Computational Mathematics with SageMath
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Lecture – 26
Local Maximum and Minimum

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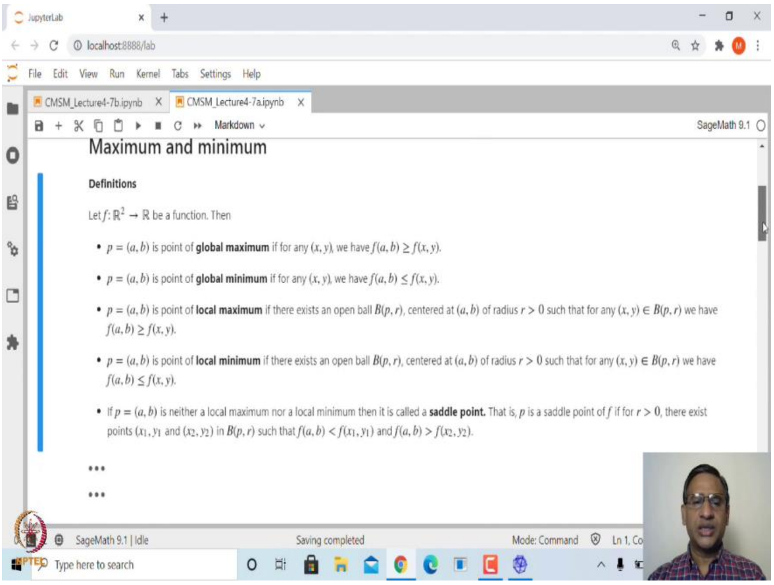

Welcome to the 26th lecture on Computational Mathematics with SageMath. In this lecture, we will look at how to find the Maximum and Minimum function of two and more variables.

Let us first start with some definition. Suppose you have a function f from R^2 to R, it can also be defined on some subset of R^2 to R or a function from R^n to R. Then a point p whose coordinates are (a, b) is called a global maximum if you take any point (x, y) in the domain of f then we have $f(a, b) \geq f(x, y)$.

Similarly, p is called a global minimum if for any (x, y) in the domain of the function we have $f(a, b) \leq f(x, y)$. However, the point p is called a local maximum if there exists an open ball $B(p, r)$, centred at p with radius r is the same as saying there exist r positive and this ball $B(p, r)$, such that for any point (x, y) inside this ball we have $f(a, b) \geq f(x, y)$.

Similarly, you can define the local minimum. For local minimum again the same thing, but this condition should be $f(a, b) \leq f(x, y)$, but in case this point p is neither a local maximum nor a local minimum, then what is the condition that you are going to get?

In that case, if you take any radius r positive and the open ball $B(p, r)$, then you can find two points (x_1, y_1) and (x_2, y_2) such that at one point function value is less than $f(a, b)$ and another point function value is bigger than $f(a, b)$. In that sense this in any small neighbourhood, the function will have smaller value and bigger value. Such a point generally we call a $saddle\ point$.

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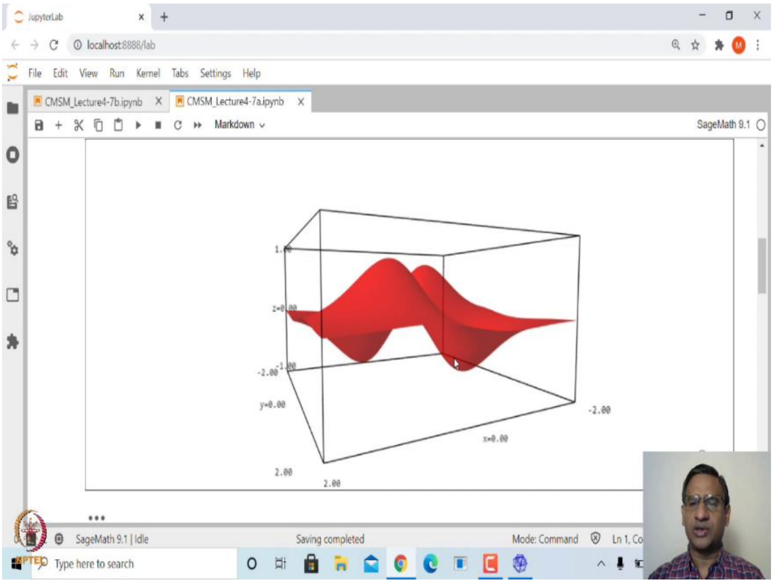

Now before we look at how to find these local maxima local minimum, first let us look at the graph of a surface. This is a surface $z = f(x, y)$. So, if you look at this graph, for example, this point and this point are a point of the local maximum. And if you look at the gradient at this point, what will happen? What is the gradient? The gradient is the direction in which function increases with maximum speed.

Now, since you are already at the point of maximum there is no direction in which function is going to increase, therefore, the gradient at this point will be 0. The gradient of the function f at this point is 0 is the same as saying first-order partial derivatives at this point will be 0; similarly, in the other case.

Similarly, if you look at this downward point, this is a point of local minimum. At this point again, the negative of the gradient will be 0. That is because at this point if we look at the negative of the gradient that is the direction in which function decreases with maximum speed.

But, at this point there is no direction in which function is going to decrease because it is already at a local minimum, then the negative of the gradient will be 0 is the same as saying gradient is 0. And, again you have that means, the first-order partial derivatives at this point will also be 0. Similarly, if you look at these two points, again a point of local maximum or local minimum, at this point if you draw the tangent plane to the surface the tangent plane will be parallel to xy-plane.

So, that tells that if the tangent plane is going to be parallel to xy-plane is the same as saying the tangent plane will have an equation $z = constant$. That again tells that the first-order partial derivative of the function at these points should be 0.

Similarly, if you look at this particular point or down below this is the only point, in this case at this point also if you try to draw the tangent plane that will be parallel to xy-plane. So, again at this point, the gradient will be 0.

However, at this point when you draw the tangent plane no matter how small a portion you take of this graph, the tangent will be will intersect the surface, but that is not the case at the point of local maximum or local minimum. This point is called a saddle point and that is what you get here at any small portion of this surface the function value will be higher than this function value at this point and it will also be lower. That is why it is called a saddle point.

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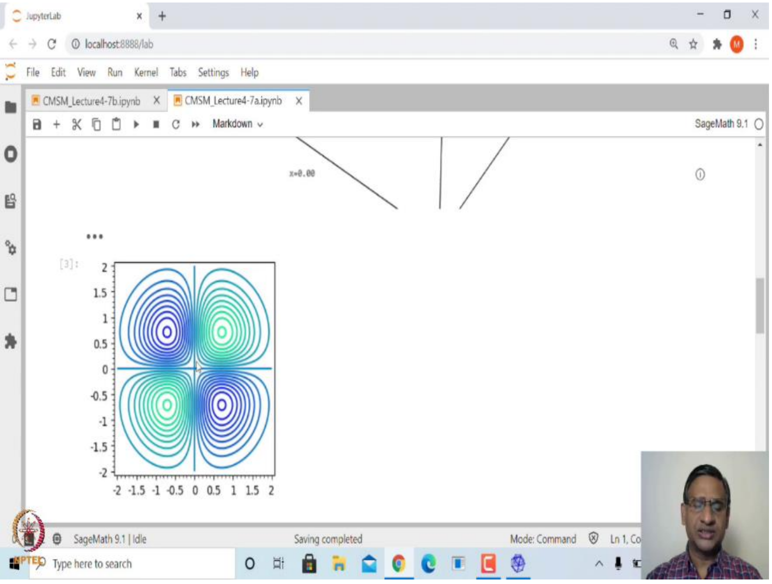

In case you look at the contour plot of the same surface this is how it looks like. In this case, you can see that one of these points this point, this point, this point, this point, these points may be the point of local maximum or local minimum because the contours near local maximum will be some kind of a concentric closed curve it will be a kind of circle close concentric circles.

And this is a saddle point where these contours will look like some kind of rectangular hyperbolas. These are the geometric meaning of local maximum, local minimum and saddle point. Now, let us look at how we can find these points.

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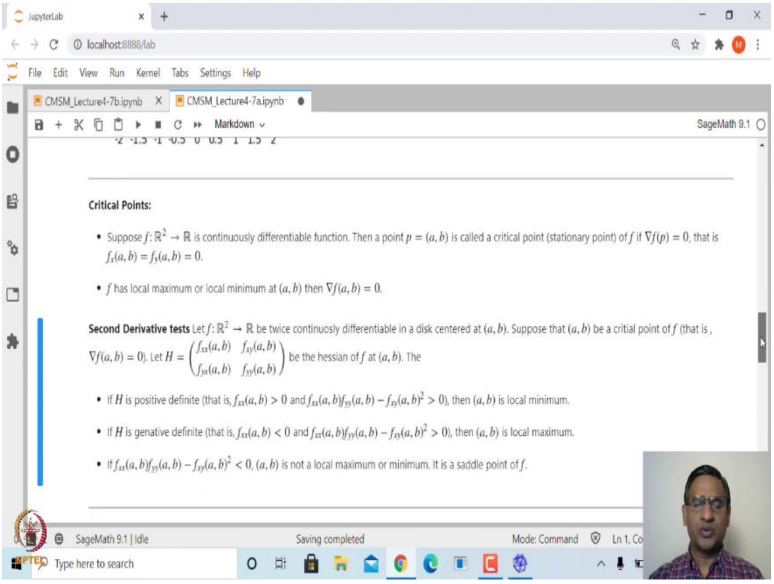

Let us start with an example, but before we look at the example whatever we saw geometrically let us just write them and then write the condition for local maximum or local minimum. First of all, those points where this gradient is 0 are called critical points. So, local maxima and local minimum and saddle points are going to come from these critical points; critical points are also known as stationary points.

At these points, in case the function has a local maximum or local minimum at any point (a, b), then we saw that $\nabla f(a, b) = 0$, i.e., the gradient at that point must be 0, which is the same as saying first-order partial derivative must be 0. That is the necessary condition for a point to be a local maximum or local minimum or a saddle point. In order to find critical points, we have to solve $\nabla f = 0$ for all those values of a and b.

And, once you have found the critical points, then you need to classify whether it is a point of local maximum or point of local minimum or a saddle point. Then you have a second derivative test very similar to one variable case. In one variable case, you must have seen that, in case, x is a critical point that is $f'(x) = 0$, then that $\nabla f(a, b) = 0$ will be a point of local maximum provided the second-order partial second-order derivative at x is negative.

If the second derivative is positive, then that point is a local minimum, that is what you have seen. Here in this case the second derivative of f is a matrix which is called Hessian of f at the point (a, b). What is the Hessian? Hessian is this matrix of second-order partial derivatives.

This is a second-order partial derivative of f with respect to x twice this is f partiality of f with respect to x, then followed by y and here this is a partial derivative of f with respect to y followed by y and this is a second-order partial derivative of f with respect to y twice.

If we assume that the function is twice continuously differentiable, in that case, this Hessian will be symmetric, because, this mixed second-order partial derivatives will be the same that is what we saw.

In that case, all eigenvalues of H will be positive and this Hessian is called positive definite if all it is eigenvalues are positive or this $f_{xx}\ (a, b)$ is positive and its determinant is also positive. Then we say that Hessian is positive definite. It is negative definite if $-H$ is positive definite. That is the same as saying $f_{xx}(a, b)$ should be negative and its determinant should be negative.

This is the definition of positive definite and negative definite. In case H is positive definite at the critical point, then that point is a local minimum. Similarly, the if at (a, b), the Hessian is negative definite, then that point is a local maximum.

Again, you can see here this is consistent with one variable local maximum and local minimum and in case the determinant of this Hessian that is $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$ is negative then such a point is called a saddle point. It is not a local maximum or local minimum. Now, let us look at some examples.

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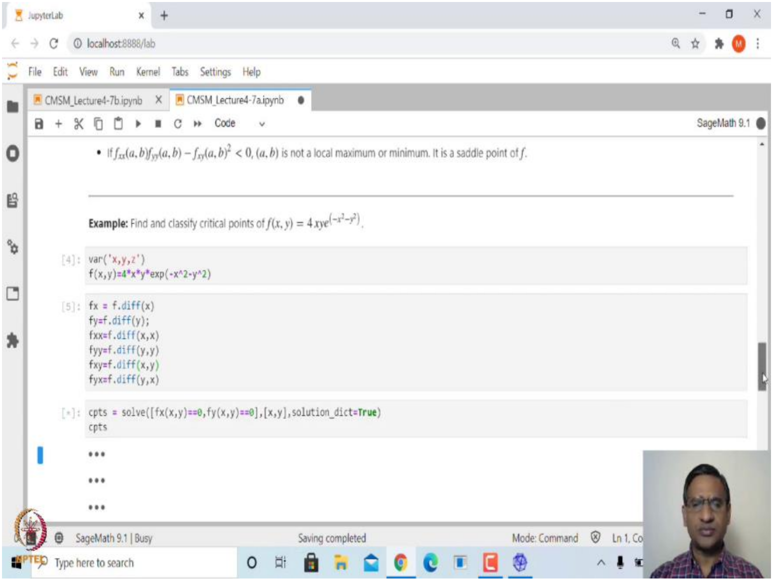

First, let us start with finding critical points and classify them for the function $f(x, y) = 4xye^{-x^2-y^2}$. Now, let us define x, y and z as variables and $f(x, y) = 4xye^{-x^2-y^2}$. And, then in order to find critical points first, we need to solve the gradient of f is equal to 0 is the same as saying the first-order partiality of f with respect to x and y must be 0.

You can even plot a graph of this function along with the contour, and we saw that these graphs of the surface and the contours are exactly graphs of that function $f(x, y) = 4xye^{-x^2-y^2}$. You can see here this function has two local minimum, two local maximum and one saddle point that is what we expect and that again is confirmed by these contour plots. Let us find out the first and second-order partial derivative.

This is the first-order partial derivative of f with respect to x, then followed by first-order partial derivative of f with respect to y, so that we are calling it as f_x, f_y. This is a mixed second-order partial derivative of f with respect to x twice and so on.

Let us evaluate this and once we have evaluated this let us find the critical points by solving $f_x(x, y) == 0$, $f_y(x, y) == 0$ for $[x, y]$ with the condition that $solution\ dictionary = true$. It will give you a solution as a dictionary. Let us run this. Once you run this it may take some time depending upon the function and complexity of that function.

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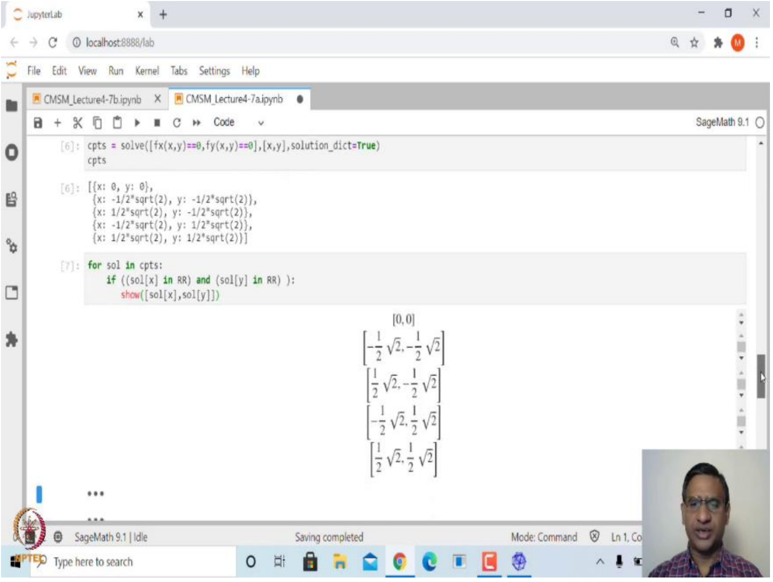

So, you can see it has found five critical points, one is $\{x: 0, y: 0\}$; another one is $\{x: -\frac{1}{2}sqrt(2), y: -\frac{1}{2}sqrt(2)\}$ and so on. Now if we want to classify these critical points; that means, at these critical points we need to find the Hessian. Let us just take one of the critical points this is just we can run a loop over all the solutions and try to print these critical points in pretty print format. These are the five critical points.

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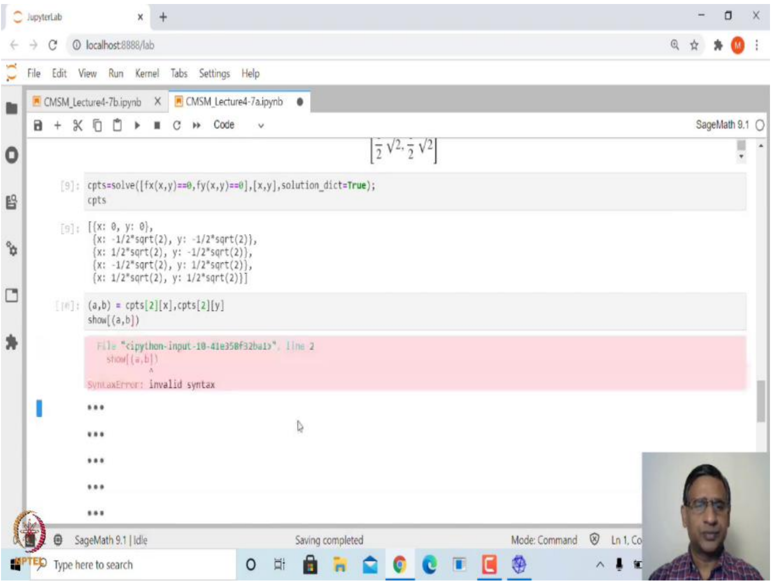

Now, this we have already done, let us not get into this. Suppose we want to find or classify a third critical point in this list i.e., $[\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}]$. Suppose we call that as in (a, b) as the third critical point. Just a second, a square bracket is missing this should be, inside this.

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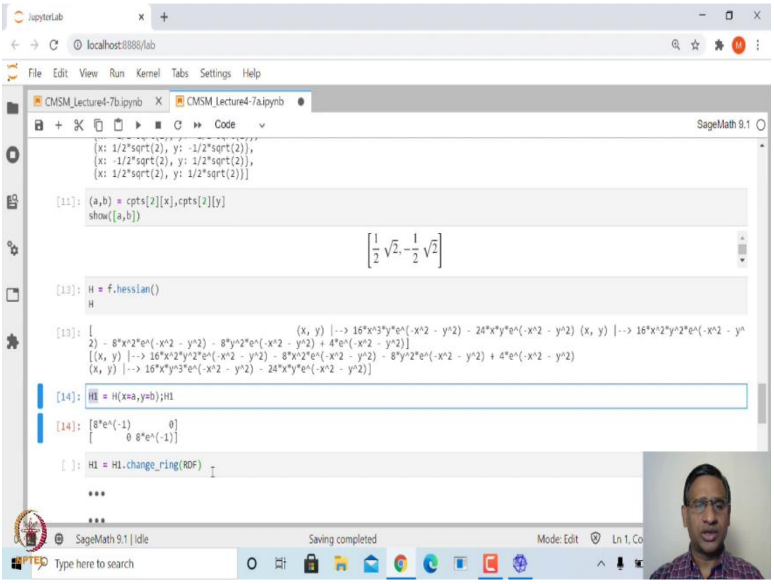

This is the third critical point. At this critical point let us look at what is the Hessian now. Hessian you can obtain in SageMath just using $f.hessian()$, this will give you Hessian. That will be a function of (x, y). Now, you can evaluate this Hessian at (a, b). Let me call that as H1; $H1 = H(x = a, y = b)$.

That is the Hessian. It is $\left[8e^{-1}\ 0\ 0\ 8e^{-1}\right]$. e^{-1} is positive; that means, this Hessian is positive definite because this term is positive and this determinant is also positive.

Therefore, this is a point of local minimum. We can also find whether this Hessian is positive definite or not, by using the inbuilt function. First, we will change the domain from which these entries are taken; this RDF means this real field. Let us change this entry otherwise, it may be a symbolic field.

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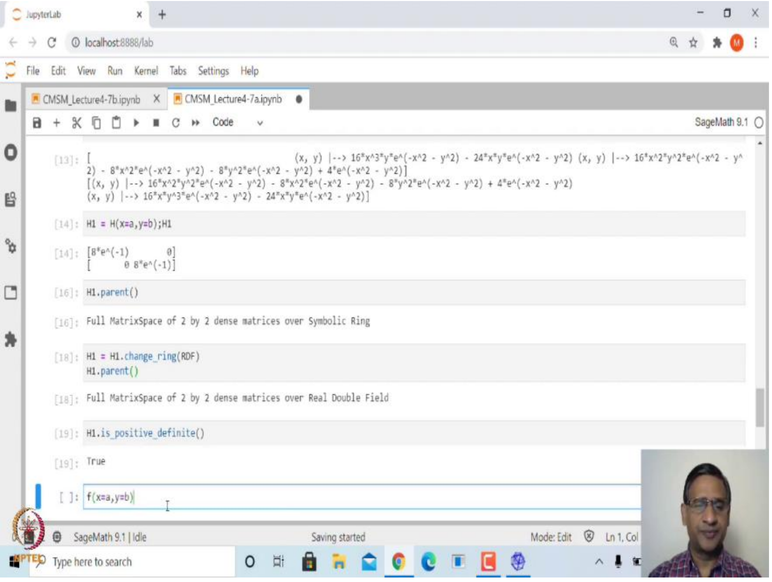

Let us check H dot I do not know whether there is a function called ring. It should be $H.parent()$, it says that this is a full matrix of 2 by 2 dense matrices over a callable function ring with arguments (x, y). Let me call this H1. It says that this is a symbolic ring, a symbolic matrix in that sense. It will not be able to check whether it is positive definite or negative definite. That is why you change the ring i.e., the entries from which it is taken.

Now if you look at what is $H1.parent()$, then this will be a real double field. Now, you can use this command $H1.is\_positive\_definite()$, this will tell you whether it is positive definite or negative definite. The value of the function at this critical point $(x = a, y = b)$ is $-2e^{-1}$, that is a negative value.

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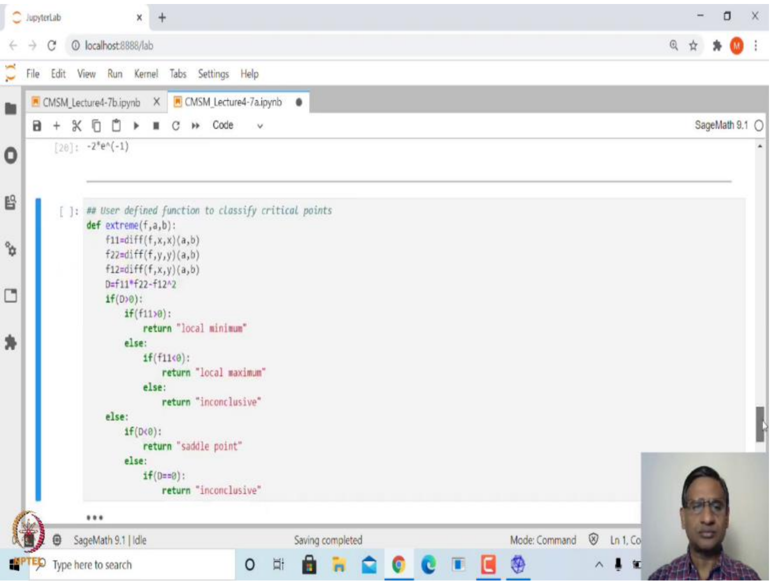

That is how you can check for a given critical point whether the Hessian is positive definite or negative definite. Now, let us look at we can create a small user-defined function to check whether a critical point is a local maximum or a local minimum. For that, we are just defining this first second-order partial derivatives.

This is a determinant of the Hessian and first, you check whether the first term i.e., the first-order partial derivative of f with respect to x twice is positive. Once it is positive then check sorry, first you check the determinant.

If the determinant is positive then check the first term in the Hessian i.e., $f11$, which is a first-order partial derivative of f with respect to x twice. If this is positive then it is a local minimum or if it is negative, then it is a local maximum. If it is not positive and not negative, then the test is inconclusive.

In case, the determinant is negative this is a saddle point and if the determinant is 0, then again, the test is inconclusive. In case this test is inconclusive one can go for higher-order partial derivatives and then try to classify, let me not get into those things.

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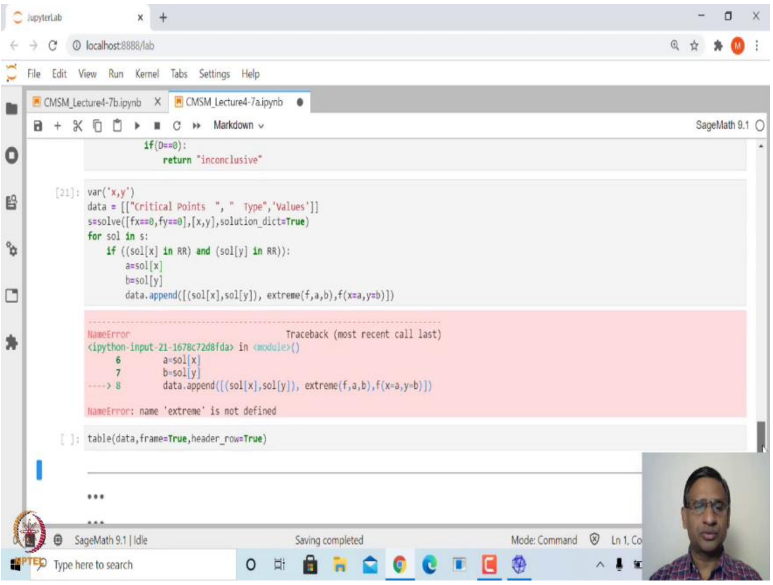

After you have defined this user-defined function to check whether a critical point is a local maximum or local minimum, then we can call this function and try to tabulate each point, its nature and the function value. This is a small loop which what is it doing?

It is creating a table of critical points, the type of the critical points and the value. Then first you are solving this in the dictionary and then for each solution inside this dictionary. First, you have to check whether the solution is real or not because if it is not real then that is not a critical point.

Then define a to be the x-coordinate, b to be y-coordinate and then append that into the data. This is the data that we are creating and then after that, you print that data. I think I did not run this user-defined function. Now, let me run this user-defined function and see what is going to be in the table.

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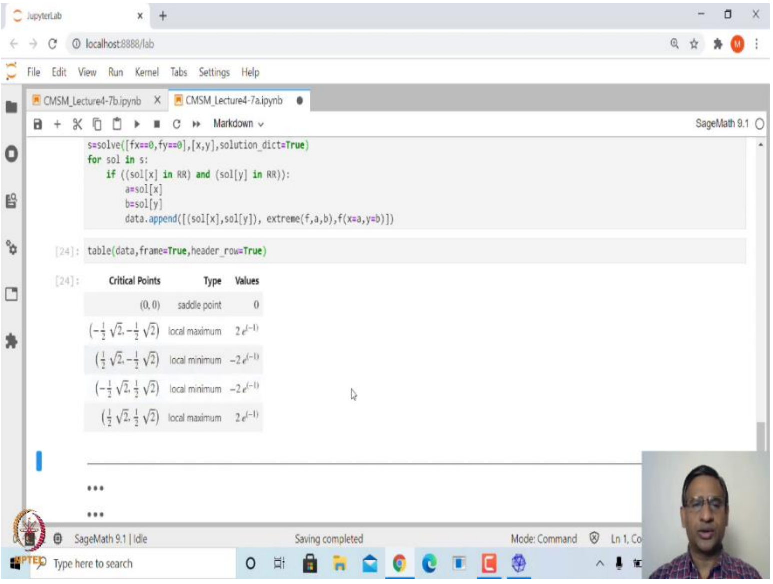

That is the table of classification of critical points. So, (0,0) is a saddle point next function value is 0, this is a local maximum the function value is this, next is a local minimum function value is this and so on. This is how we can classify critical points in the function of two variables and you can extend this to the function of three variables, except that in this condition you need to now check on positive definiteness or of the Hessian.

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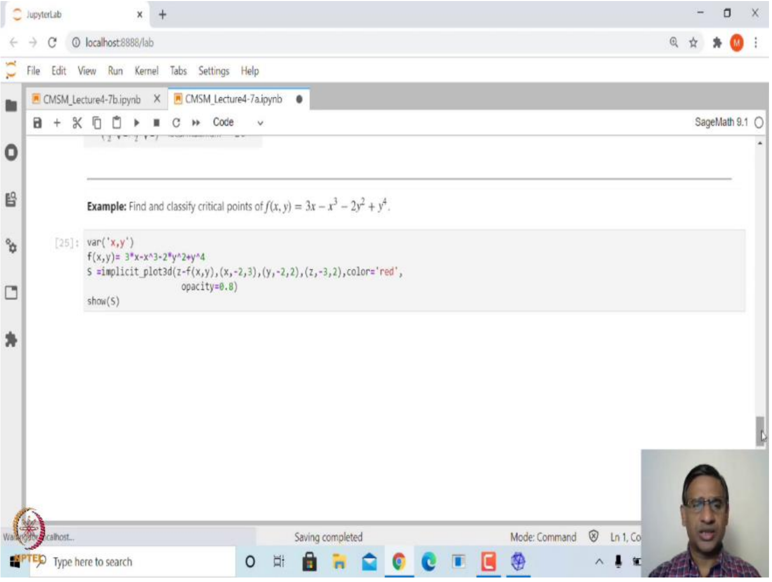

Next, let us look at one more example. This is the function $f(x,y) = 3x - x^3 - 2y^2 + x^4$.

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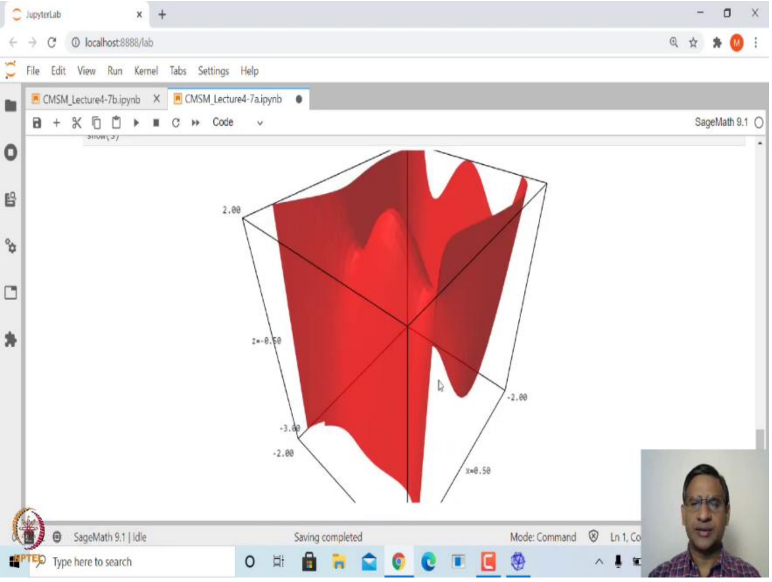

If you try to plot a graph of this function that is a surface, this is how it looks like. In this case, you can see here there is one point of local maximum, there are two points below is a point of local minimum and this is going to be a saddle point. This is one saddle point here, there are other saddle points inside this. This is another saddle point; this is another saddle point. So, it has three saddle points, one point of local maximum and two local minimum.

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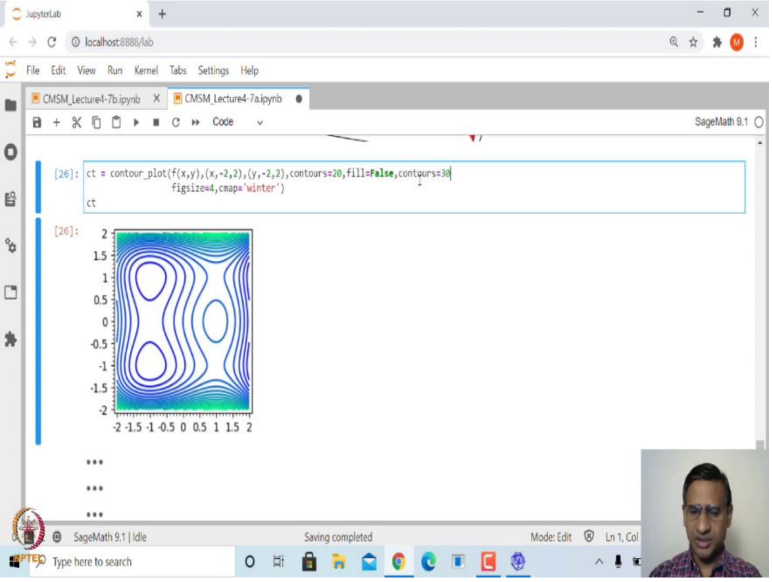

Similarly, if you try to plot the contours that should confirm that it has three saddle points, one saddle point will be somewhere here, another saddle point is somewhere here and these are going to be a point of local maximum or minimum.

In this case, we saw that this should be a point of local minimum and this is a point of a local maximum. If you want you can increase the contours. Let us say contours equal to, let me make it 30, it may take more time contours equal to 30. There twice I have written these contours.

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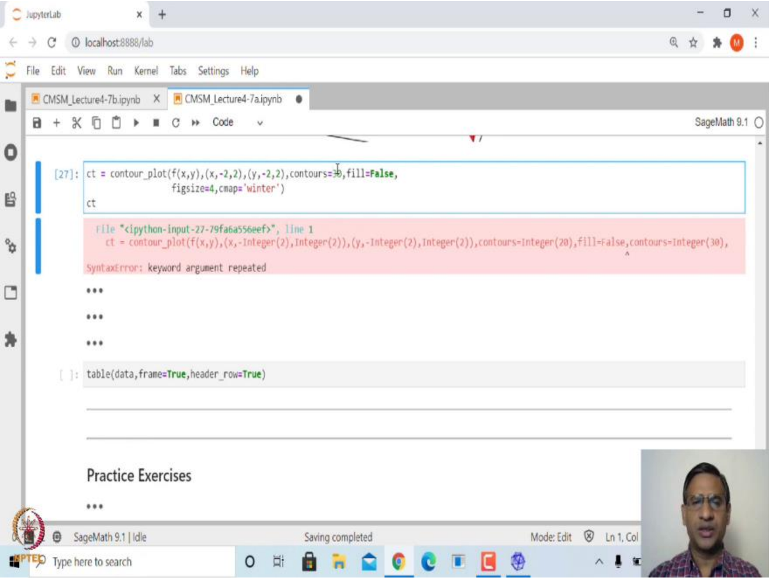

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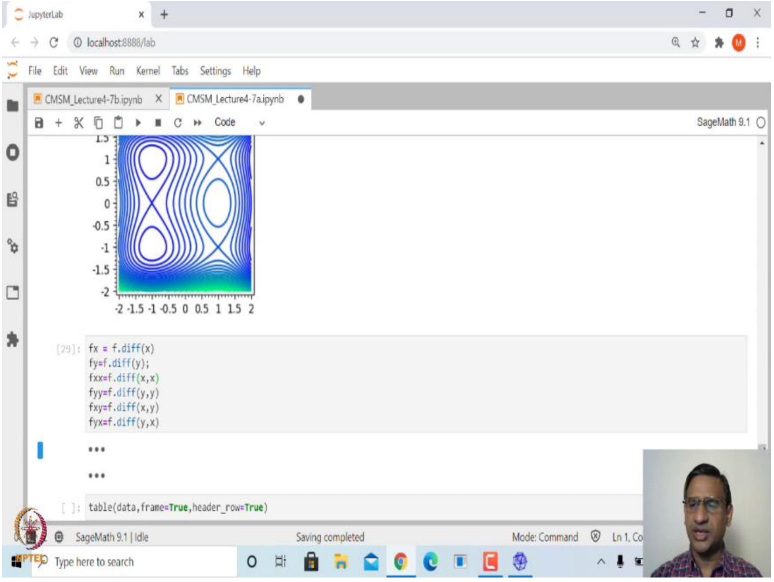

This instead of 20, let me make it 30 and then. You can see here these you get more contours. Now, let us classify them. Find all these first and second-order partial derivatives.

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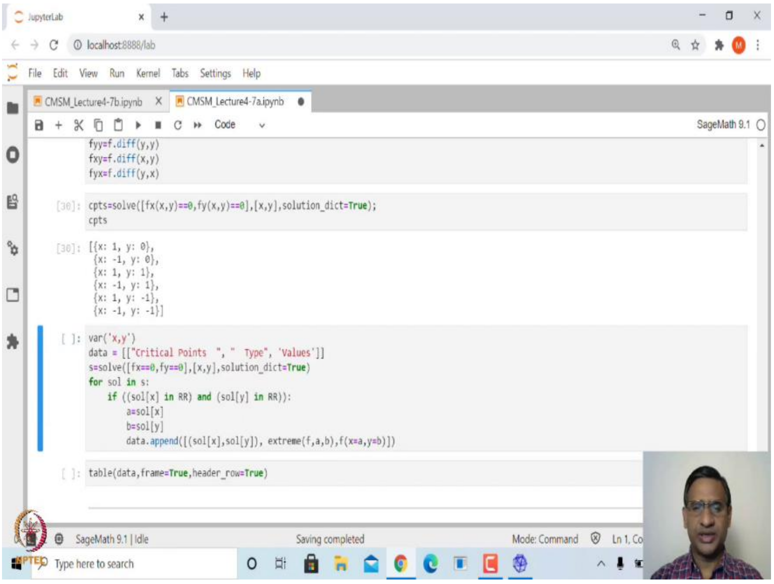

Then let us find the critical points by solving this gradient equal to 0. Again, you can see here this confirms that it has five critical points $\{x: 1, y: 0\}$, $\{x: -1, y: 0\}$, $\{x: 1, y: 1\}$, $\{x: 1, y: -1\}$, and $\{x: -1, y: -1\}$. Now, in order to classify this, we will again tabulate the critical point, its nature that is whether local maximum, local minimum or saddle point and the function value by just running again this loop and calling this the extreme function is a defined function that we have created.

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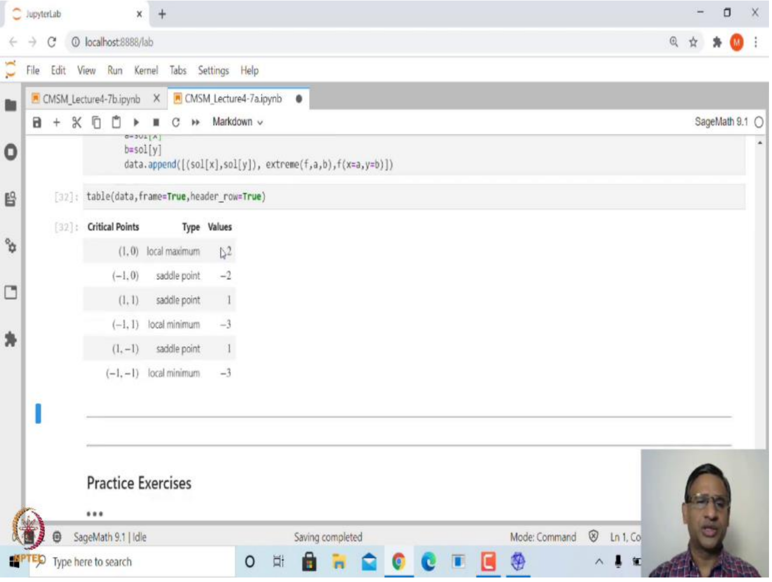

Once we do that and then let me create the table of this. You can see here this $(1,0)$, is local maximum with function value 2, this is saddle point with a function value -2, this $(1,1)$ is again a saddle point, and you $(-1,-1)$ is a local minimum. There are two local minimums and one local maximum and three saddle points. That is what we saw from the graph.

You can do more examples. One of another very standard problems that you may find. You can try to find out relative or absolute maximum and absolute minimum. In that case, apart from finding the local maximum, local minimum in that given domain, you also need to evaluate the function along with the boundary point and compare the value of the function with the local maximum and local minimum.

In the function value which is the smallest among local maximum, local minimum and the function value at the boundary point will be an absolute minimum. Similarly, you can find the absolute maximum. It has several applications. We will look at its application in what is called least square problems in the next lecture.

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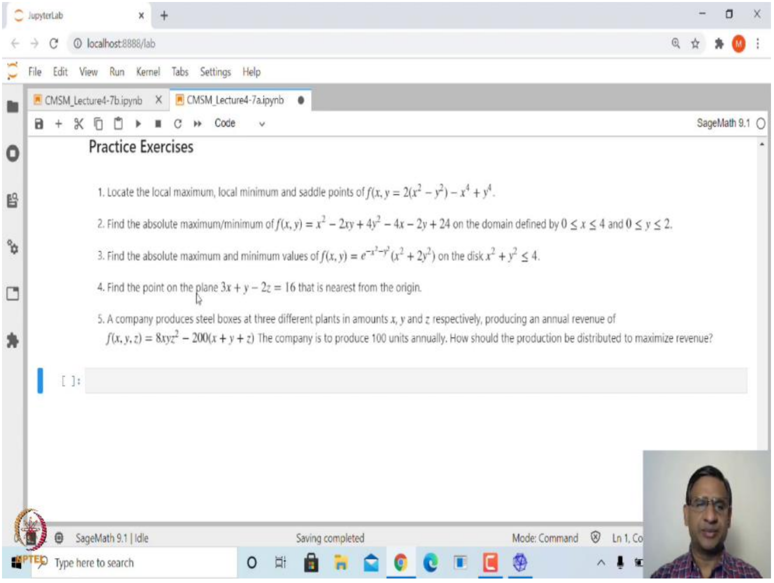

Let me leave you with a few exercises. These are straightforward exercises. This is a function whose critical points you need to classify and this is another function defined on this closed and bounded domain and you need to find the absolute; spelling mistake, absolute maximum, minimum. As I said, you need to first find the local maximum, local minimum and then also find the function value at the boundary point and then compare. This is, find the point on the plane that is nearest to the origin.

That is the same as saying if you take any point on this plane, look at the distance, you need to minimize the distance. Distance from this any point (x, y) will be $\sqrt{x^2 + y^2 + z^2}$. But then the square root is not a differentiable function, so, you need to minimize the square of this distance function that is $x^2 + y^2 + z^2$ and this is what you need to minimize. That is a very easy problem and this is again a problem in three variables.

You need to find the partial derivatives with respect to all these three variables x, y, z. Find the critical points and then check whether the Hessian is positive definite or negative definite to classify.

Thank you very much.