**Computational Mathematics with SageMath**
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**Vector Spaces**
**Lecture – 31**
**Vector Spaces in SageMath**

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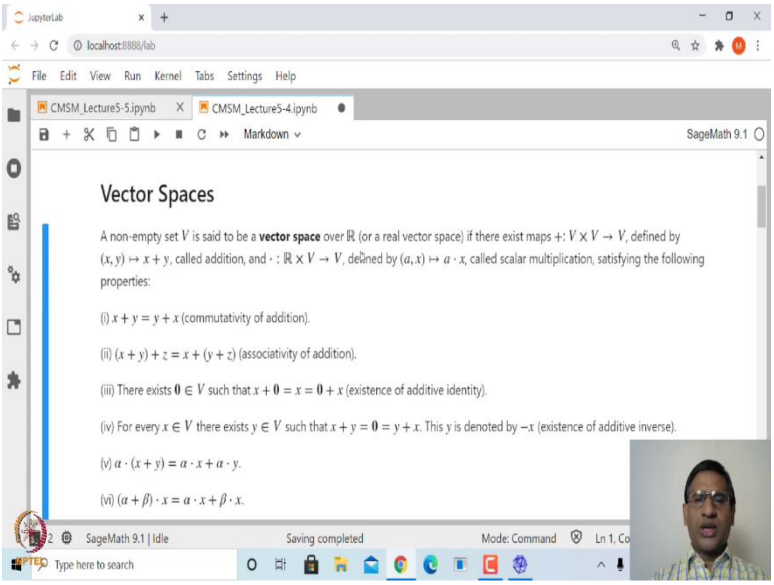


Welcome to the 31st lecture on Computational Mathematics with SageMath. In this lecture, we will look at how to define Vector Spaces in SageMath, and explore various concepts in vector spaces. So, let us get started.

So, before we look at how to define vector spaces in SageMath, let me just recall definition of vector space. So, if you have a non-empty set V, we say that this set is a vector space over R, or it could be any field, if there exists two maps: one we call as plus, from V cross V to V.

So, if you take any x comma y, the, it goes to x plus y, that is the notation, and we call this as addition or vector addition, and the other one is dot or multiplication, that is a map from

R cross V to V, and this takes any ax to a times x, or a dot x. This again is just a notation, and we call this as a scalar multiplication, and it should satisfy the following properties.

First, this addition should be commutative, this addition should also be associative, and there exists an element which we denote by 0 in V, such that when you add 0 to x, and it should give you x itself. Similarly, because of the commutativity, this is same as, 0 plus x is same as x plus 0. So, 0 plus x should be equal to x, which is also equal to x plus 0. So, this if it exists, this 0 is called additive identity. So, that means, we are saying that there exists additive identity in V.

Similarly, if you take any x in V, you can find another y in V, you can find a y in V, such that x plus y is the additive identity, and which is same as y plus x. This y generally we denote it by minus x, and we call this as negative of x, or additive inverse of x.

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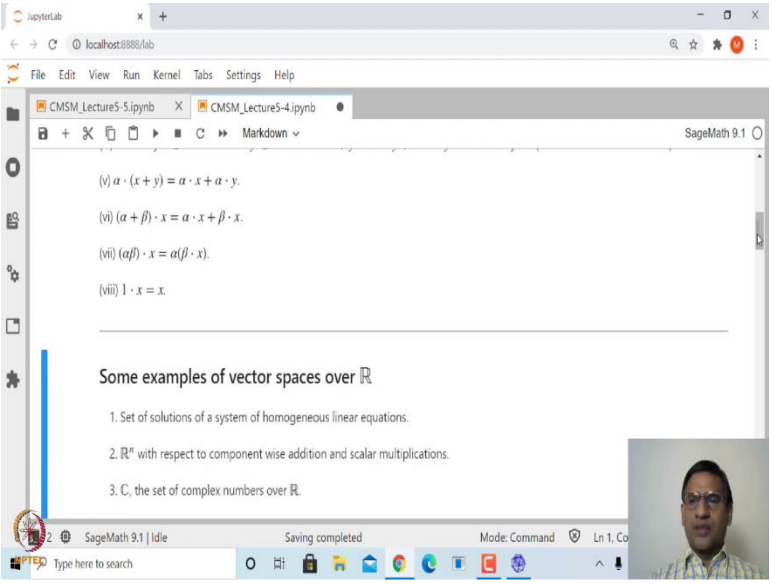


And the other properties relates the two operations, which is alpha times x plus y, should be alpha times x, plus alpha times y, and this would happen for every alpha in R, and every x, y in V.

Similarly, alpha plus beta times x should be equal to alpha times x, plus beta times x, and this should happen for every scalar alpha and beta in R, and x in V, and alpha into beta times x should be equal to alpha times beta x, and the last one is also very important, and this is, it says that 1 leaves every vector in x unchanged. So, 1 times x would be equal to x for every x in V. So, that is a definition of vector space. Now let me just also mention some examples.

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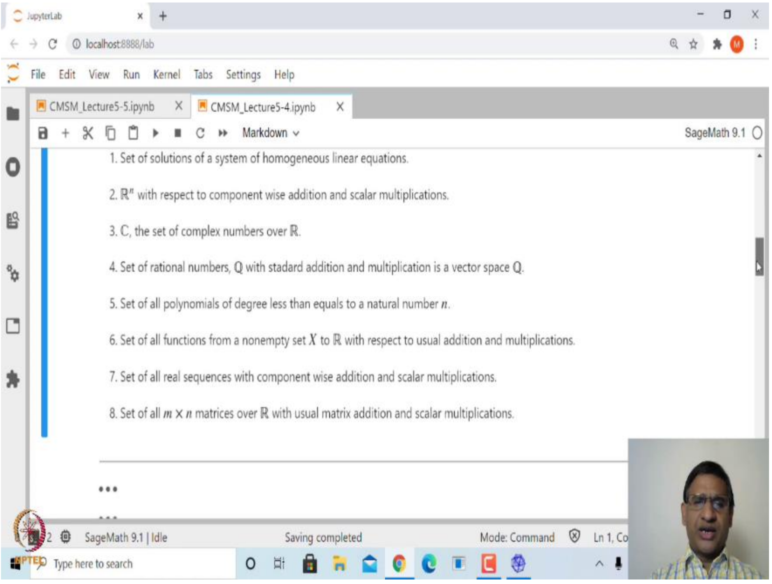


So, if you take for example, R itself over R, that is a vector space; if you look at set of all solutions of a homogeneous linear equations, that forms a vector space, because if you add any two, if you have two homogeneous solutions, let us say x and y, their addition is also a solution of the same homogeneous equation, and scalar multiple of homogeneous solution is also a solution.

In general, Rn is a vector space over R, and here the addition is taken as component wise, and multiplication again is component wise, scalar multiplication is component wise. Similarly, C, the set of complex number, can be made vector space over R with respect to standard addition and scalar multiplication. Set of rational number is also vector space over itself, over set of rational number, because set of rational number is a field.

So, you can talk about vector space Q as a vector space over Q. However, Q may not, will, will not be vector space over R with respect to the standard addition and multiplication. Set of all polynomials of degree less than equal to a natural number n is also a vector space over R, and again the set adding two polynomials in the standard way, the add, the, the coefficient wise.

Similarly, the set of all functions from a non-empty set to R is a vector space over R, and here if you look at, if you have f and g, two functions, then f plus g at any x is defined as f(x) plus g(x), alpha times f at any x is defined as alpha times f of x. So, with respect to this addition and scalar multiplication, set of all functions from x to R is a vector space over R.In fact, this is a class of vector space. This is not just, many vector spaces arise in this way. You can talk about set of real sequences with component wise addition and scalar multiplication. In fact, this is a particular case of the sixth one, because if I take x to be natural number, then set of all functions from a natural number to R is set of all sequences. So, this is also a vector space over R. Similarly, set of all m cross n matrices over R with usual matrix addition and scalar multiplication is a vector space, and there are many other vector spaces you can talk about this. So, we will, these are some standard examples, but you can find many more examples, right?

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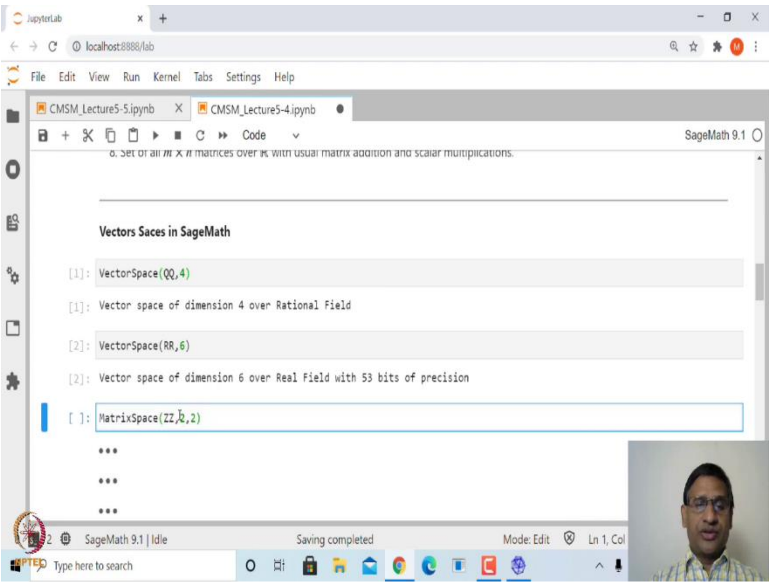

Now, let us look at how we can define vector space over R, or over some other field in SageMath.

So, the easiest way of defining a vector space is to use vector space function, and mention the, the field here. QQ means rational field, and4 means it is 4-dimension. So, it is QQ to the power 4, this is, this is Q to the power 4, this is a vector space of dimension 4 over rational field.

Similarly, instead of QQ, if you mention RR, that will become R to the power 4, instead of 4, I can mention anything. Let us say, if I mention 6, this is a vector space of dimension 6 over real field. Of course, we will come, come to formally defining what is dimension of a vector space, but this, when you ask for what is this object which is created in sage, it gives you, it is a vector space over real field with the 53 bit precision, and its dimension is 6.

Similarly, you can define matrix space. You could also, instead of RR, you can say CC, you can even mention some other domain also. So, this matrix space ZZ 2 comma 2, this is set of all 2 cross 2 matrices over Z, ok?

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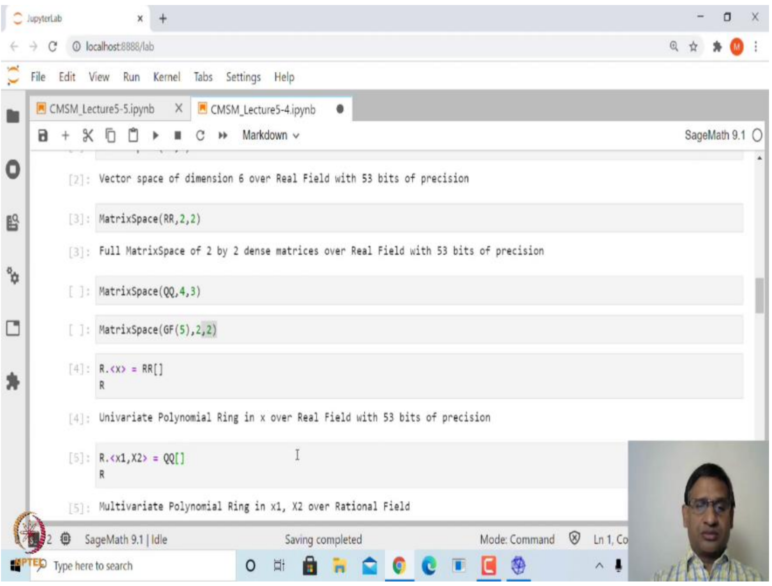

So, this you can mention over RR, this is set of all 2 cross 2 matrices over Z.

Similarly, you can you can have matrix space over QQ of 4 by 3. You can have matrix space over finite field. This is a finite field GF 5, this is FP 5. So, the elements in this are going to be 0, 1, 2, 3, 4 as equivalence class right, and this is 2 cross 2 matrices.

You can have a polynomial over x, polynomial in x over real number. So, this is a polynomial ring over R. You can also define polynomial over two variables x1, x2 over rational fields. So, these are some definitions of rings and vector spaces, and you can work with these vector spaces.

So, mainly we will be working with, let us say vector space Rn over Q, or vector space Rn over vector space Rn, vector space Qn, and things like that. We will also look at some examples on polynomial. Set of all polynomials of a particular degree, of degree less than equal to n, and things like that, ok?

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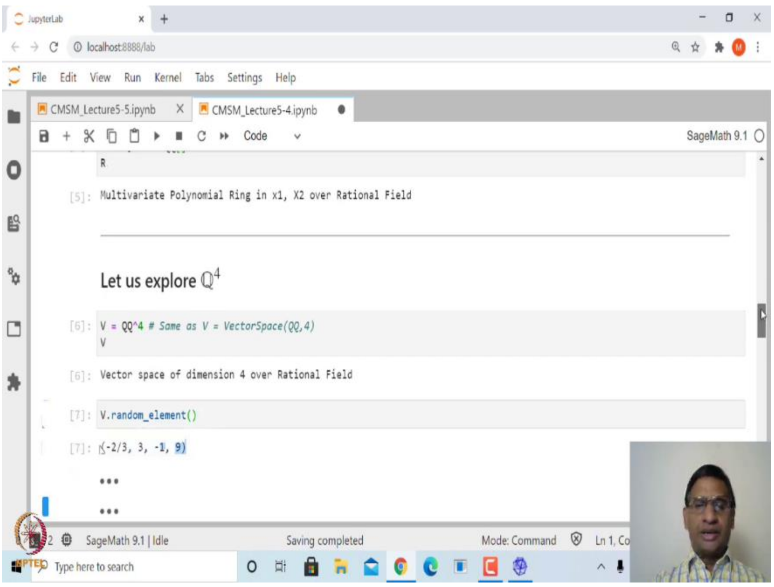


Now let us look at how we can explore these vector spaces. So, let us work with Q to the power 4, Q4 over Q as a vector space. So, how do we define? One can define V as vector space QQ 4, or you can simply write V is equal to QQ to the power 4 both are the same.

So, if I ask for what is V, it will tell me it is a vector space of dimension 4 over rational field, right?

So, and you can find a random element in, in V, Q4, by V dot random element. So, if I execute this, if I run this, this is a random element ok, right?

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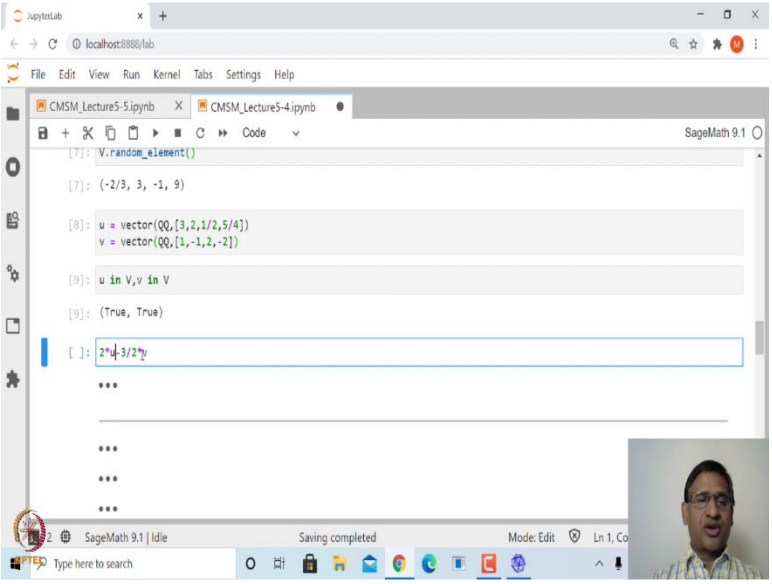


So, similarly, you can, you can define a vector inside this vector space. So, by the way, any element in a vector space is called a vector, and the element in the, the field on which V is a vector space is called scalars.

So, suppose if you define a vector. We will define exactly in the same way as we saw earlier, vector over QQ. So, this here, you need to mention what is the, the field from which this, the scalars are taken, and then mention so many components in square bracket v and u. So, we have defined two vectors over QQ, and we can check whether u and v, which we have defined, whether they lie in V, capital V.

So, you can simply say u in capital V. If the answer is true, it will give you a true answer, and similarly v in capital V. So, if I say u in V, and v in capital V, this both gives you answer true. That means, u and v both are in Q. You can take any scalar multiplication of u and v, or take linear combination scalar linear combination of u and v.

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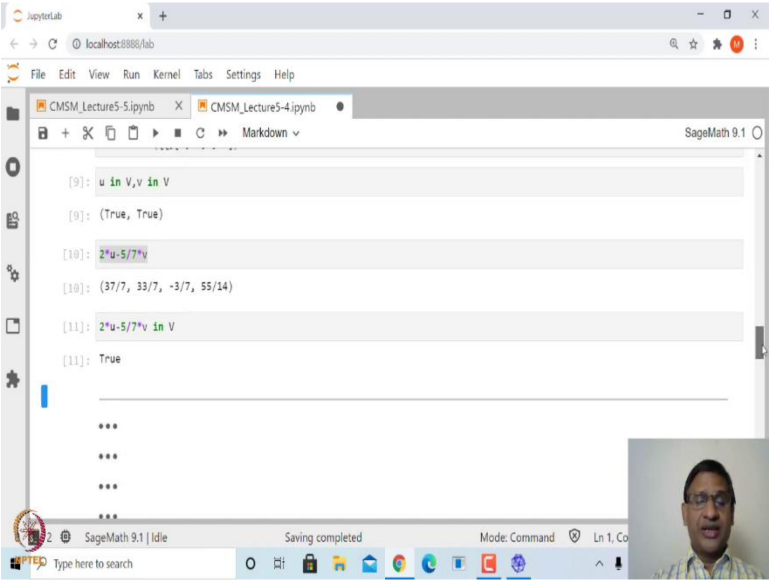


So, for example, if I say u, 2 u minus 5 by 7 v, it will give me this particular vector. So, any scalar linear combination can be obtained in this way. You can check whether, when we have done this scalar linear combination is this in, in, in V. So, we have done 5 by 7, and we can check whether this lies in V. The answer is true. So, any scalar linear combination of vectors also lies in vector, because that is a, that is a defining property of this addition, right?

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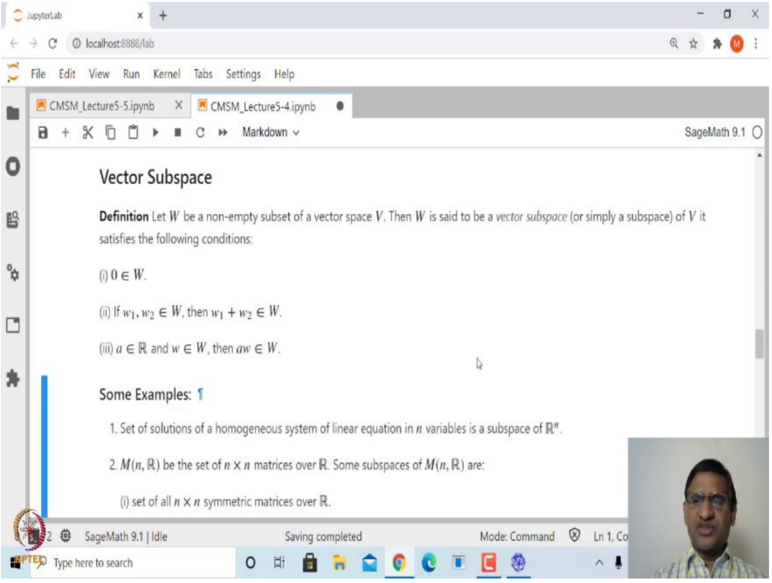


Next, let us look at what you mean by vector subspaces of a vector space. So, if you have a, some subset let us say W, let us begin with non-empty subset W. We say that W is a vector subspace, subspace of capital V, the vector space V again over the same field.

So, unless I mention, we will assume that V is a vector space over R, and we say that W is a subspace, vector subspace, if actually W itself is a vector space over the same field R, right? Or we, this, either we say that vector subspace or just simply subspace. So, what are the properties it should have? So, either you say that W itself is a vector space over RR, or this satisfies the following properties.

One, the, the additive identity of V, that should also that should be in W, and whenever you take any two elements in W, then their sum should be in W, and if you take any element in W, and then take any scalar in R, in, a in R, then a times W should be in, in W. So, here, though in the definition we denoted by a dot W, we will keep writing aW instead of a dot W. That is again a just a notation for our convenience, right?

So, if you have a non-empty subset which satisfy these three properties, then we say that it is a vector subspace of V, and you can check that once you are given these three properties for W, other properties are basically properties that addition and scalar multiplication satisfy, and since those properties are satisfied in W, in exactly in capital V, it will also be satisfied for elements in W.

So, W is a vector space over R. So, what it says is W is just vector subspace. It simply means that W should be non-empty first of all, it should contain identity element, and it should be closed under addition, and closed under scalar multiplication. That is what it says, right?

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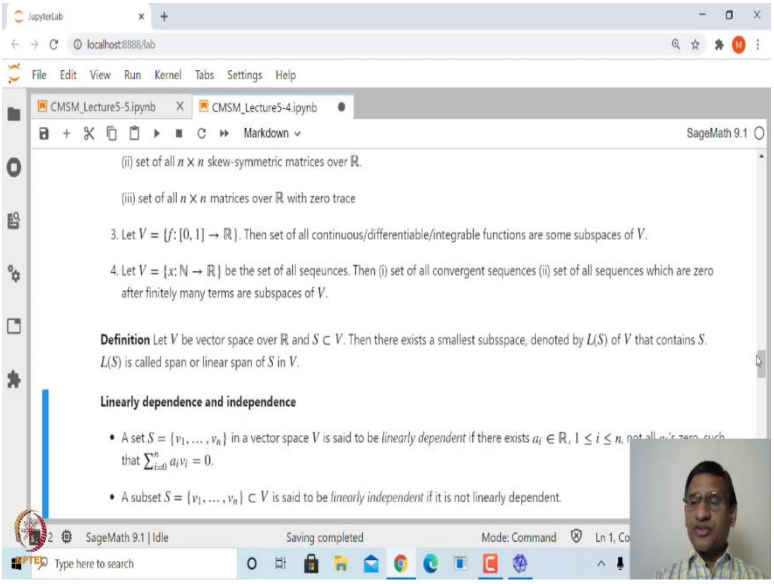



Let us look at some examples. So, for example, if you look at set of all solutions of homogeneous system of linear equations, this is actually a vector subspace of Rn. In fact, every vector subspace of Rn arises as a solution of some homogeneous system of linear equations.

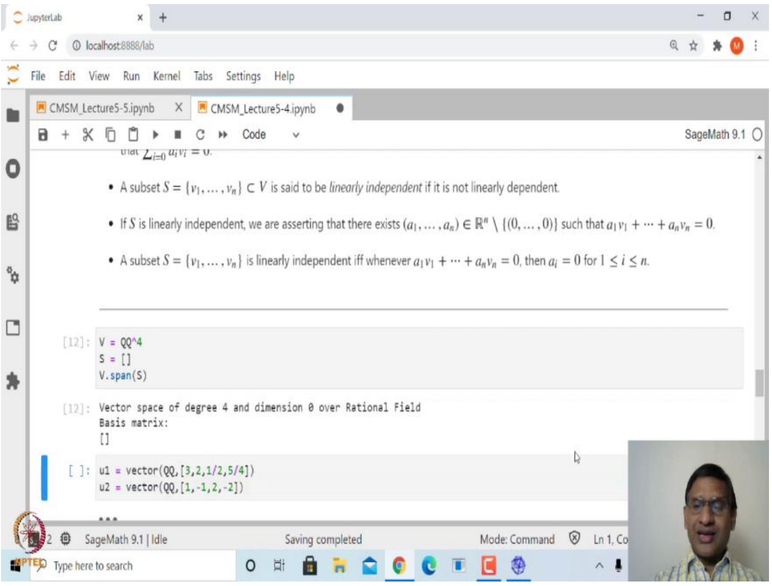
If I look at, for example, set of all n cross n matrices M, and R, then you can find many subspaces. For example, one of them is set of all symmetric matrices over Rn cross, and symmetric matrices. Similarly, n cross n skew symmetric matrices over R, or set of all n cross n matrices with trace is 0, trace is 0, is same as saying sum of the diagonal entries is 0, and you can find many more.

Similarly, if you look at set of all functions from, let us say, closed interval [0, 1] to R, we, this is a vector space over R, and for example, if you look at, restrict to only two continuous function that says sub-vector, subspace of V, similarly differentiable function, set of all differentiable functions from [0, 1] to R is a vector subspace. Set of all integrable functions is a vector space.

In fact, set of functions, which, let us say, vanishes at some point, let us say at 0, will also be vector subspace of V. Similarly, if you take set of all sequences, this is a vector space, and you can take a set of convergent sequences, that is a subspace of this. You can take set of all sequences converging to, let us say 0, that is a vector subspace. Set, set of all sequences which are 0 after finitely many terms, that is also vector sub space.

So, there are few examples. Of course, you can find many more, ok? Now let us also define what is meaning of linear span of a non-empty of a subset of V. So, suppose if S is a subset of V, then you can find a sub subspace of capital V which contains S.

In fact, V itself will be a subspace which contains V, S, but that is not interesting. What you can find, is a smallest subspace of V that contains S, and generally we denote that by LS linear span of S, and we call this as linear span of S, LS we call as linear span of S. (Refer Slide Time: 16:53)



So, similarly, I am sure you are aware of the concepts of linearly independent, dependent. So, if you have a set of vectors, let us say v1, v2,…, vn as set S, we say that it is linearly dependent, if you can find scalars a1, a2,…, an such that a scalar combination of vi, let us say a1*v1 plus a2*v2 plus dot dot an*vn, this is equal to 0, and these, these ai's are not all 0.

Of course, if I take everything 0, this scalar linear combination will be 0 itself for any vector, but that is not what we are saying. We are saying that there exists scalars a1 to an, not all 0, at least one of them is nonzero, such that this linear combination is, is 0, right?

And this, it actually amounts to saying that one of the vector in, in S can be written as scalar linear combination of the remaining vector. That is what it means, and in case a set is not linearly dependent then we say that it is linearly independent, and this.

So, if you look at, for example, linearly dependent, what it says that you can find point a1, a2,…, an in Rn minus the origin, the R minus origin means this, this at least one of the component is nonzero such that this sum is 0. So, if it is not true; that means what? You cannot find any point a1, a2,…, an in Rn minus origin with this property.

So, in, in particular, if you negate this, this statement, if you negate this statement, what it means is that if I have any scalar linear combination a1 to a, a1 v1 plus dot dot an vn is equal to 0, and that should imply, if it is linearly independent, it should imply that all ai's are 0. So, generally this we take as a working definition; however, this is the more natural definition. So, we will work with this definition, right?

Now once we have defined these concepts, let us look at how we can work with these concepts in SageMath. So, let us start with again V to be QQ to the power 4, you can start with RR to the power 4, that is R4, and suppose I have S to be non-empty set. This is empty set, then we can define span of S over V, and this says that it is a vector space of degree 4 of dimension 0.

So, this is actually, this actually contains a singleton 0 this, though it looks empty, but it actually it is a singleton 0 set. If I take, let us say, two vectors u1, and u2 in, in capital V, and let me define S to be span of u1, it will be span of u1.

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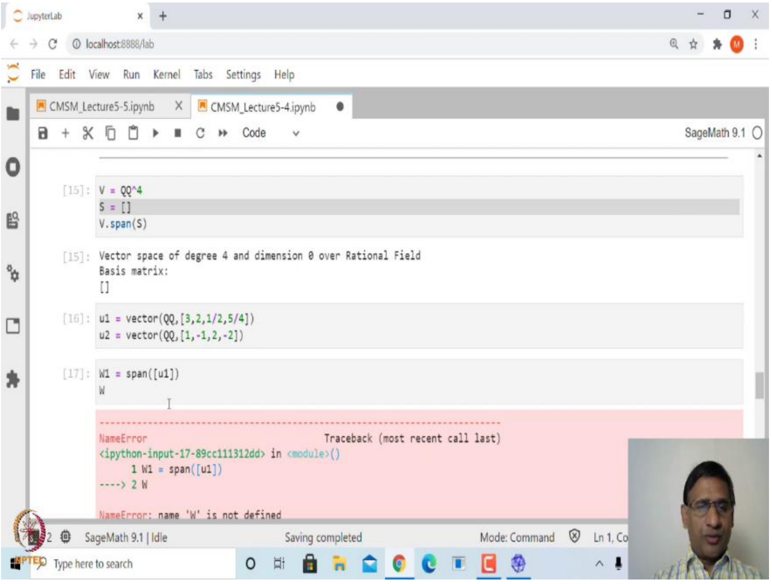


So, this, you can check that this is nothing but all, set of all scalar multiple of u1. W will be set of all scalar multiple of u1.

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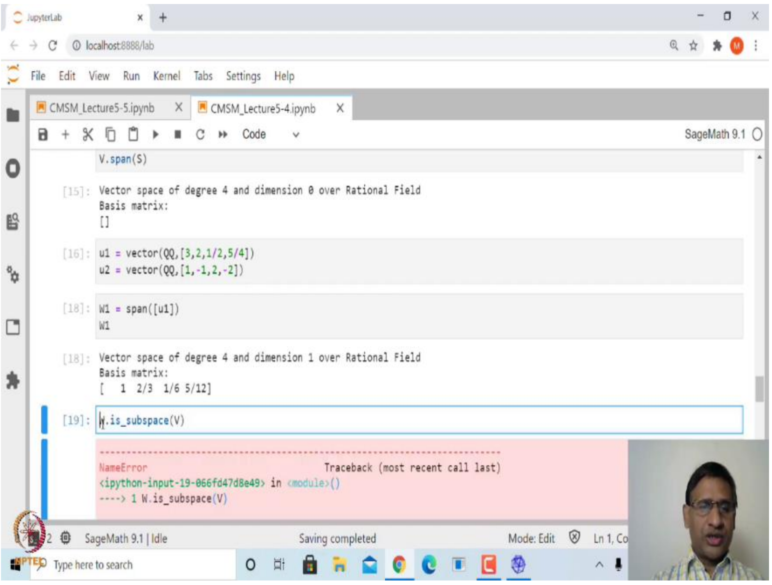

So, I think we have not, just a second, W is not defined. This, this is W1, not W. So, this is, this is a vector space of degree 4, and dimension 1 over rational field, ok? Spanned by this, these are all the scalar multiple of these, right? Similarly, can I, can you can check whether W is a subspace of V?

So, how do I check? W dot is underscore subspace of capital V, and the answer. Again, that is not W, it is W1. This is the answer is true.

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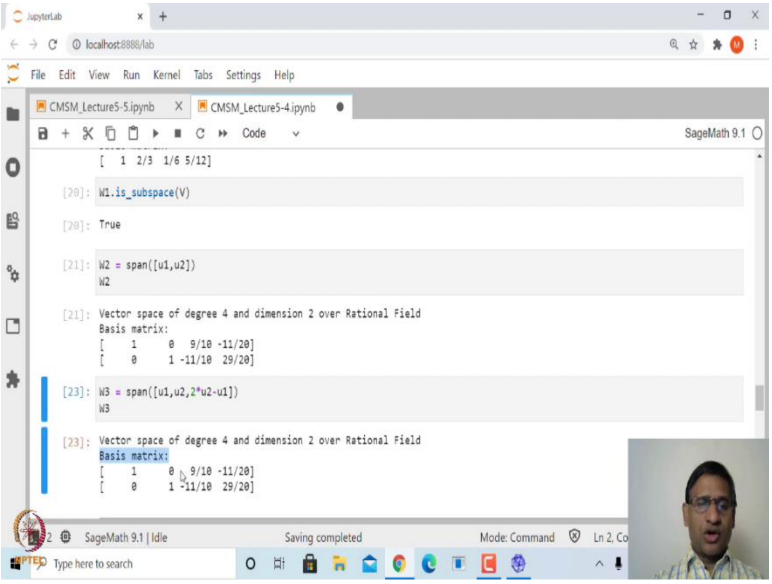


So, W1, which is a linear span of the singleton set u1, that is a subspace of V. Similarly, you can define span of W, u1, and u2. So, that is the set u1 and u2, and I am calling this as W2. So, W2 is a vector space of degree four, dimension 2, over rational field, right, and we can again check whether this is a subspace. Similarly, I can say, let us say linear span of span of u1, u2, and2 times u1 minus u2.

So, what will it give? This, let, let me call this as W3. So, that W3, if you can see here W3 and W2, both actually are same. So, it also reports what is basis matrix, or basis of this vector space subspace which you have generated. We will talk about basis later, but this is how the sage reports the subspace spanned by a set.

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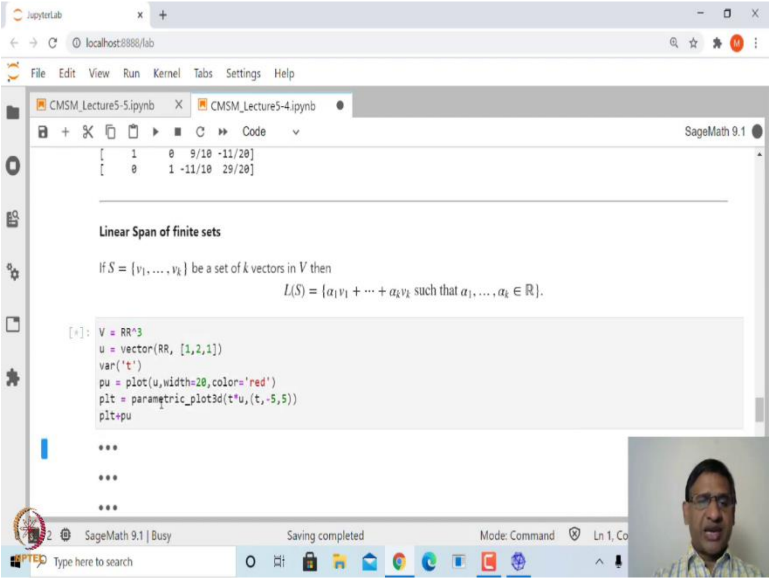


You can for example, if you look at what is linear span of a finite set. So, though this LS the sage has given by default, we do not know how this has been computed, but if you look at, if S happens to be finite set, linear span of S, one can show that this is nothing but set of all vectors of the form alpha 1 * v1 plus dot dot alpha k * vk where v1, v2,…, vk is, they are the elements of S such that alpha i is R in k that is the how you define linear span.

So, that is why this, if you have a singleton v1, then it will be just alpha 1 v1 such that alpha 1 lies in R, that is a set of all linear combination of, or scalar multiple of v1. If I have two, then a set of all scalar multiple, scalar linear combination of v1 and v2, alpha 1 v1 plus alpha 2 v2, right? So, you can even try to plot this. Try to visualize these linear linear spans.

So, for example, let us look at, if I have a vector space capital V, which is R3, and take a vector u over RR, 1, 2, minus 1 that is a vector, that is the element in V you can check, and suppose we look at, try to plot this vector, and try to plot all the scalar linear combination of u, that is scalar multiple of u, and then what you expect will be a straight line passing through this.

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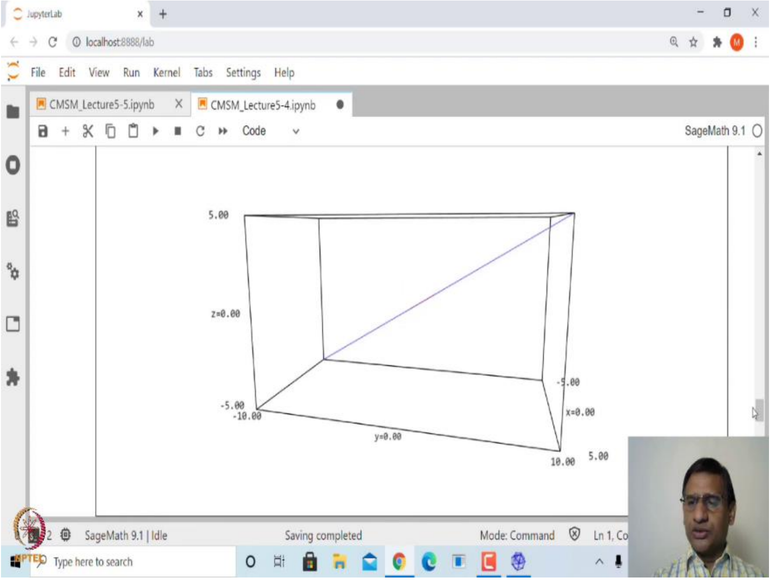


Let me rotate this, and see here. So, that is the vector space. Let me make it slightly bigger, so that. So, you can see here, that is the vector u, and this straight line passing through this is the subspace containing, containing u. So, it is linear span of u.

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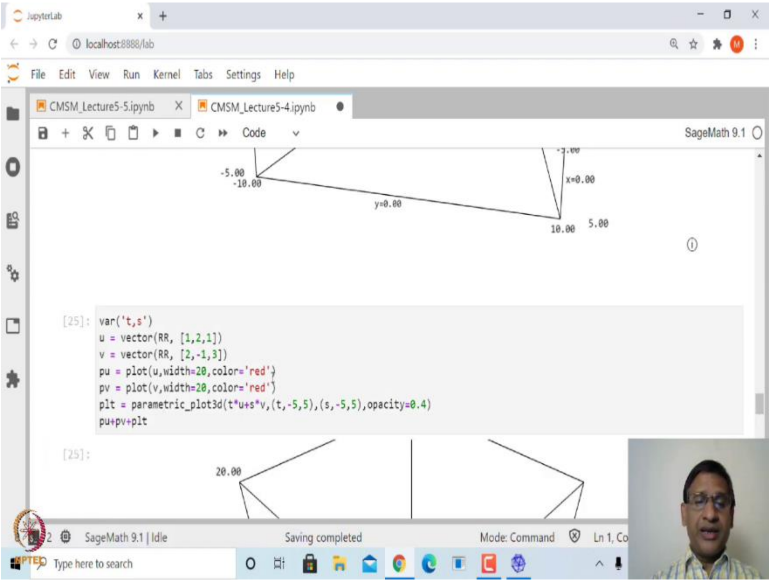

Similarly, if you have two vectors, let us say one is u, which is 1 comma 2 comma 1, other one is v which is 2 comma minus 1 comma 3.

And if you try plot these two vectors u and v, and then plot set of all scalar linear combination of u and v, that is t times u plus s times v, where t varies between, let us say t can take all real number, s can also take all real number, but when it comes to plotting, we have to restrict t and s on some finite domain.

So, if I plot all these things etcetera, you will see that will be actually a plane passing through u and v.

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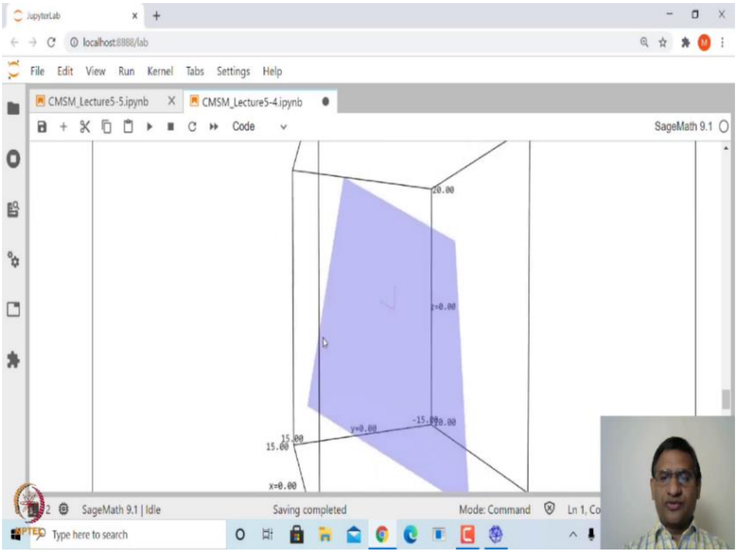


So, let us just see, let me rotate this, you can see here there are two, two vectors u and v, somewhat not visible that nicely, but this is u and v, and this is the subspace spanned by these. So, subspace spanned by two vectors in, in, in R3 is going to be plane of course, in case second vector v is scalar multiple of u, then it will be just a straight line.So, it could be a plane, it can be also a straight line, right?

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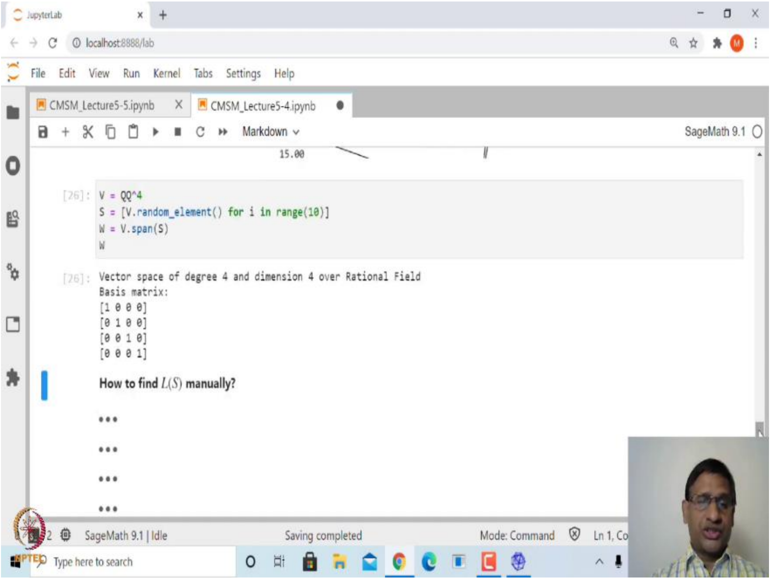


Similarly, you can generate linear span of any set number of vectors. So, suppose if I take V is equal to QQ to the power 4, and let us take some 10 random elements in V, and consider that set as S, and then we can define what is linear span of this S, that will give you a vector space of degree 4 of dimension 4.

So, this, in fact, is entire space V, right? Now, let us look at, as I said, this linear span, when I say V dot span S, the SageMath knows how, what is to be done, but suppose we have to generate this on our own. Let us say, we, you are given this question in the, in the exam. How do you find it?

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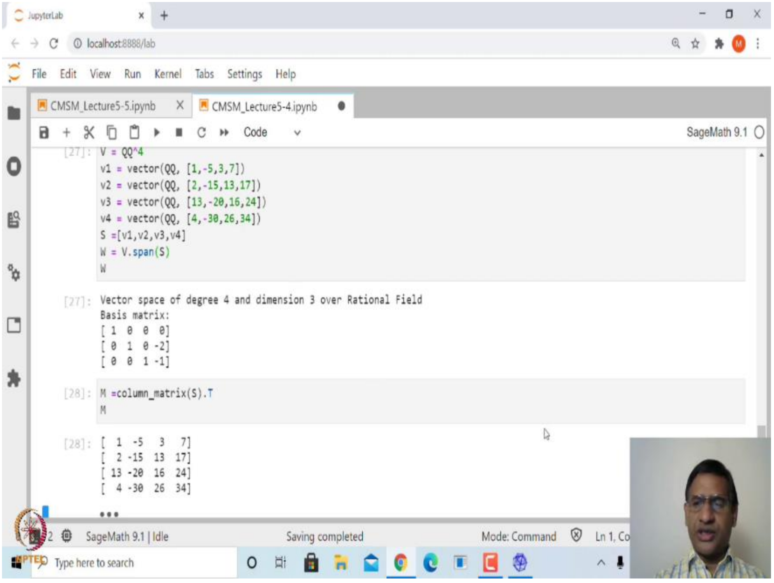


How do you find it? So, let us start with an example. Suppose I have four vectors v1, v2, v3, v4 in, let us say, QQ to the power 4.

So, let me define capital V to be QQ to the power 4, and they have four vectors, and then S is the set s1, s2, s3 s, sorry v1, v2, v3, v4, and then we know V dot span gives me the linear span, that is using inbuilt function. So, it says that it is a vector subspace of dimension 3, and this is a basis matrix. Now suppose we need to find on our own this, this one. How do we do that?

So, the way one does is, you, you look at these vectors v1, v2, v3, v4, and put it in a row matrix, and then apply RREF, and all these nonzero rows, you will constitute basis for this LS. That is what you, you need to do. So, how do I generate a row, row matrix? So, there is no row matrix, but you can generate column matrix.

So, if I say column underscore matrix of S, this M will become a column matrix, and then take transpose of this. So, this M is now a matrix whose rows are v1, v2, v3, v4. You can just look at, this is the first row, this is second row, third row, and fourth row. You can compare with v1, v2, v3, and v4, right? So, this is a matrix.

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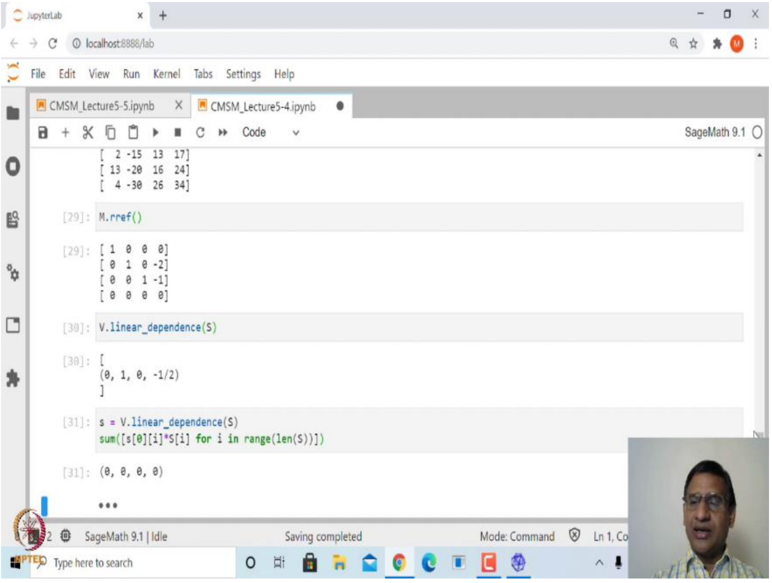


Now, apply RREF on this matrix. So, when you apply RREF on this matrix, and take only the non-zero rows, and that is what you can see here. There are only three nonzero rows, and these three nonzero rows are exactly same as what you got from the inbuilt function. So, RREF is very useful you can find row, you can define linear span of any set of vectors using RREF.

In fact, this, this is also a, this, this M which we have given, that is a matrix, and what we have obtained from RREF, that is known as row space of M, that is a vector subspace. It is spanned by row vectors, and that is called row space of this matrix M, right? Similarly, you can define column space.

So, you can take matrices, matrix, the column as vectors, and then find out what is column space, that is the inbuilt function to find row space, and column space, ok? Now you can check whether set of vectors are linearly independent or not. We have already defined what is meaning of linearly independent. So, how do we do that? You can simply say V dot linear underscore dependence.

So, when I say linear dependence, in this case it gives me component 0,1,0,minus half. This is same as saying 0 times v1, plus 1 times v2, plus 0 times v3, minus half times v4 is equal to 0. So, this is not linearly dependent. These are linearly, this is not linearly independent, but it is linearly dependent. So, in case you get empty square bracket, then it will be linearly independent.

So, for example, as I said this, this whatever output you have got, it means that 0 times v1, plus 1 times v2, plus 0 times v3, plus minus half times v4 is equal to 0. So, let us just check that. If I put this output in S, if I put this output in S, and look at S[0], S[0], see it is giving the output as a list of tuples. So, sometimes it may give you two tuples, actually.

So, this is a list of tuple, and so, the zeroth element will be this tuple, and then s0 of i times S of I, that S of i is ith element in the set S, and then for i in the range of length of S, here we have said S is equal to, S has length 4. So, we can say range 4 also, and when we sum this, and then after that you get 0, 0, 0.

So, we have shown that this is a set of scalars a1, a2, a3, a4, not all 0, such that a1 times v1, plus a2 times v2, plus a3 times v3 is 0, and hence this set S is linearly dependent.

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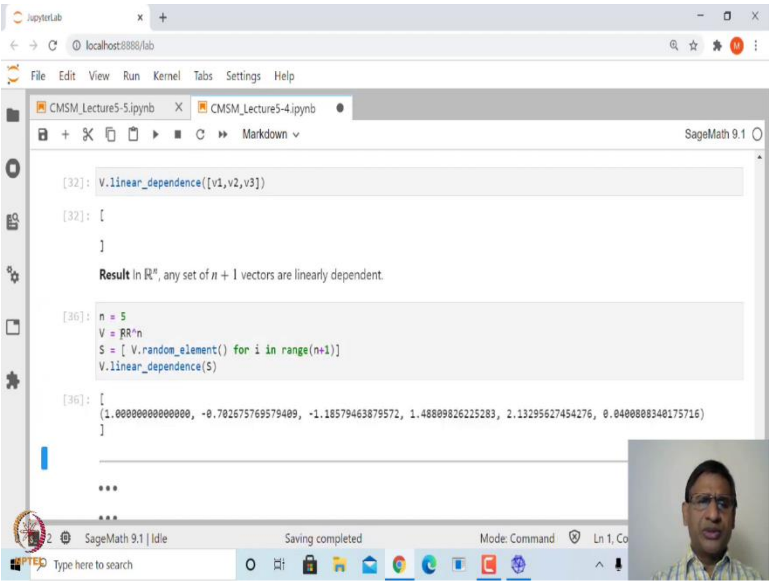


If I, if I take only v1, v2, v3, and ask whether this set is linearly dependent, then you will get an empty set, because v1, v2, v3 are linearly independent.

So, that is what I said, when, when you want to check, when, when some set is linearly independent or dependent, you just apply this function linear underscore dependence, and then in case you get empty set, then it is linearly independent, otherwise it is linearly dependent, right?

And I am sure you know that any n plus 1 vector in Rn are linearly dependent. This is just the, solving system of linear equations. So, if you have n plus 1 linear equations in n variables, you can have n solutions, which will have a non-zero, that is what it means, ok?

So, if I very try to verify this, suppose n is equal to 5, V is equal to R to the power 5, and take any random 5 elements in this, and check whether this is linearly independent. The answer, most often, this will be true. This is linearly independent. But it is possible that sometimes you get linearly dependent, that is possible because we are taking random, and it also means that randomly if you select any five vectors it is very high chance that it will be linearly dependent, right?

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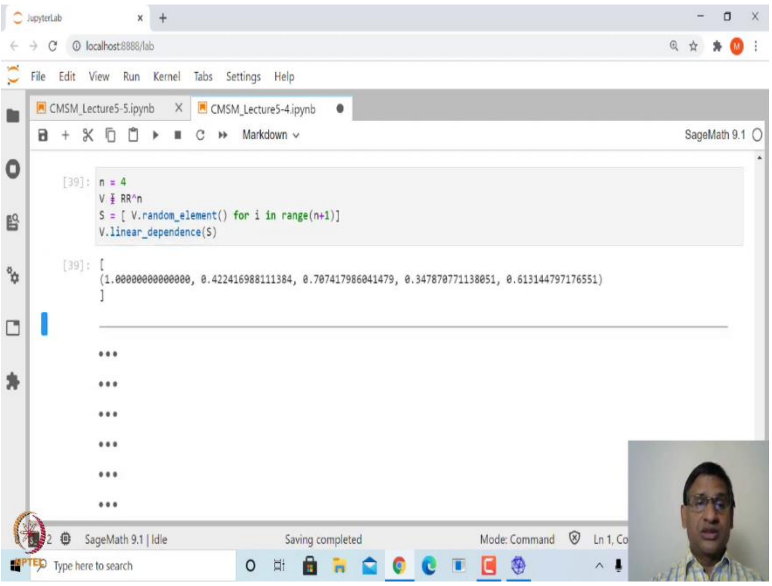


Of course, this will be linearly independent if I take four vectors, and check, then sometimes it may come linearly dependent. But five vectors will always be linearly dependent that is what it means. You can keep on trying this, but five vectors in R 4 will always be linearly dependent right, ok?

Now, let us define what is meaning of basis and dimension of a vector space.

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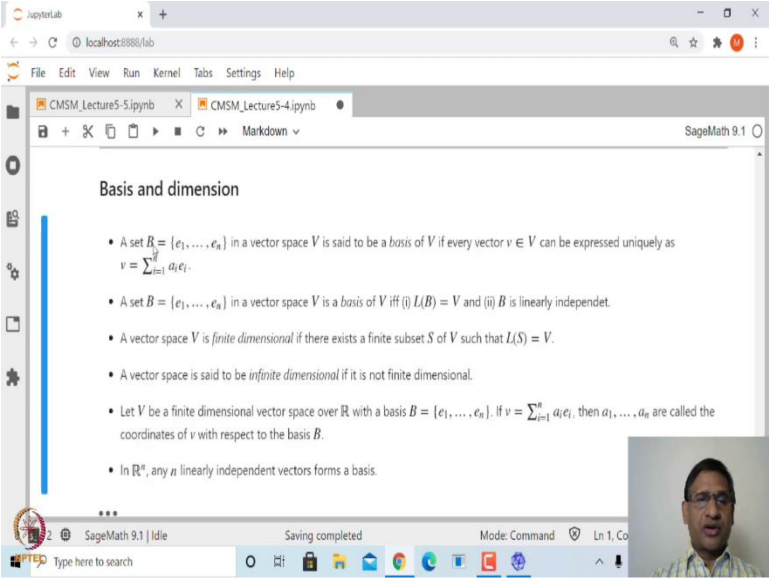


So, if you have a set of vectors B, e1, e2,…, en, we say that it is a basis of V, if every vector v in capital V can be uniquely expressed as linear combination of e1, e2,…, en. This is unique. This is same as saying, now of course, v can be expressed as linear combination, and this ai should be unique. That is same as saying if I have another linear combination, let us say summation bi*ei equal to 0, then ai must be equal to bi. And in case you have a set B which is e1, e2,…, en we say that it is a basis, if linear span of B is whole capital V, and it is linearly independent. So, this definition, using this definition, you can show that a basis is a set which is, which spans V, and it is a linearly independent set, and vice versa.

So, this is a simple result, and in case you have a vector space such that it can be spanned by finite set S, then we say that V is a finite dimensional vector space, and one can show that the any, if you take any two bases on a finite dimensional vector space, it, the number of elements is same, and that number is called the dimension.

A vector space is said to be infinite dimensional if it does not have, if it is not a finite dimensional, and for example, set of all functions from, let us say, any non-empty set x to R is a infinite dimensional vector space, and if you have a finite dimensional vector space over R, and if, if it is, B is a basis, B is, let us see e1, e2,…, en, and then we know that v, v can be written as scalar linear combination of a e1 to en, and this is unique.

So, these coefficients e1, e2,…, en are called coordinates of v with respect to this basis, and in Rn, the, any n linearly independent vectors forms a basis, right?

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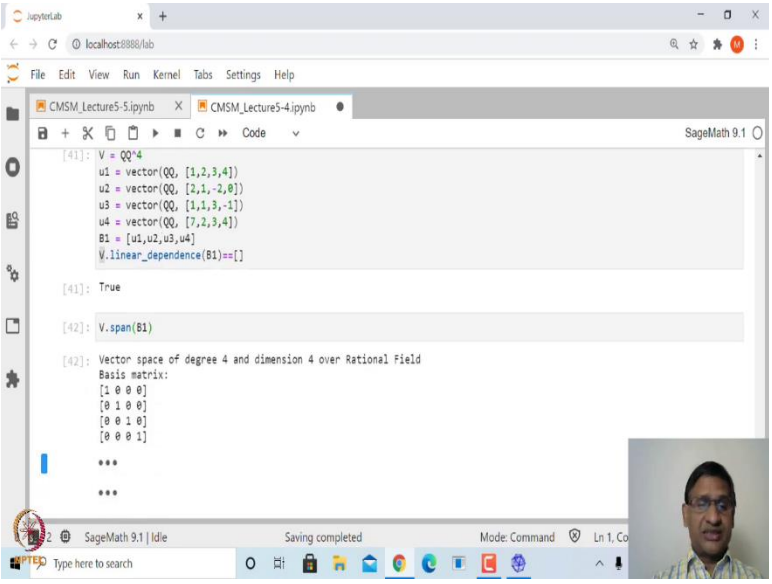


So, let us look at some example. Suppose I have a vector, capital V is equal to QQ to power 4, and take any four vectors here, u1, u2, u3, u4, and let us say B1 is u1 to u4 the set, and we, let us check that this is a linearly independent, linearly independent set, the answer is true.So, this, this B1 is linearly independent, and therefore, it will form a basis. Similarly, you can find what is the span of this B1? You can see that this will span entire space. This is space of degree 4 of dimension 4 here, degree 4, it simply means that it has four components.

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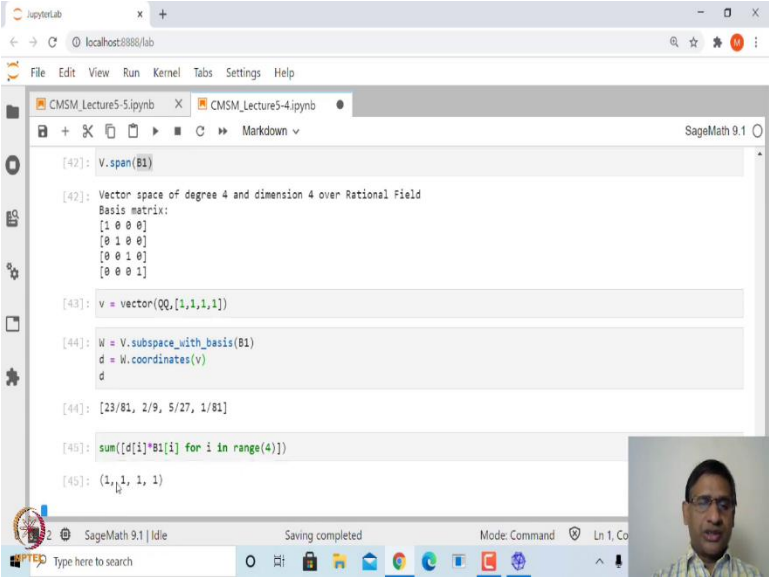
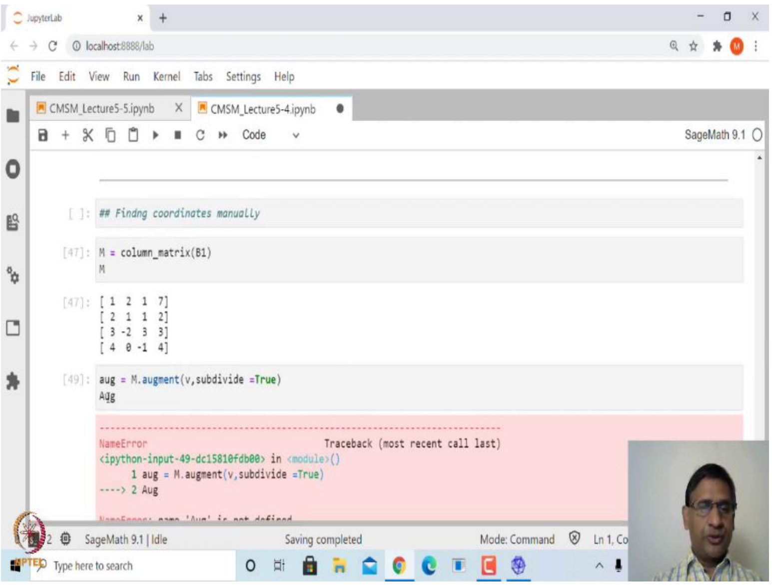



Similarly, if I take any vector v then let us say v is 1, 1, 1, 1, and we can find coordinates of v with respect to this basis capital B. So, how do I do that? First, we need to define a W subspace with B as, B1 as basis. So, this is how you define B, and then say W dot coordinates, and in the bracket you write v. That will give you coordinates of this vector v with respect to this basis B1. That is how you find the coordinates.

So, these are the coordinates. You can, you can check that, for example, this times u1, plus this times u2, plus this time u3, plus this times u4, this should be equal to v. Let us just check that sum of di into B1i should be equal to v, that is what you get. So, that is the the coordinate. (Refer Slide Time: 36:30)



But how do you find the coordinate manually? So, v is a vector, and you want to find coordinate of v with respect to B; that means, suppose the coordinates are d1, d2, d4. d1, d2, d3, d4; that means, d1 times v1, plus d2, d1 times u1, plus d2 times u2, plus d3 times u3, is, should be equal to v, right? And now v1, v2, v3 and v4 are vectors. So, vectors, generally we think of as column matrix.

So, what you have is d1 times the first column, that is u, u1 plus d2 times second column u2, plus d3 times u3, plus d4 time u4 is equal to v, and we have seen that if you have Ax equal to b this, and x is x1, x2,…, xn then Ax equal to b is equivalent to first column times x1, plus second column times x2, plus dot dot n column times xn that is equal to v. So, that is what exactly you, you have here.

You have d1 times u1, plus d2 times u2, plus dn times en. So, this d4 times e4 is equal to v, is same as saying, I can take a column matrix whose first column is v1 second column is, sorry u1, second column is u2, third column is u3, fourth column as u4, and multiply by, let us say d1 , d2,…, d4 column matrix. That should give me v. That is same as saying, you are, you want, you want to find d1, d2,…, d4 is same as saying you solve this system of linear equations.So, the way to find is, we can define a column matrix of B1. So, this what is M? This is, first column will be the vector u1, second column is u2, third column is u3, fourth column is u4, and then how do I solve? We want to solve Mx is equal to v. So, we just append v in capital M. Augment v in capital M, that is the augmented matrix.

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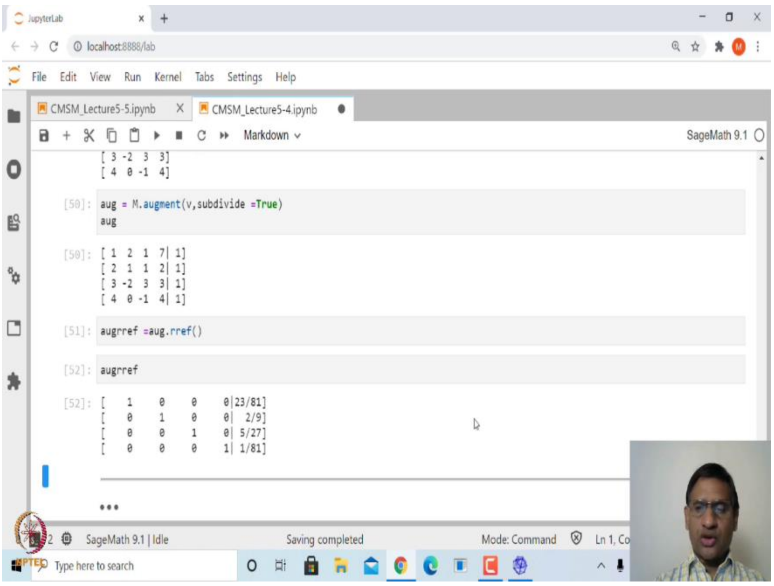

And then let me just see, Aug is not defined, as a small, aug. So, this is a matrix, and then apply RREF to this. When I, when you apply RREF to this, and then see what? You get the last column will be the, the solution of Mx equal to v, and this will give you the coordinate. If you compare this coordinate with what we got, the coordinates d, you can see that these two are the same.

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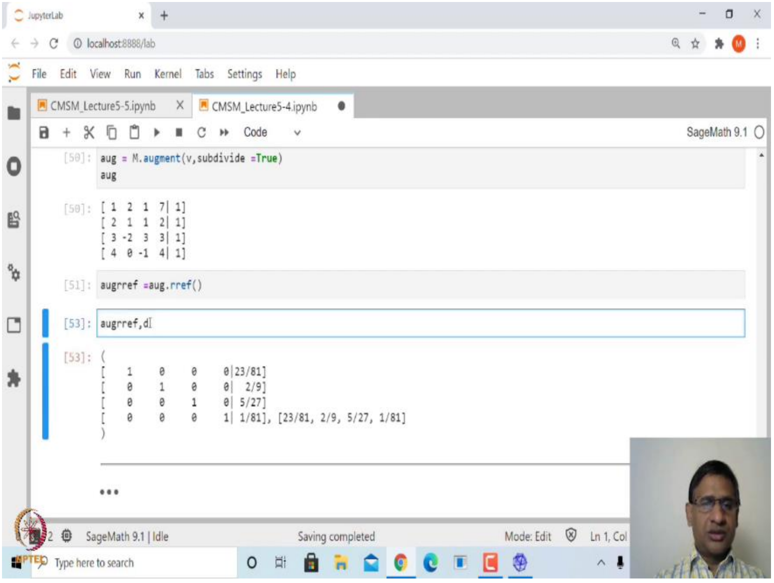


So, let me show you both together. So, this, and another one is d, right? So, yeah, I think this, let me print it separately.

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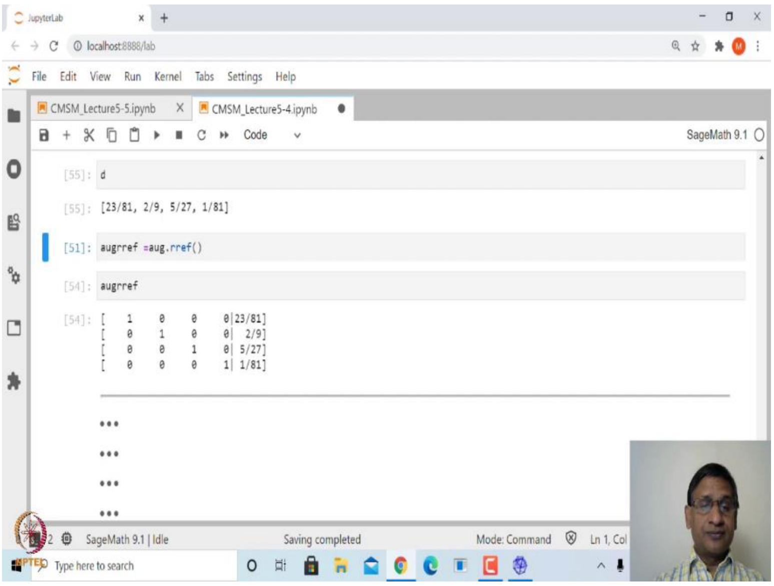

So, if I say d, here, yeah. So, this column is same as this vector. So, that is how you can find coordinates manually, or using inbuilt function coordinates, right?

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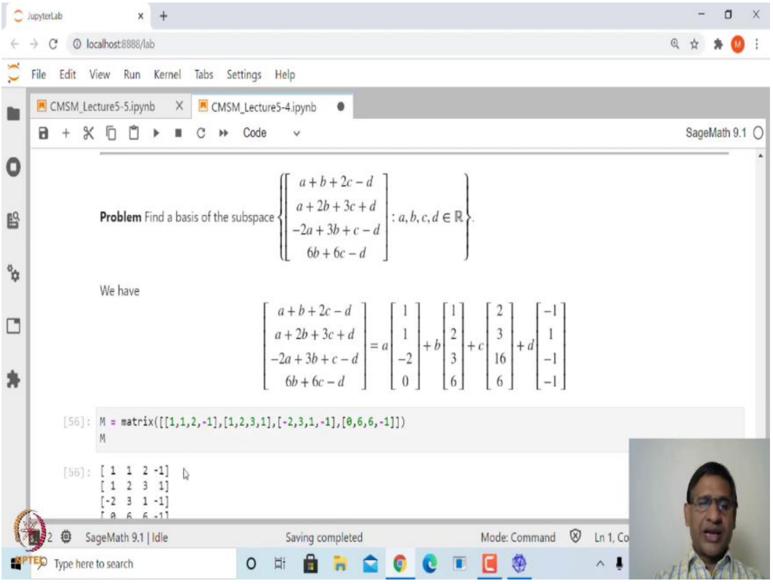


Let us just take one simple exercise. Suppose, you are given a problem to find a basis of subspace which is denoted by this set of all vectors of the form a plus b plus 2c minus d, a plus 2b plus 3c plus d, 2a minus 3b plus c minus d, and 6b plus 6c minus d, such that a, b, c, d are real numbers, you want to find this subspace basis of, of this subspace. We know that such, such a thing will form a basis.So, how do I do that? If you look at, for example, this any vector in this, typical vector in this, you can write this actually as a times some column vector. So, column vector will be, that is coefficient in the first component here, b times column vector, c times column vector, d times column vector. So, that means, it is, it is a subspace spanned by these four column vectors. So, we are, all we need to find is linear span of these four column vectors, and that is quite easy. So, we defined a matrix whose, whose rows are these these vectors, and take the, take the RREF of this, take RREF of this matrix, and you can take RREF of this row matrix.

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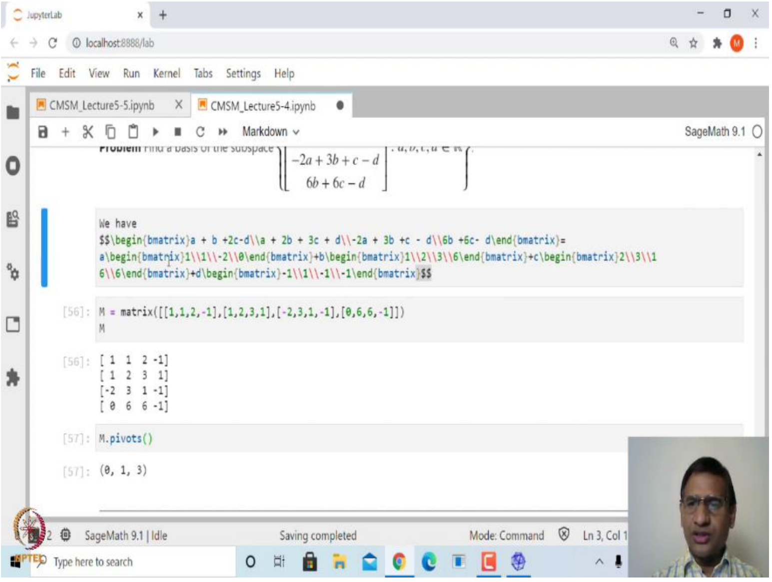


So, if you notice here, we, we are defining the, this row wise this 1, 1, 2, 1.

So, this is we are taking as row wise. So, you can also find pivots of M. So, pivots in this case is 0, 1, and 3. 0 means the first column vector, that this is, this vector, 1 means the second vector, and 3 means the fourth vector. So, first vector, second vector, and fourth vector forms a basis for this subspace. That is what it means.

So, this is one way of solving this, this problem. You could have also obtained basis by taking this row space, and generate. You can apply RREF, then it will give you basis in reduced form, not in this form, but now this, in this case, only these three vectors, first, second, and fourth are linearly independent. Second one, actually if you look at, second one is nothing but the sum of, there is small typo here. Instead of sixteen somewhere it has become 16 where is, where is that?

Let me again show you this. So, this would be 6 actually, it is not 16. So, where is 16? Yeah, this one, right?

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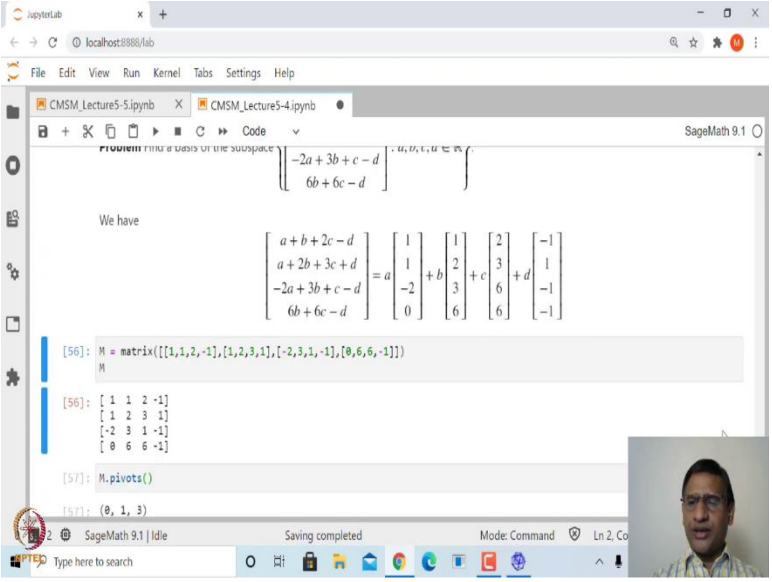


So, this is how you find, show that, or you can find basis for this subspace, ok? So, next time we will look at more examples of subspace, and some more examples or concepts related to coordinates of vectors with respect to a given basis.

Thank you very much.