Computational Mathematics with SageMath
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Lecture – 32
Basis and dimensions of vector spaces in SageMath

Welcome to the 32nd lecture on Computational Mathematics with SageMath. In the last lecture, we looked at how to define vector spaces in SageMath, and worked with various concepts like linear span, coordinates of a vector, finding basis of linear span, and things like that. Let us look at some more examples of vector spaces, and some other concepts in this lecture, ok?

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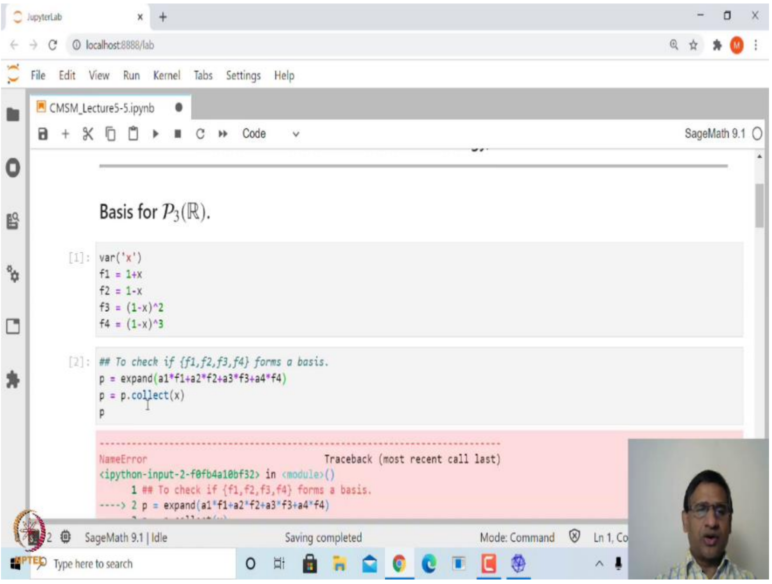

So, let us look at, suppose we want to work with set of all polynomials of degree less than equal to 3, this is P3R is set of all polynomials of degree less than equal to 3 over R. So, this is a vector space over R, and, suppose, let us say we define 4 vectors in this. f1 is equal to 1 plus x, f2 is equal to, f1 is 1 plus x, f2 is 1 minus x, f3 is 1 minus x cube, and f4 is 1 minus x to the power 3, and we want to check whether this is a basis.

If so, we can find, we can find coordinate of any vector in this. That means, any third degree polynomial with respect to this basis. So, let us just check. How do I check whether this is a basis or not? So, first of all we, we need to show that this is linearly independent, and I am sure all of you know that this is a vector space of dimension 4. So, any 4 vectors which are linearly independent will form a basis.

So, how do I check whether this is linearly independent? So, if I take a1 times f1, plus a2 times f2, plus a3 times f3, plus a4 times f4. If this is equal to 0, and we can get a1 equal to 0, a2 equal to 0, a3 equal to 0, a4 equal to 0, then we will prove that this f1, f2, f3, f4 are linearly independent.

So, let us find out what is a1 times f1 plus a2 times f2 plus a3 times f3 plus a4 times f4, and let us expand that expression, and when we say collect p dot collect. So, I think this we have not run.

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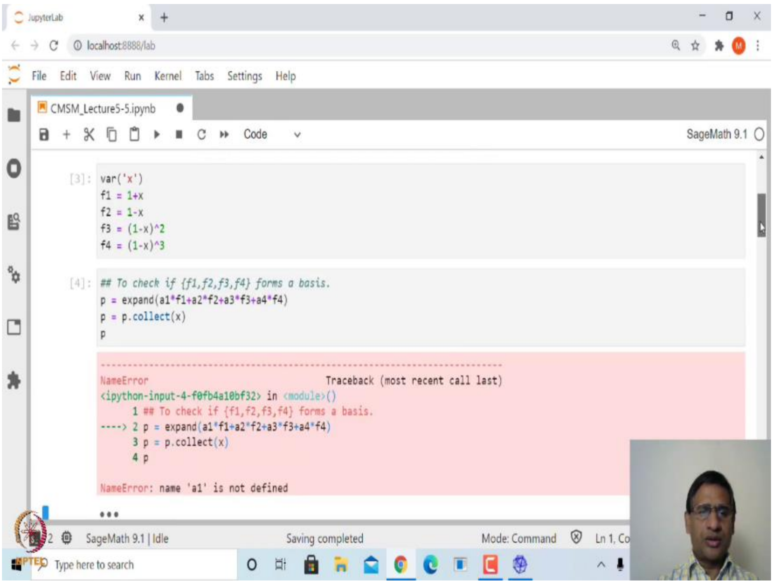

If when we say p dot collect, it will, we need to declare a1, a2, a3, a4 as variables.

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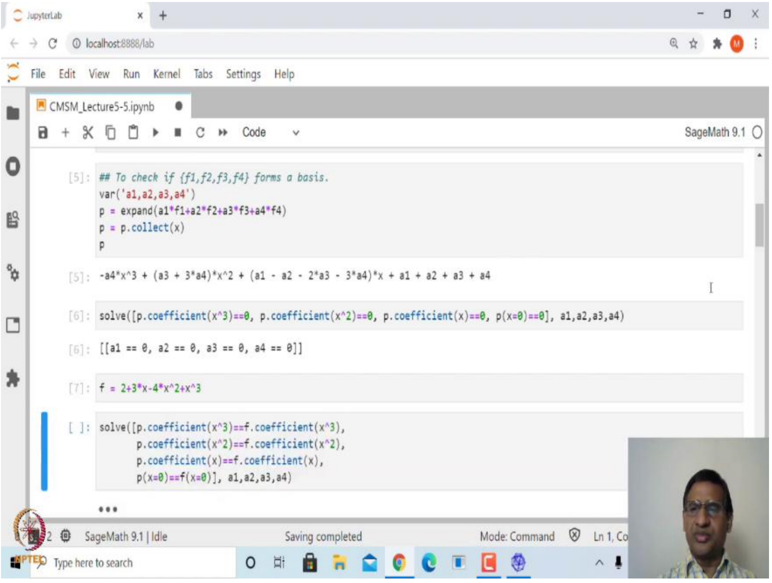

So, let us say var, in the bracket a1, a2, a3, a4, right? So, when you say collect, these coefficients, these, these coefficients are collected. Now, what you have is, this is equal to 0, 0 means 0 polynomial in degree 3. So, the first term the coefficient of x3 should be x to power 3 should be 0; that means, a4 is 0.

Similarly, since a4 is 0 the coefficient of x square is this. So, when a4 is 0, then at a3 is 0. So, a3 and a4 are also 0, and from this, coefficient of x, a3 and a4 are 0. Therefore, a1 minus a2 is 0, which is same as saying a1 is equal to a2. And now, the constant term, which is also 0, which is a1 plus a2 plus a3 plus a4 is equal to 0. That would mean, since a3 and a4 are already 0, a1 plus a2 is equal to 0.

But, a1 is already equal to a2. This is same as saying 2 a1 is 0, and hence a2 is 0. So, this shows that this set of vectors are linearly independent. However, you can, you can do this checking, using inbuilt function solve. So, you, how do you do that? You will collect the coefficient of x cube, and equate it to 0, coefficient of x, x square equate it to 0, coefficient of x, x equate to 0, and when you say p of x equal to 0, that is the constant term, that is also equal to 0, and solve this for a1, a2, a3, a4.

You should get this as 0, 0, 0, 0, solution, right? So, that is how you can check whether set of polynomials in P 3 R is a basis, is linearly independent or not. So, it took a little while, and we get this is linearly independent, and hence this forms a basis. Now, suppose I have a vector which is 2 plus 3 x minus 4 x square plus x cube, and we want to find coordinate of this vector, we want to find coordinate of this vector that are, this, with respect to the given basis f1, f2, f3, f4.

So, how do I do that? If coordinates are, let us say a1, a2, a3, a4. That means, we have, we should have a1 times f1 plus a2 times f2 plus a3 times f3 plus a4 times f4 this should be equal to 0. So, that is same as saying you just simply, we have already defined p as summation ai*fi. So, you collect the coefficients of this ai f, summation ai*fi, and equate it to the coefficient of f. That is what you, we are doing.

So, coefficient of p, x cube coefficient of p, equate it to x coefficient of f, x square coefficient of, coefficient in p equate it to the x square coefficient in f, and so on, and, when you do that, you will get the solution.

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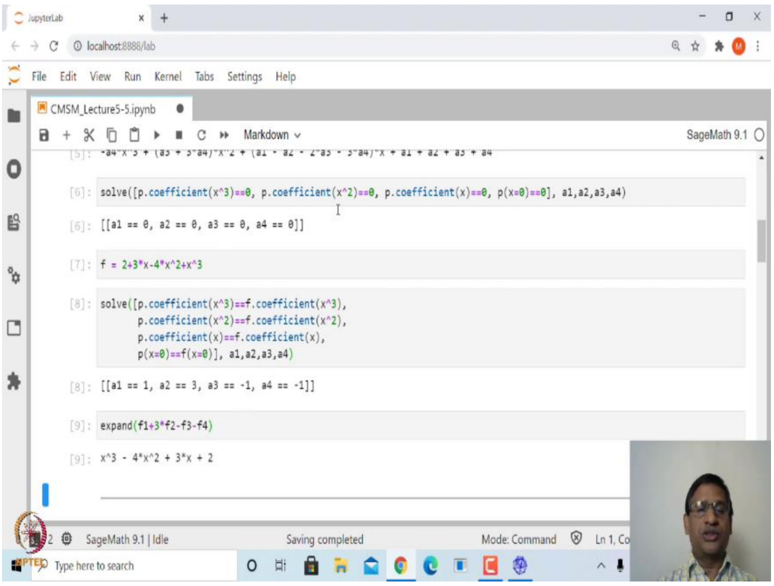

So, a1 is 1, a2 is 3, a3 is minus 1, a4 is minus 4. So, if you try to, for example, check what is f1 plus, 3 times f2 plus, minus 1 times f3, minus 1 times f4, this should be equal to f, right?

So, we have found the coordinates of this polynomial with respect to this basis. So, that is very simple exercise.

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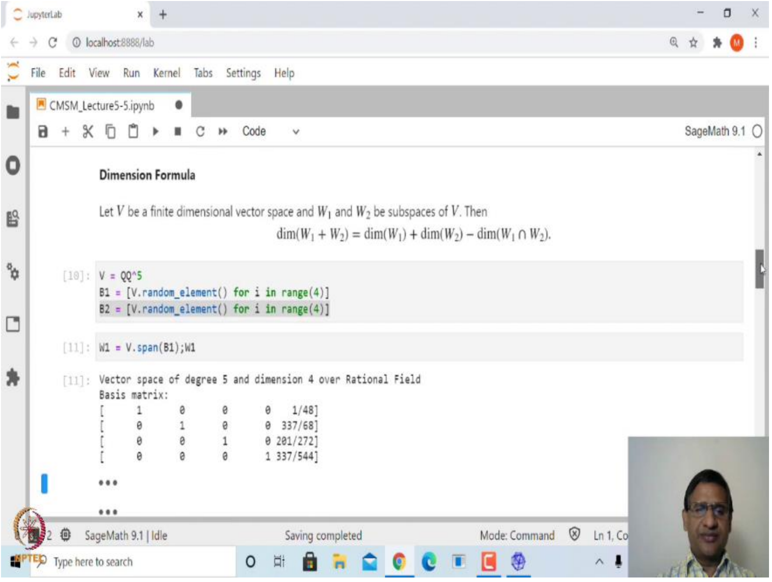

Now, let us look at, you must have studied what is called dimension formula. So, if you have 2 vector subspaces, let us say W1 and W2 of V, you can generate more subspaces. One of them is W1 intersection W2, is a sub space of V.

Similarly, W1 plus W2 is also a subspace of V, and dimension of W1 plus W2 is related to dimension of W1 plus dimension of W2 minus dimension of W1 intersection W2. So, let us verify this dimension formula, and showing this is all, is actually a standard theorem in linear algebra, I am sure you must have seen this, ok?

So, let us start with V to be Q to the power 5, QQ to the power 5, and let us take 2 sets B1 and B2, of having 4 elements, 4 random elements in this. So, it is possible that B1 generates 4-dimensional subspace of V, B2 also generates 4-dimensional subspace of V.

So, let us define this, and let us define W1 to be the linear span of B1, that is the, that is a vector subspace of dimension 4; W2 also a subspace of V generated by B2, that is also a subspace of dimension 4. So, dimension of W1 and dimension of W2 are 4 each, and, let us find out what is intersection.

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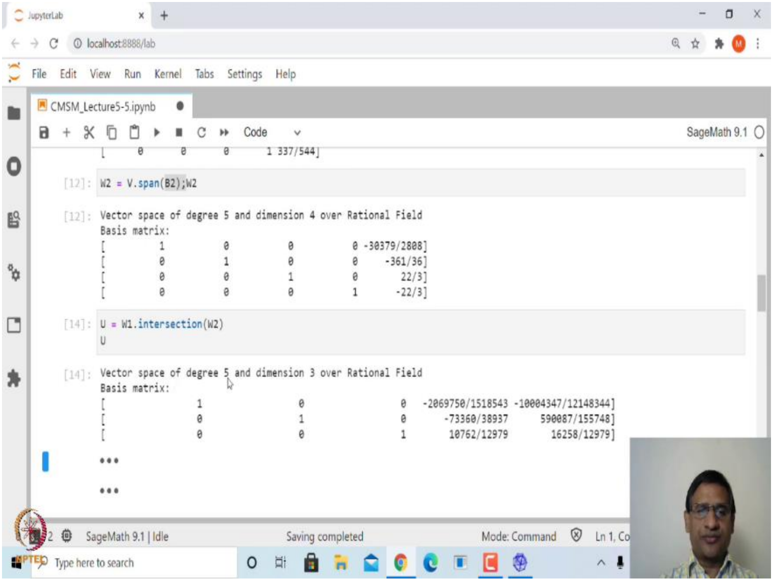

So, the intersection is, can be the, of W1 and W2 can be obtained using this intersection function. So, W1 dot intersection W2 will give you intersection of this.

So, if you want, I will, I will call this as, let me, let me call this as U, that is the intersection. So, if I say what is U? This gives me vector space of dimension 3. The intersection is vector space of dimension 3, and what is W1 plus W2?

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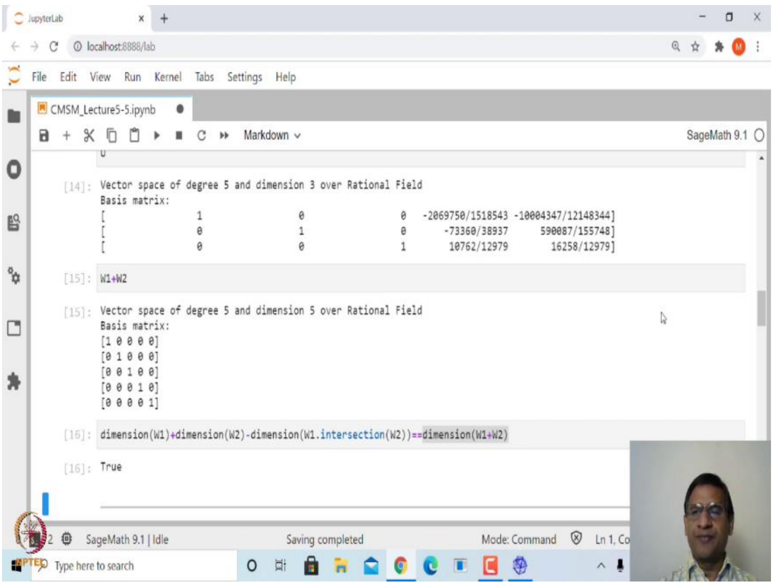
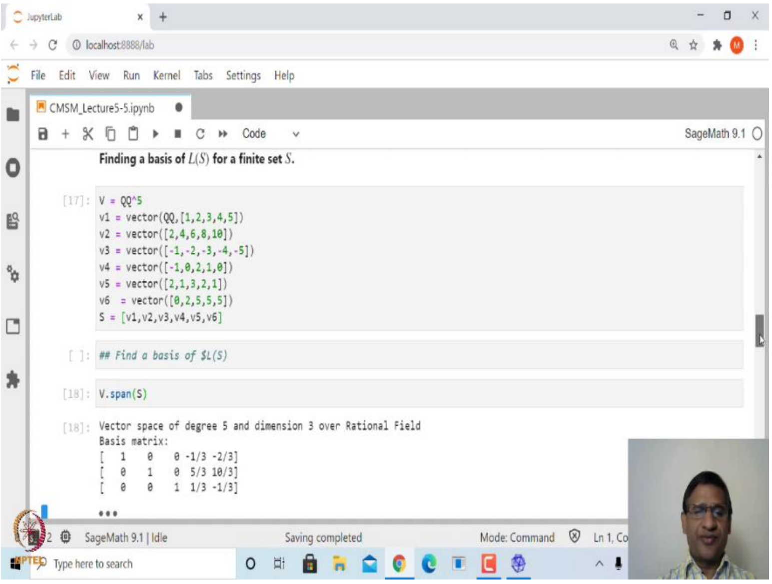


W1 plus W2 can be obtained by just W1 plus W2, and this is actually a vector space of dimension 5. So, this is entire space, in fact, and then, let us verify the dimension formula.

So, dimension of W1 plus dimension of W2 minus W1 dimension of W1 intersection W2, is it equal to dimension of W1 plus W2? The answer should be true. So, this is verification of dimension formula. Of course, you can ask how this W1 plus W2 is generated, how is this W1 intersection W2 generated, right?

So, I will leave that as an exercise. Just take some example, and take some W1 and W2 subspaces, which could be linear span of some vectors, and see how you can generate W1 plus W2, and W1 intersection W2, right? (Refer Slide Time: 09:20)



Next example, let us look at finding a basis of linear span for a finite set, ok? This actually we have already seen, but let us look at it again. Suppose, I have QQ to the power 4, and I have 6 vectors v1, v2, v3, v4, up to v6, and S is the the set containing v1, v2, v3, v4, v5, v6. And, if I want to find the basis for this, what we can do is, we can, we can take a, of course, we can take linear span of this, and find the basis using inbuilt function.

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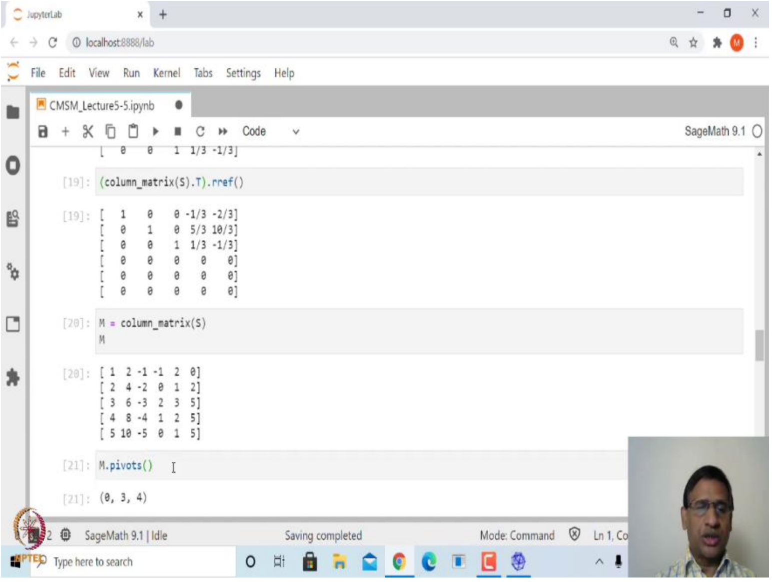

However, if you want to do manually, what you can do is, you can, you can look at these vectors, and form a matrix, column matrix whose first column is v1, second column v2, and so on, and, then apply RREF. When you apply RREF, you can see here, this, this, this column, this column, this column, first 3 columns, they form actually identity matrix.

So, in fact, so, actually we are generating this as a row matrix where row, rows are vectors in S. First row is v1, second row is v2, and so on, and then apply RREF. This will give you the, the basis, and which is same as what you got using inbuilt function. So, in order to find basis of a vectors space LS, you simply take vectors in rows, and apply RREF, this we have already seen.

However, you can also do it with column matrix. So, when you do the RREF over column matrix, then you can, you can find the pivots of this. So, pivots, in this case, pivot of this column matrix. So, this is 0, 3, 4; 0 means first column, 3 means fourth column, and this is fifth column. So, first column, that corresponds to v1, this, this is the fourth column, corresponds to v4, and this is the fifth column, which is v5.

So, that in this case, that means, basis for this is going to be v1, v4, and v5. Those are the linearly independent vectors, and every other vectors are linearly dependent on these 3.

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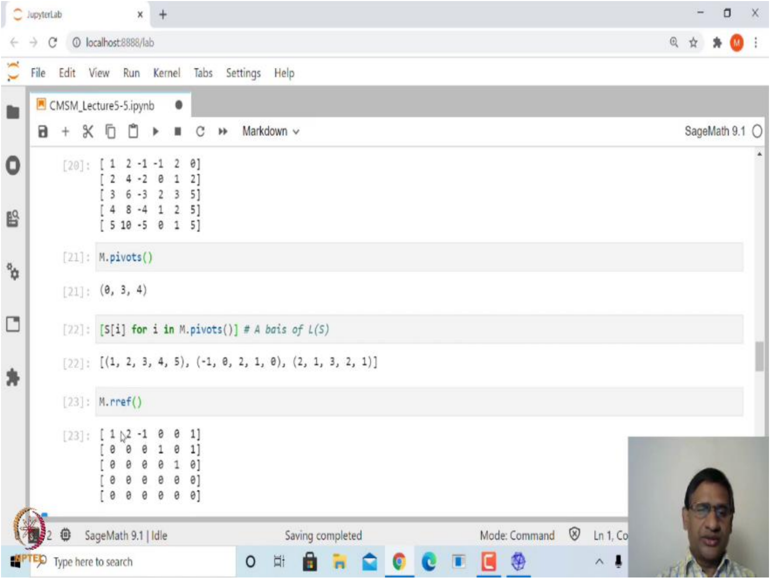

In fact, you can, you can try to check each of this pivot, i running over this pivot, and Si over i, this, these are the 3 vectors which are linearly independent, and it forms a basis for LS, right?

You can apply RREF to M, and then you will see that first column, fourth column, and fifth column forms an identity matrix. So, these are the pivots, pivots are 1; second column is twice the first column so, that is not. Third column is also minus 1 times the third, first column. So, it is the third column; fourth column is, is linearly independent with the first column.

So, first column, fourth column, and this fifth column, they form a basis. The last column is actually linear combination of first and third, sum of first, not first and third, first and fourth, right?

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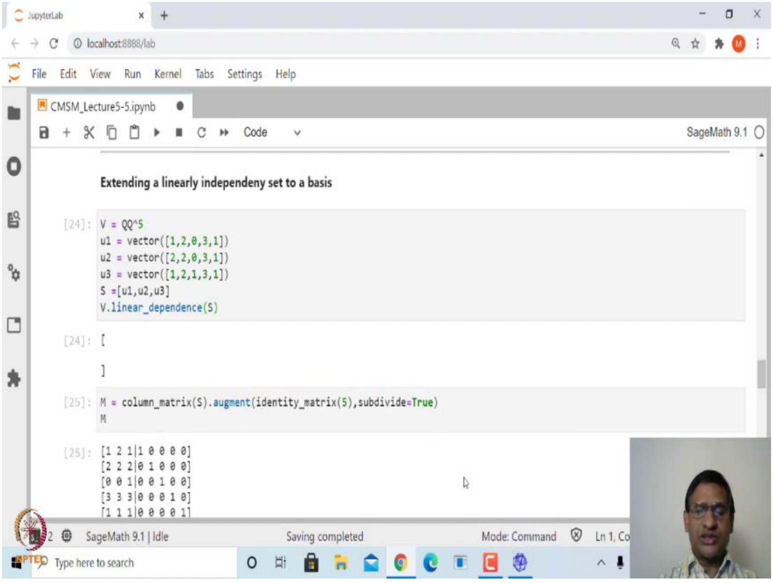

Let us look at another example. This is again very, for example, this problem, this you must have seen in the theorem, that any linearly independent set can be extended to a basis.

By the way, this previous problem which we, we saw is what? It is a actually a, if you are given a set of vectors in, in vector space V, then from that you are able to generate a basis for the linear span, that is what you have seen. Now, we are looking at, you start with a linearly independent set that can be completed to a basis of the vector space. Of course, we are talking about everything in finite dimensional.

So, let us say V is equal to QQ to the power 4, u1, u2, u3 are these 3 vectors, and let us check whether they are linearly independent. Yes, they are linearly independent. So, you want to extend this to a basis of Q5. So, how do I do that? Generally, in the theorem, what you would have seen is that you take linear span of this u1, u2, u3, and take a vector u4 which is outside this linear span.

Now, how do I want to select something which is outside this in some arbitrary space? It is not, which is not very convenient way of selecting, right? So, and then after that, once you have selected u, u4 which is not in the linear span of u1, u2, u3, u4 you can show that u4, u1 to u4 will be linearly independent.

So, again you take a linear span. In case it is not the complete fold vector, then you select something which is outside linear span of u1, u2, u3, u4. So, you keep on continuing this, that is what the proof of the theorem. So, in this case, what we can do is, we can look at this vectors u1, u2, u3, u, u3, and look at this standard basis e1, e2, e3, e4, e5.

So, if e1 is in the linear span of u1, u2, u3, there is no problem you can take u2, e2. If e2 is also in linear span do not, leave it. So, you keep on checking, not every e1, e1, e2, e3, e4, e5 will be in the linear span. Otherwise this whole thing will span the entire V, which is not correct.

So, therefore, the u4 will come from one of this ei, similarly u5 will also come from one of this ei. So, we can take this as a clue, and what we can do is, we can augment, take this column matrix u1, u2, u3, and augment identity matrix to to this matrix, that is same as saying you are augmenting e1, e2, e3, e4, e5.

Let us look at what is this M, right? So, M, the first column is u1, second column u2, third column u3; this is e1, e2, e3, e4, and, now find pivots of this, right?

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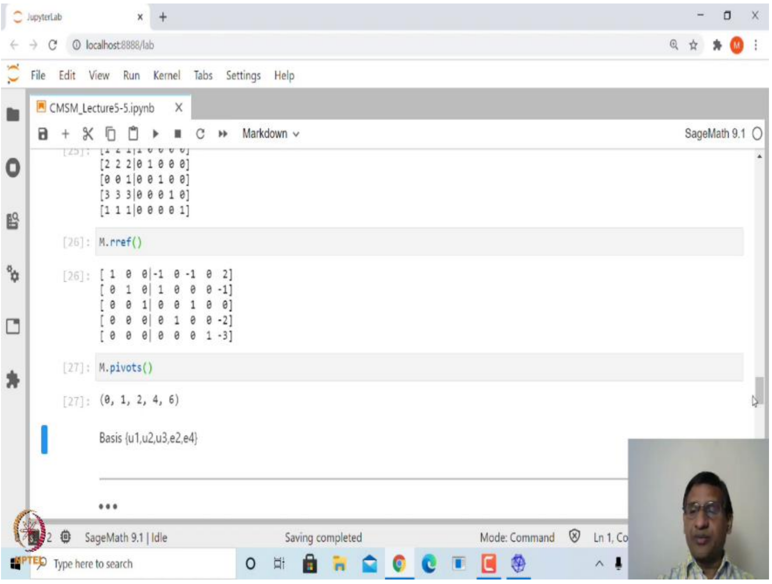
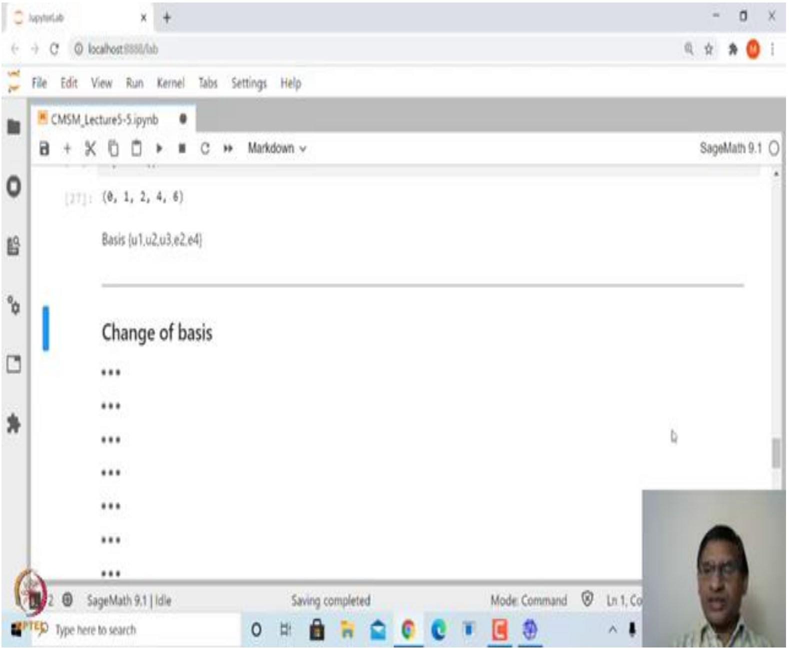


You can, you can just, you can find RREF. That will also give you pivots. So, pivots are first 3 columns here, and then this is not the case, this is the one.

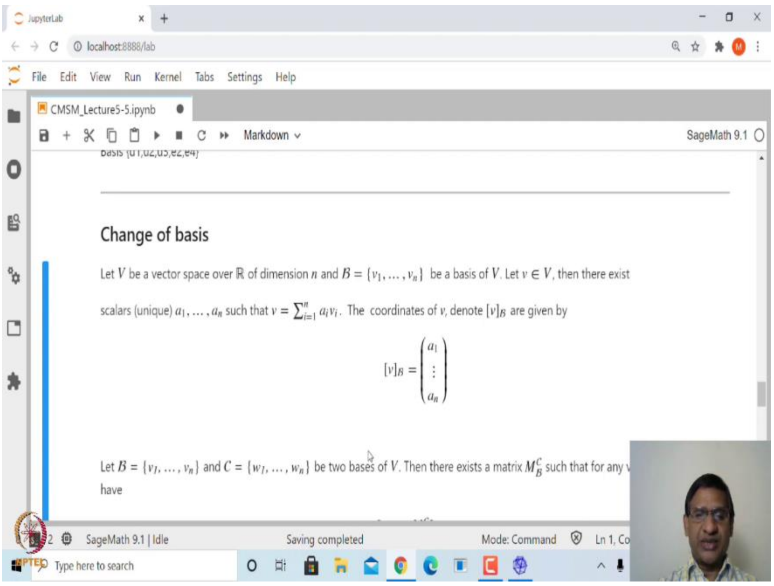
So, this corresponds to e1, this corresponds to e2, e2, this is also not there, this is e4. So, the pivots are 1, 2, 3, 4 is not there, 5, and 7. So, these are the pivots, and therefore, what will be the basis? Basis will be e1, sorry u1, u2, u3, and then e2, and then e5, e5, e, e4 because this is e1, e2, e3, e4.

So, u1 to u3, and then e2, and e4, that is what you have. You can just find the pivots. So, the pivots are 0, 1, 2, 3, and 4, and 6. So, that that corresponds to u1, u2, u3, e2, and e4.

That is the basis which is obtained by completing this linearly independent set of vectors to a basis, right? (Refer Slide Time: 16:55)

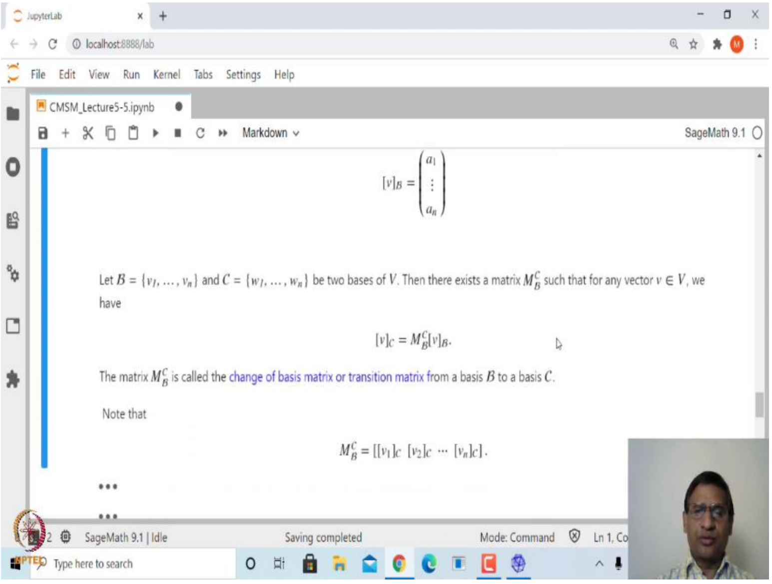


Another thing which you can say, suppose you are given 2 basis, and take any vector, you can find coordinate with respect to one basis, coordinate with respect to another basis, and how are these coordinates related to thess bases b1 and b2? Let us say b1 and b2 are two bases of vector space V. That is what we want to look at. (Refer Slide Time: 17:19)



So, if you have, let us say two bases B, and another basis let us say C; B is v1 to vn, the C is w1 to wn, these are the two bases.

And, suppose I have a vector small v, which is, whose coordinates are a1 to an, these are the coordinates. So, v is summation ai*vi. That means, ai's are coordinates. So, that is, we can write as a column vector, coordinate vector. (Refer Slide Time: 17:48)



Similarly, with respect to the basis C, I have these coordinates, let us say v underscore C.

And, then one can show that this, the coordinate of v with respect to beta, and coordinate of v with respect to, let us say C, they are related by a matrix which we, let me denote this by M underscore B C, that is called a change of basis matrix or transition matrix. So, this is v C is multiple of a matrix multiplication with this.

Now, how does one obtain this, that is, matrix? So, that is quite actually easy. So, what you have to, one can do is, this, this, this matrix will be a matrix whose first column will be nothing, but coordinate of v1 with respect to the basis C.

Coordinate of v, second column, will be coordinate of v2 with respect to basis C, and so on, coordinate of vn with respect to C. That is the, that is what you need to do, and v1, v2,…, vn are what? They are the, the elements of the basis B.

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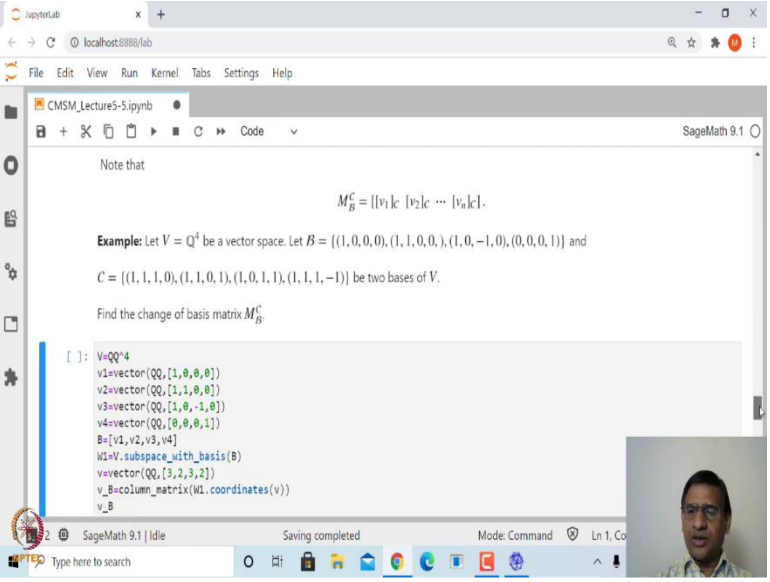
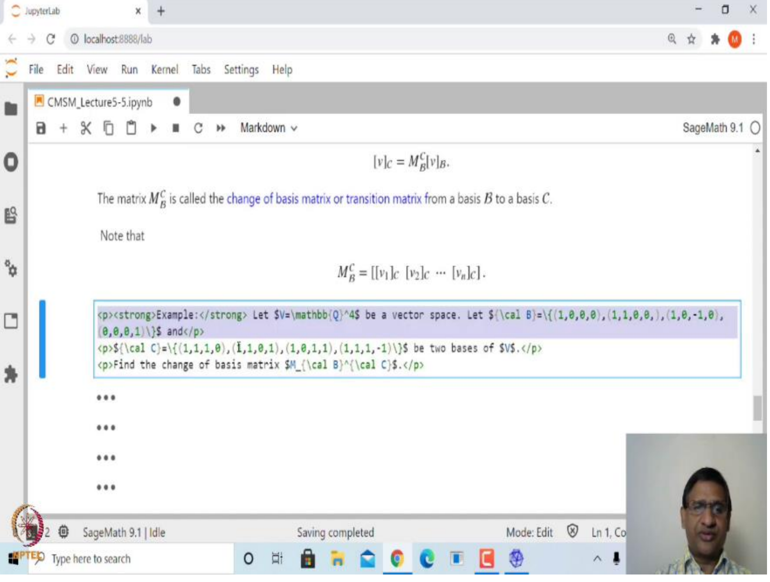


So, let us look at this, an example. So, let us say beta is these four vectors in Q4, and this you can check, this forms a basis. (Refer Slide Time: 19:09)



Similarly, let us say C, these are again some 4 vectors, it forms a basis. We want to find this transition matrix.

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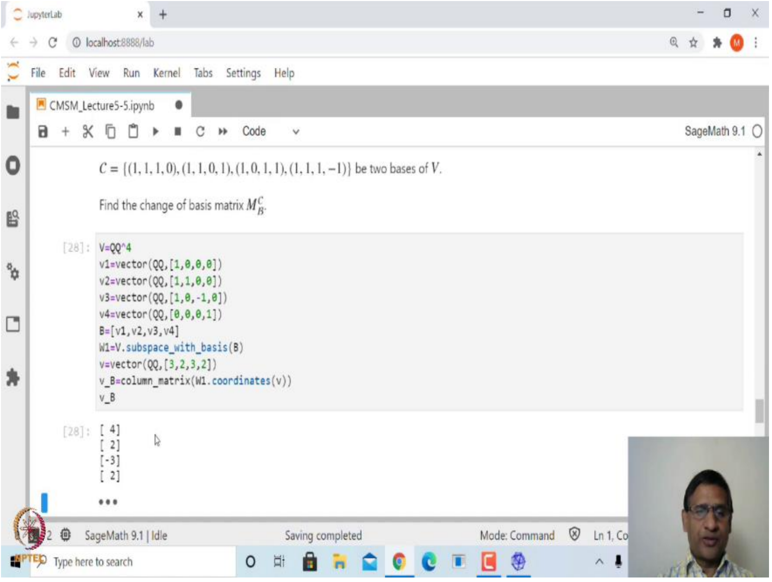

So, how do we do that?

In, in order to do this, let us declare V as Q4; v1, v2, v3, v4 is this, capital B is this set, and let us generate a subspace with this B as a basis, that is W1, and I have a vector v which is 3, 2, 3, 2, that we want to find coordinate of v with respect to this beta B. So that we can obtain, let me call this as v underscore B, and that is the, the, the column matrix with these coordinates.

So, let us see here. So, that is the, the column, the coordinate vector of v with respect to the basis capital B, ok?

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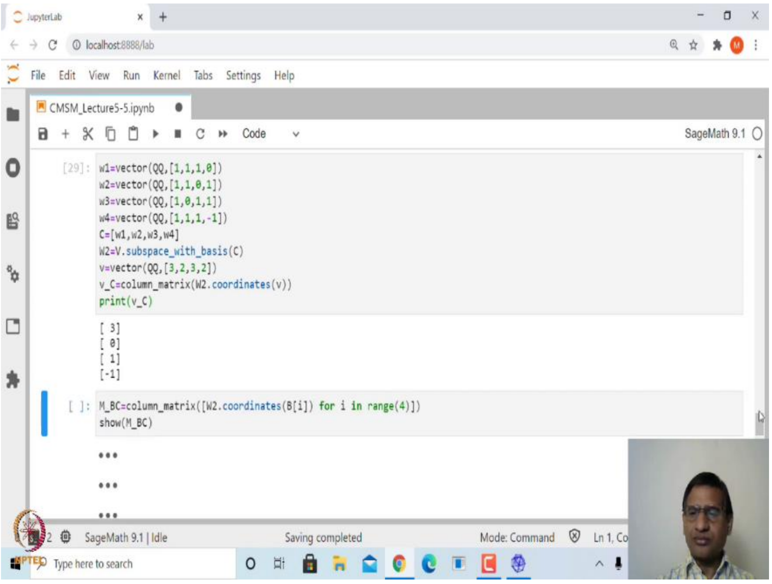

Similarly, let us find basis of v, same vector v, with respect to basis C which is w1, w2, w3, w4. These are w1 to w4, and exactly the same way, let us find the the coordinates. So, these two are the, the, this is coordinate of v with respect to basis beta; this is coordinate of v with respect to coordinate C, and with respect to basis C, and let us see how they are related.

So, as I said, how they are related? So, that is, is, a column matrix, whose first column is coordinate of v1 with respect to basis C. So, that means, you, how do I find that? We have already seen that it can be obtained by using RREF. So, that is what we will look at. But, first let us take the coordinate of Bi with respect to basis C. So, that means, you are finding coordinate of Bi with respect to subspace W2, which has basis C, and then take all these vectors, write this as column matrix.

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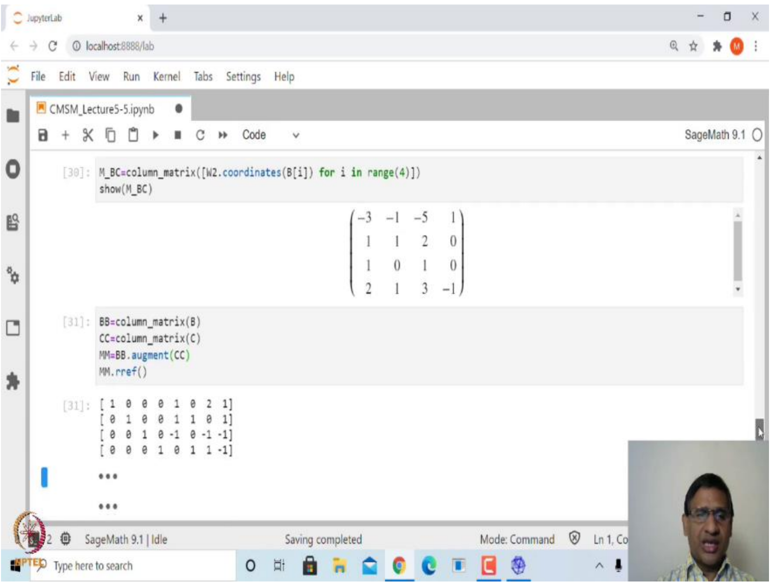

So, that is your matrix, transition matrix, or change of coordinate matrix. This is what you get. Now, the same thing you can do it with, by take, by augmenting, let us say, the matrix inside B, B you augment this this C, C. So, what we are doing in this case? You are finding the coordinate of each vector in C with with respect to basis B, B, that is what you are doing.

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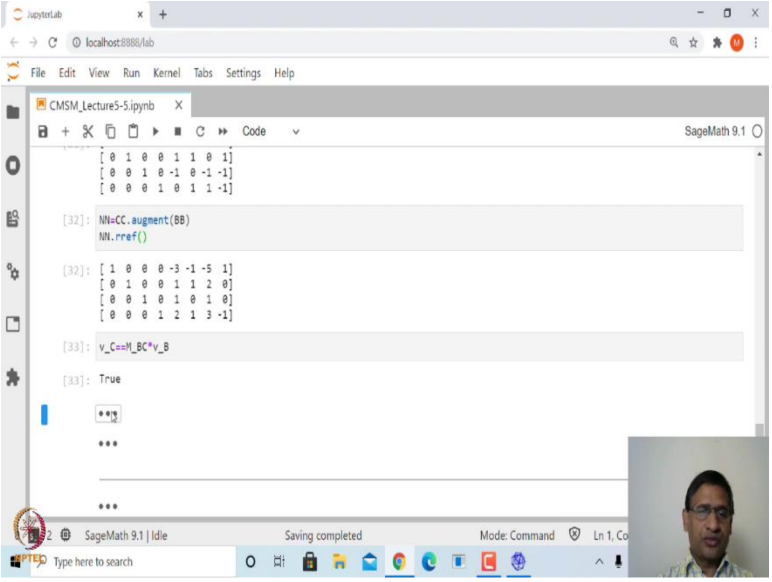

Similarly, if you take if you augment B, B over C, C, C, that means, you are augmenting the v1 to the set of this, the matrix whose first 4 columns are coming from basis C, and then apply RREF. Then the last 4 columns will be nothing, but this, this matrix which you have obtained. So, that is your M, C B, right? So, this, you can use RREF, or you can use inbuilt function.

Now, let us check the coordinate of v with respect to C, is it equal to the matrix, and this transition matrix M BC times coordinate of v with respect to capital B? The answer is true. So, we have verified this result, that the coordinate of v with respect to 2 bases are related to this change of matrix, change of basis, that is called transition matrix, and of course, you could also see what is v B will be equal to this inverse times v C. So, that you can check.So, in fact, you can just verify.

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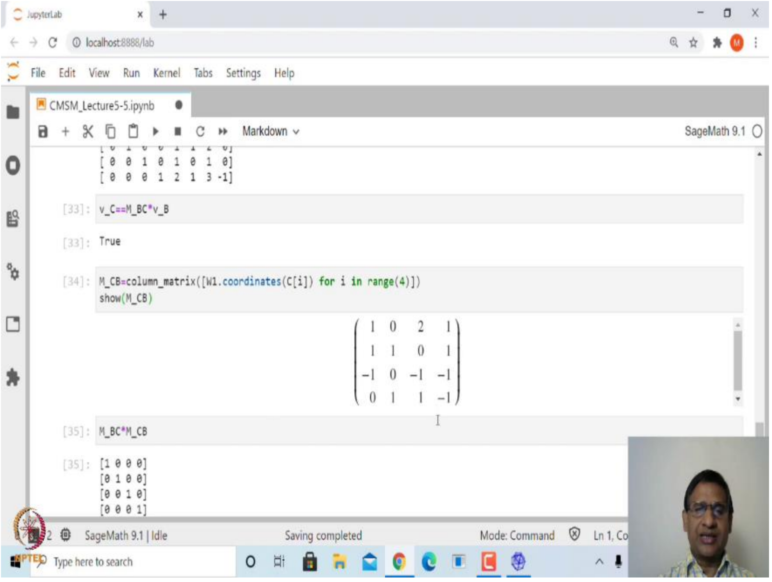

So, first you can find the coordinate of C with respect to the basis B, and let us call that matrix at M CB, and then M CB, and M C, M BC, they are inverse of each other; they are inverse of each other. So, if I take the multiple of these two, I should get identity matrix, right? So, that is how they are related.

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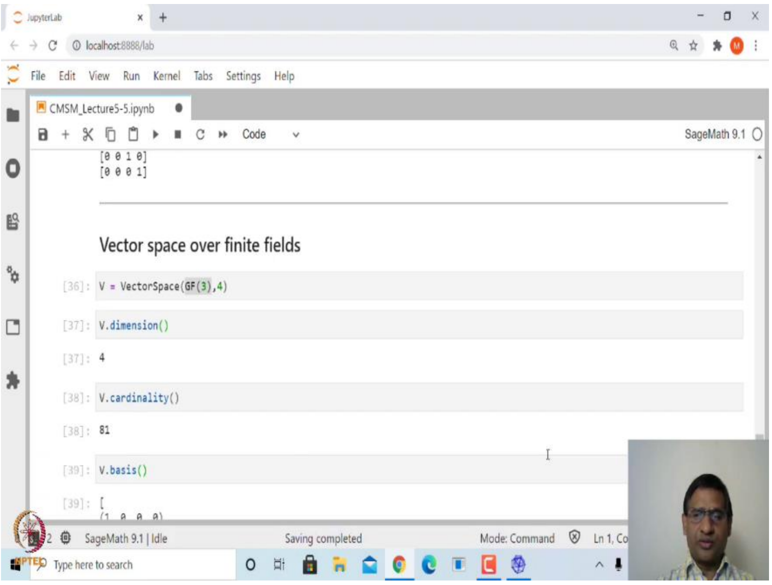

Now, let us look at one last topic. Suppose you want to define a vector space over finite field. So, let us take an example. Suppose I have a vector space defined over F p 3 of dimension 4. So, that is capital B, capital V, and so, we can ask for what is dimension of this? It will be 4, and, you can even find out in this case what is the cardinality.

So, it, any vector will have 4 components, and these 4 components will come from these elements in GF 3, which will take value 0, 1, 2 or 3. So, the cardinality should be 3 to the power 4, which is 81.

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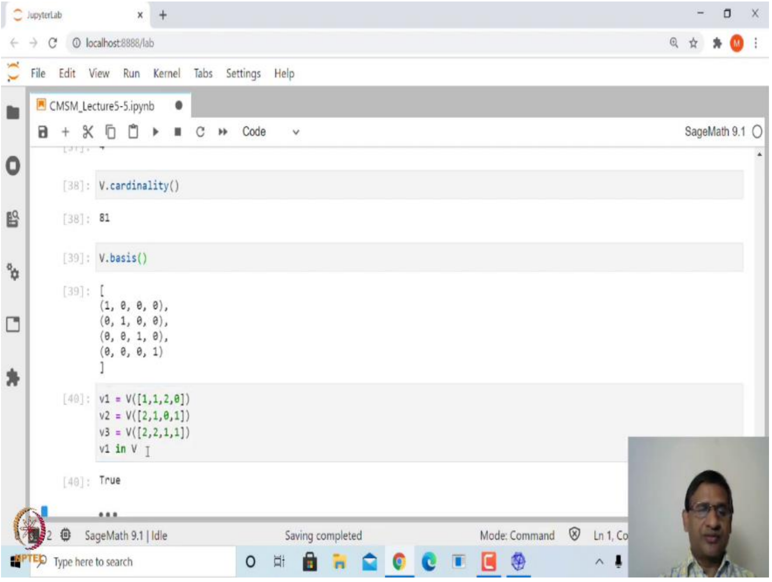

Similarly, you can find basis of this. So, basis in this case, will give you the standard basis 1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0, and 0, 0, 0, 1, right?

Now, suppose I have 3 vectors, some arbitrary 3 vectors, v1, v2, v3, v4, and we can check this is how we can define vector in capital V. Capital V, in the bracket you write, in the round bracket, you give the, the components in square bracket, and you can check whether v1 lies in capital V, whether v2 lies in capital V, and the answer should be true.

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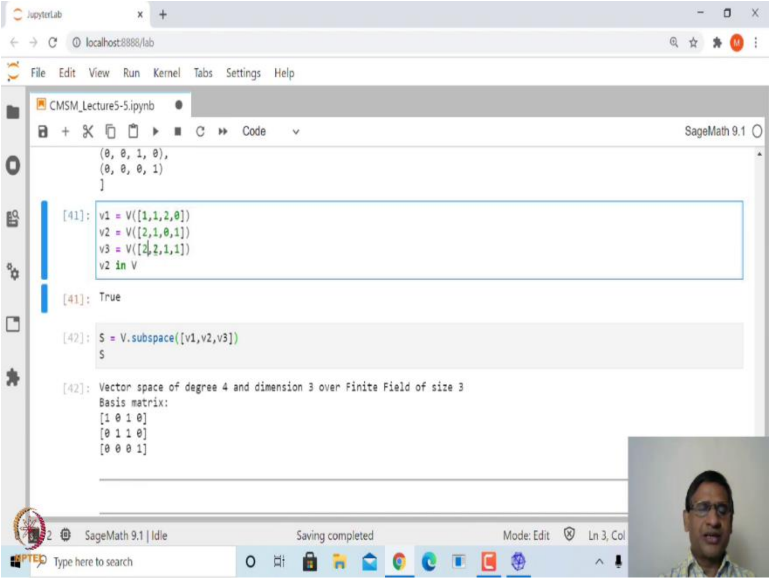

So, that is how you can define vectors in this space.

Now, for example, you can ask for subspace spanned by v1, v2, v3. So, again, exactly with the same way. This is a 3-dimensional subspace spanned by this. So, for example, in this case, let me, let me change v3. I will make this as twice. So, this is 2, this is 2, 4, would mean 1, this would be 0.

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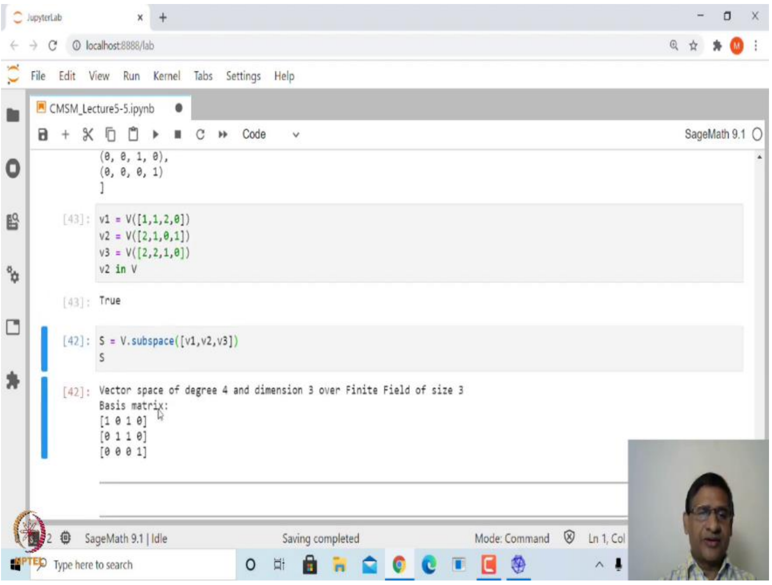

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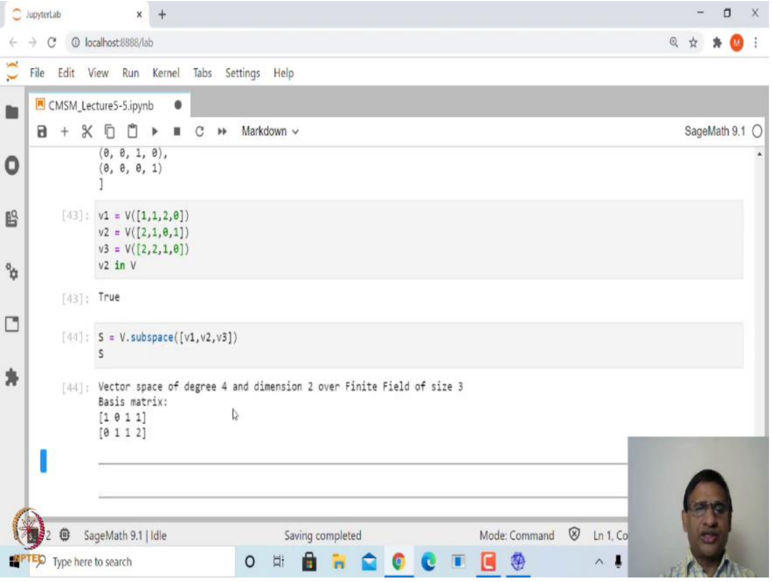

So, if I, if I say this, and then find linear combination, you can see it now, it is a subspace of dimension 2 of this.

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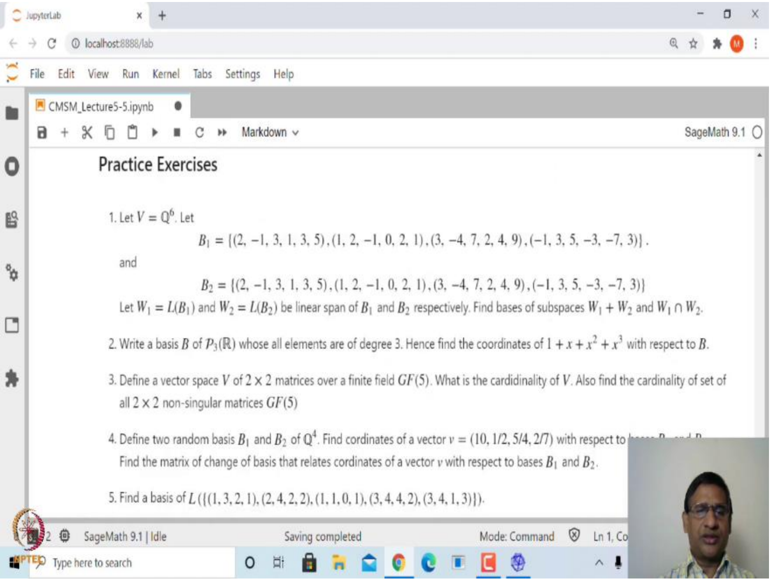

Similarly, you can work with matrix space, and try to find its basis, dimension, etcetera, right? So, let me leave you with these simple practice exercises. First exercise is actually to verify this dimension formula, which we have already done. We did it in, I think Q5. Now, I am asking you to do it in Q4, right? Now, you, you can write a basis of P 3 R whose every element is of degree 3, and then find coordinate of 1 plus x plus x square plus x cube, with respect to that basis.

The next problem is, find a vector space V of 2 cross 2 matrices over GF 5. So, again matrix space will be used. You can find the cardinality of this set, that is how many elements are there. You can also find cardinality of set of all 2 cross 2 non-singular matrices. These are called G ln 2. Next, define 2 bases B1 and B2 of Q4. Find coordinate of, let us say, this vector, with respect to each of this, and find the transition matrix, and then see how they are related.

And, the last question is, find a basis of L of this. So, that is again quite easy. So, these are very simple exercises, ok? So, next time we will look at the concepts like linear transformations, and things like that.

Thank you very much.