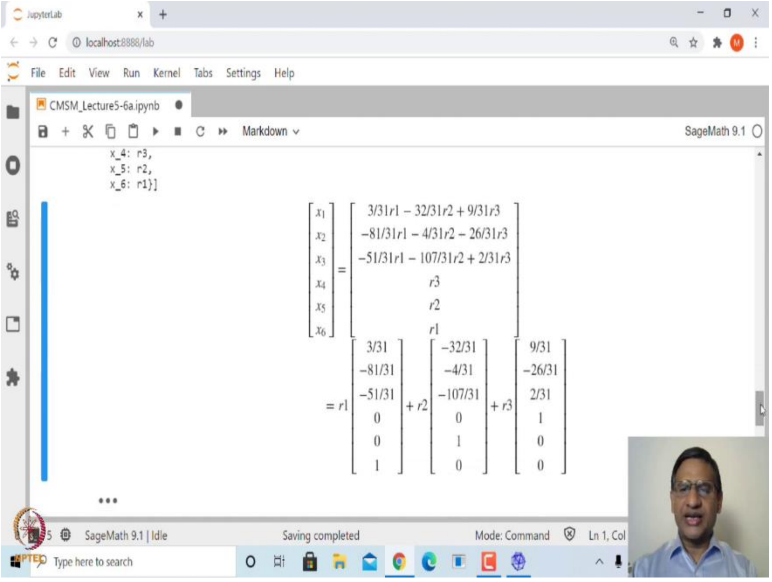

Now, this column matrix, we can write as linear combination of r1, r2, r3, and r1 multiplied by this vector, r2 multiplied by this vector, r3 multiplied by this vector. So, basically, this, any vector in this null space of A is nothing but a linear combination of these 3 vectors. So, this shows that this null space is three-dimensional subspace of Q to the power 6, and it is spanned by these 3 non-zero vectors.

Now, you can, you can, and these these 3 vectors will form a basis. But if you look at this basis, if I look at these as a basis, and what we obtain, basis using inbuilt function as right underscore kernel, they are slightly different. But this actually is obtained using RREF in reduced form. So, let us, let us apply RREF to these 3 vectors. Let us think of these 3 vectors as row vectors, and apply RREF to this.

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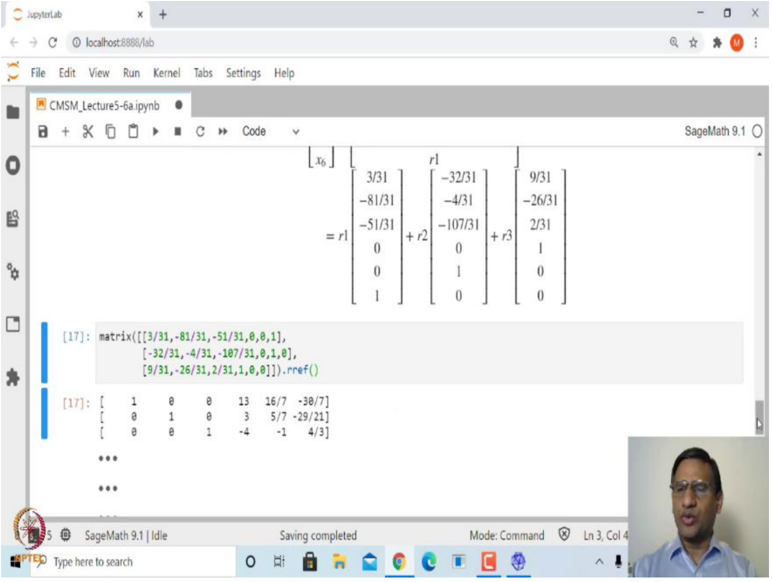

So, let us look at this as a matrix, this is first row, this is second row, third row, and apply RREF, and then you will see that this gives you basis for this null space of A, and this basis is same as what you obtained using inbuilt function. So, you can see here again, that this RREF is very useful operation, and all the computations that we have done in most of the cases can be obtained using RREF.

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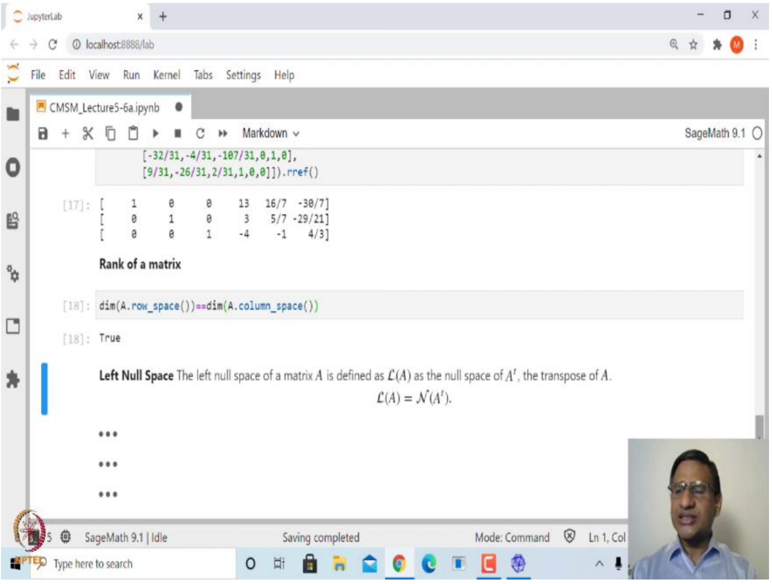

So, we already defined what is meaning of rank of a matrix. If I take any matrix, then the dimension of the row space is always same as dimension of the column space, and that is what is called rank of a matrix. So, in this case rank is 3, right?

Rank is 3, is same as saying, you have only 3 linearly independent columns, or you have 3 linearly independent rows. Remaining columns can be written as linear combination of these 3 columns. Similarly, the, the other columns which are not a part of linearly independent rows, other rows which are not part of linearly independent rows, can be written as linear combination of the remaining rows.

So, it actually amounts to saying that if we have a matrix m by n which has rank r; that means, you should bother about only some r linearly independent rows, others do not matter. There is another subspace associated to a matrix that is called left null space.

And it is defined as, left null space is, is actually nothing but the null space of A transpose. So, here A was a 5 cross 6 matrix. So, A transpose is going to be 6 cross 5 matrix, therefore null space is going to be that, the left null space of A is nothing but the null space of A transpose.

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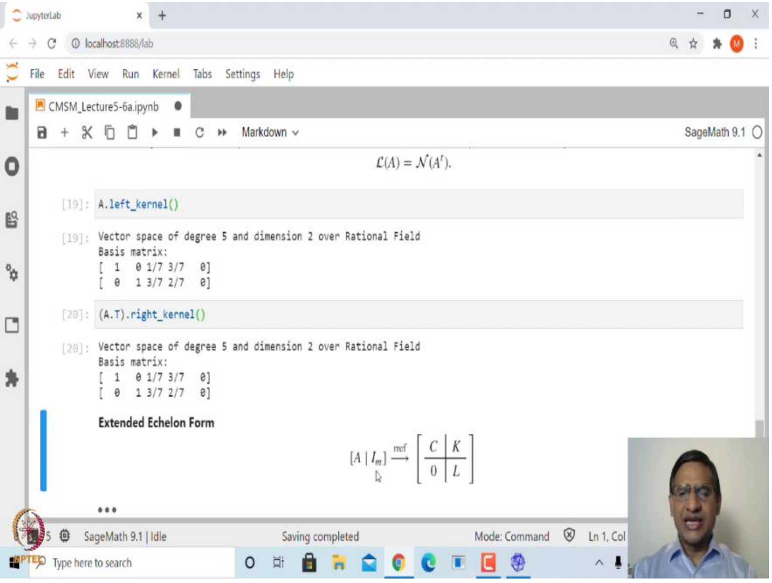

So, let us see how we can obtain this in sage. So, sage has an inbuilt method. Once you have created this object A, then you can say A dot left underscore kernel. So, the null space was right underscore kernel, and null, left null space is left underscore kernel, and we know that how we can obtain using, by solving this system of homogeneous equation, just now we did for null space, the same thing one can do it for the left null space, right?

So, in this case, the left null space has dimension 2, and it is spanned by these 3 vectors. So, and how do we obtain that? You can, you can take A dot transpose. That means, you take a transpose of A, and then find the null space of A. That A right underscore kernel, you will see that these two are the same. So, that is what is called left null space of A, and you can obtain this either using inbuilt method or using your own, I mean, or using RREF, right?

And actually, all these 4 subspaces can be captured using another echelon form, which we call as extended echelon form, and what is it? If you have a matrix A which is m by n matrix, then you append appropriate identity matrix. In this case, m, that is m by m identity matrix to A, and then apply RREF. When you apply RREF, now let us divide this into 4 sub matrices C, K, 0, and L.

So, this C is going to be the non-zero rows, and this will be, will be 0 rows, and then these things.

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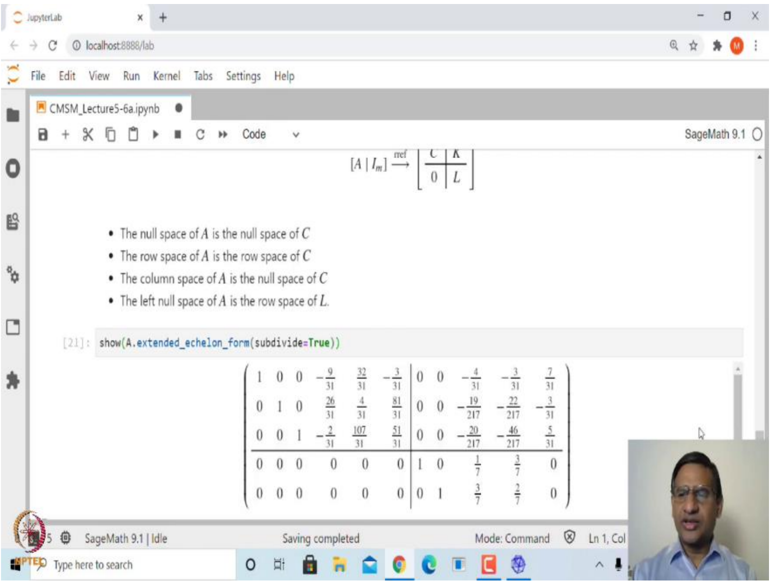
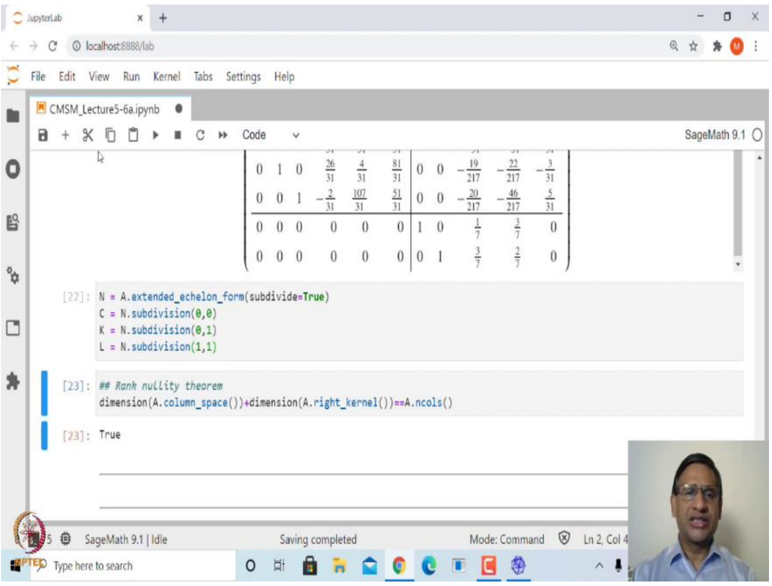


So, one can show that in this case, this, these matrices C, L, and K are the one which will also determine these 4 subspaces. So, null space of A is nothing but null space of C. So, the C captures null space of A.

Similarly, row space of A is same as row space of C. So, null space and row space are captured in this reduced matrix C. Similarly, column space of A is also same as column space of C.

So, actually row space, column space, and null space of A are captured in this reduced row echelon form of this matrix A. That means, this non-zero rows, and the left null space of A is nothing but row space of L. So, directly you get the, the basis of this left null space, which is, this is in L.

So, now let us look at how we can do it. So, we have a matrix A, and then we create extended A dot extended underscore echelon form, and also say that sub divide equal to true, so it will show all this C, K, L. So, let us run this. So, when you run this, you can see here this is the, the matrix C, this is the matrix L, and this is the matrix K. Now, if you have already noticed, these two rows of L is nothing but basis for the left null space, ok? (Refer Slide Time: 18:42)



So, you can obtain these C, K, L using, using what is known as sub division. So, if I say, suppose this extended echelon form is stored in capital N, then N dot sub division (0, 0);

0, 0 is this one, K is equal to N dot sub division (0, 1) is this, and L is N dot sub division (1, 1).

If I say N dot sub division 1 comma 0, this will give me this 0 matrix. So, that is how you can obtain, or extract these sub matrices. These are also called block matrices. So, this block, this block, this block can be extended from, from this, and then you can, you can verify the other properties which I have mentioned here, ok? So, that I will leave it as an exercise.

Now, if you look at any matrix A, which is m by n matrix, then you can look at the dimension of the column space, that dimension is called rank of a matrix. Similarly, look at the dimension of the null space, that is dimension of the right underscore kernel, that is called nullity, and if you add these two, what you get is the number of columns, number of columns.

So, therefore, for any matrix, sum of the rank plus nullity is equal to the number of columns of A, and this is what is called rank-nullity theorem for a given matrix. So, you can verify this, that is true, and you can take any random matrix, and try to verify this.

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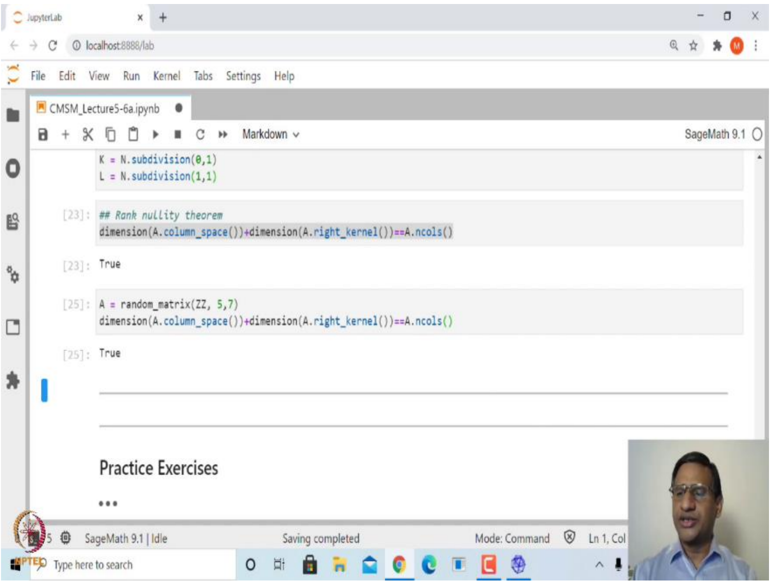

So, for example, let us, let us see, I have a random matrix, let us say A is equal to random matrix. Let me say it is 5 by 7 matrix over ZZ, this is A, and let us see whether this relation is true for this random matrix. It says it is true, right? So, this is verification of this rank-nullity theorem. This is again very useful result in linear algebra.

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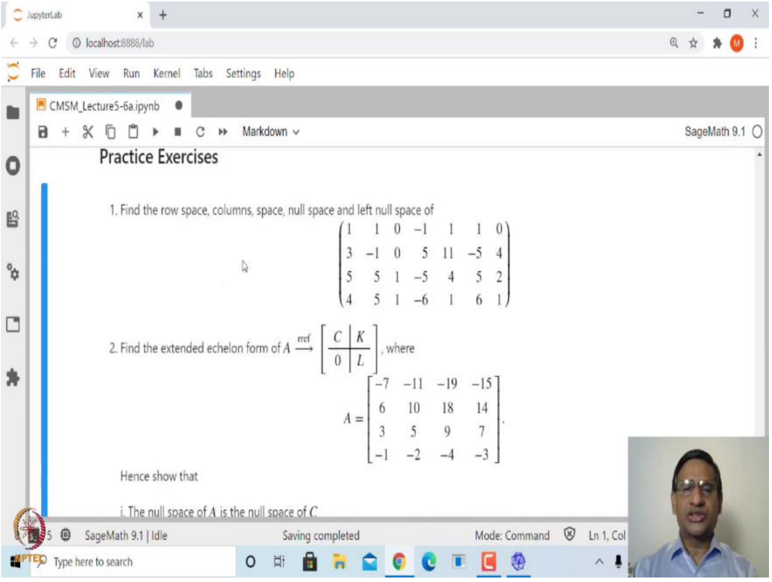

Now, let me leave you with some simple exercises. So, first one is to find row space, column space, null row space, column space, null space, and left null space of this matrix. Similarly, apply extended echelon form to this, this matrix, and find out all these C, K, L, and verify these results, or these properties which I mentioned.

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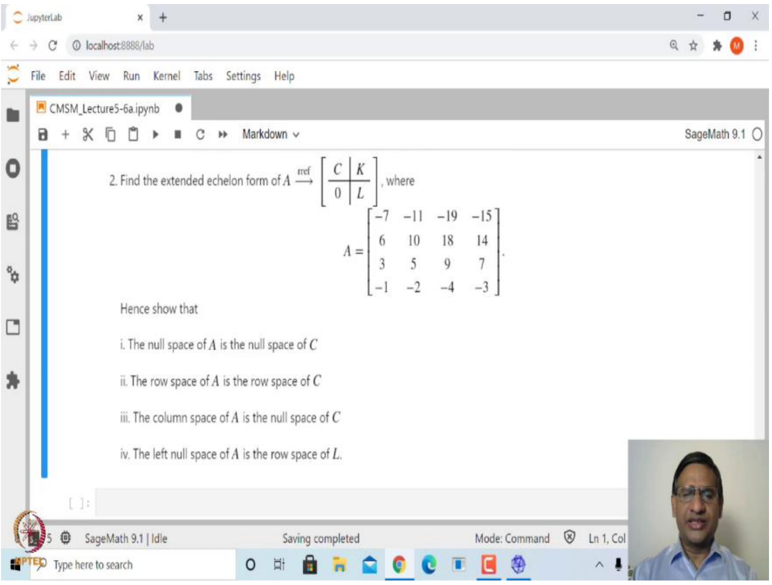

Thank you very much.