Computational Mathematics with SageMath
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Lecture – 24
Limit and Continuity of real valued functions

Welcome to the 24th lecture on Computational Mathematics with SageMath. In this lecture, we will first looked at some calculus on polar coordinates and then, we will start multivariable calculus. We shall look at how to find the limit of functions of two variables and three variables, and then we should also look at what is the meaning of continuity and all these things etcetera.

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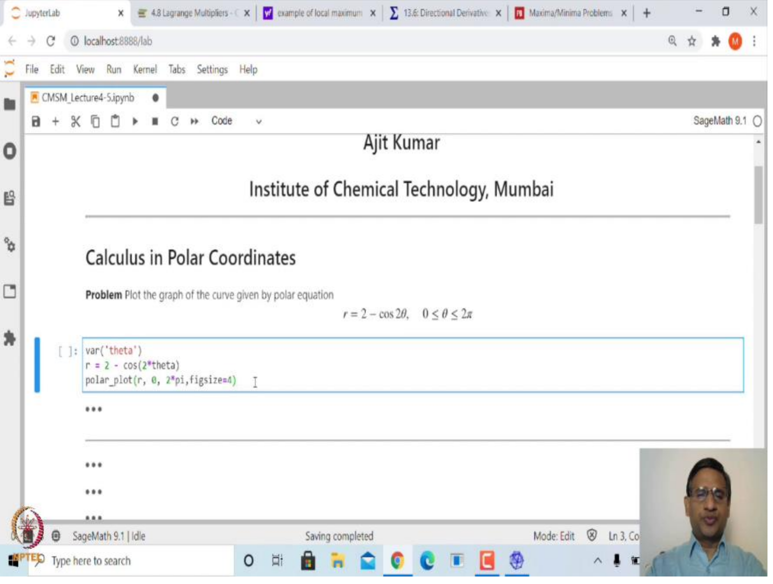

Let us get started. First, suppose you are given a function in polar coordinates let us say r is equal to $f(\theta)$. For example, here $f(\theta) = 2 - cos(2\theta)$ and θ varies between 0 to 2π. We have seen how to plot the graph of such functions. How does one plot graph? you need to use a polar underscore plot and when you run this polar underscore plot of r and θ varies between 0 to 2π.

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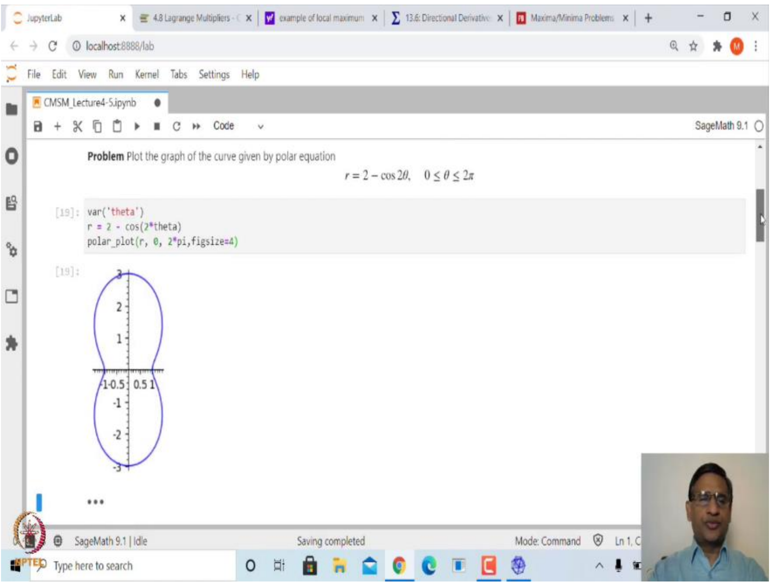

The fig size is 4. This is what you get: this is the graph of the function.

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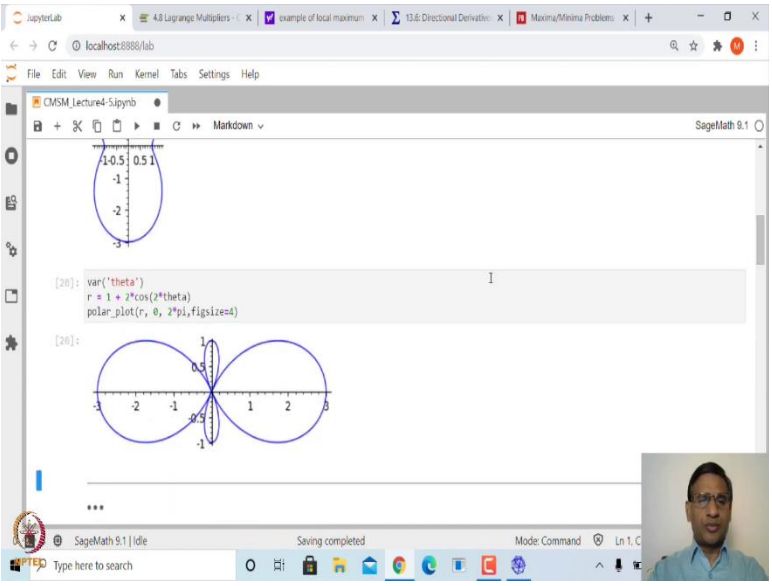

Next, suppose you have another function let us say $1 + 2cos(2\theta)$; again θ varies between 0 to 2π. Its graph looks like this; this is somewhat more complicated.

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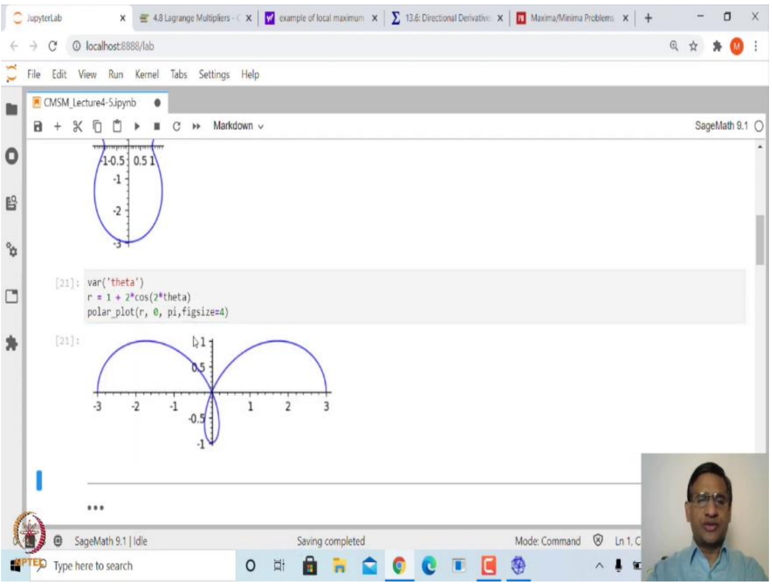

For example, changing θ range i.e. if I say θ varies between 0 to π itself, you will get only half of the graph. Let us say we will make it again 2π. That is the graph of $1 + 2cos(2\theta)$.

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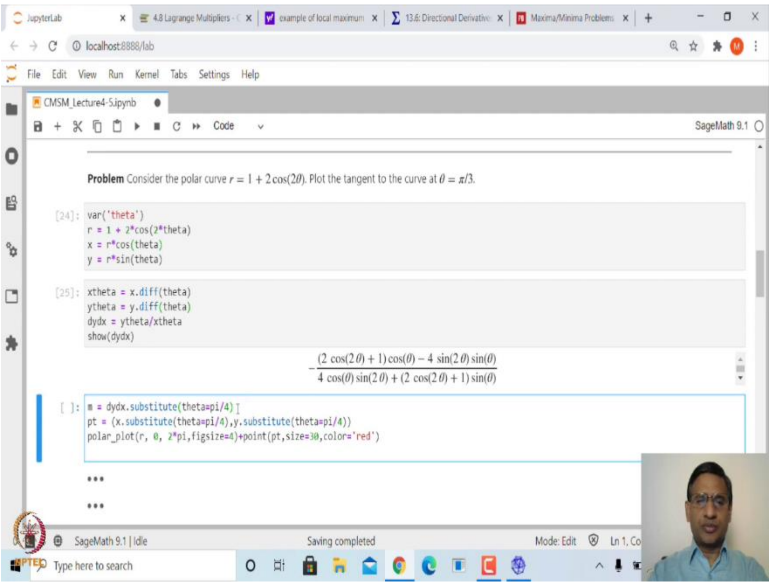

Next, let us look at how to plot tangent to a curve $1 + 2cos(2\theta)$at θequal to $\pi$by 3. What we can do? We have r is equal to $f(\theta)$ and where we know that in polar coordinate, we have $x = rcos(\theta)$ and $y = rsin(\theta)$. in Cartesian coordinate if you substitute $x = rcos(\theta)$ and $y = rsin(\theta)$ this function will get converted into polar coordinates.

Now, what you can do is you can find the derivative of x with respect to θ, the derivative of y with respect to θ. I have denoted it here by $x(\theta)$, $y(\theta)$and then you can take the derivative of y with respect to θthat is $\frac{dy}{d\theta}$ and then you can ask it to show. That is the derivative of this function r is equal to $f(\theta)$ at $\theta$and you can find out the value of this derivative that is the slope of the chord at θequal to $\frac{\pi}{3}$.

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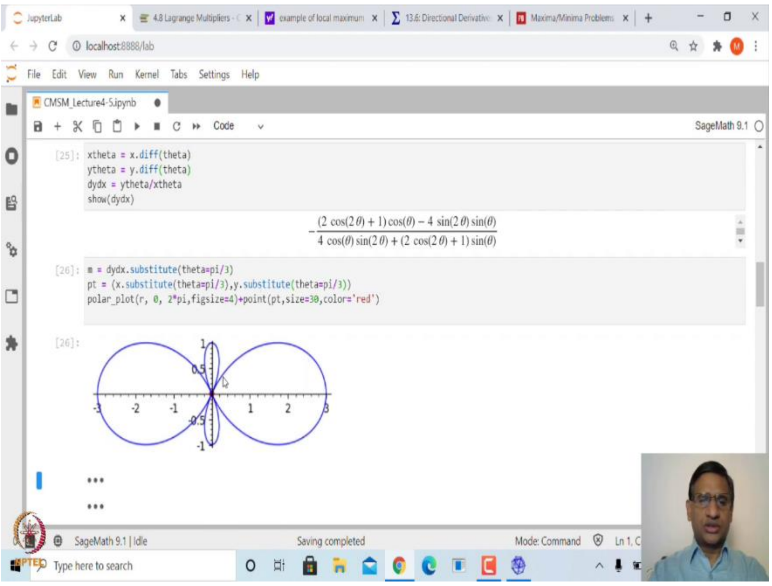

Let us make $\frac{\pi}{3}$ everywhere, then what are we doing? So, pi by 3 is somewhere here. It is the origin.

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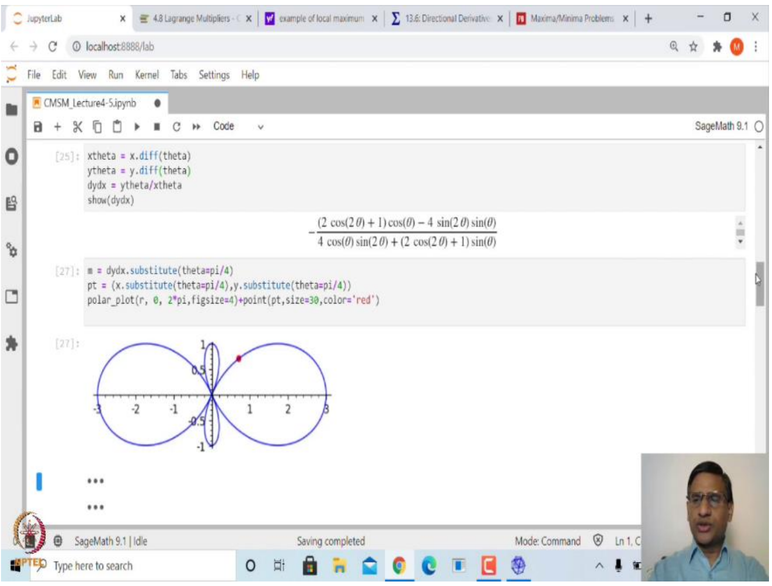

What are we doing? $\frac{\pi}{3}$ is somewhere here. It is at the origin. Let us make it at $\frac{\pi}{4}$ everywhere. We are plotting the slope at θ equal to $\frac{\pi}{4}$ and the point at this point. This is the point at $\frac{\pi}{4}$.

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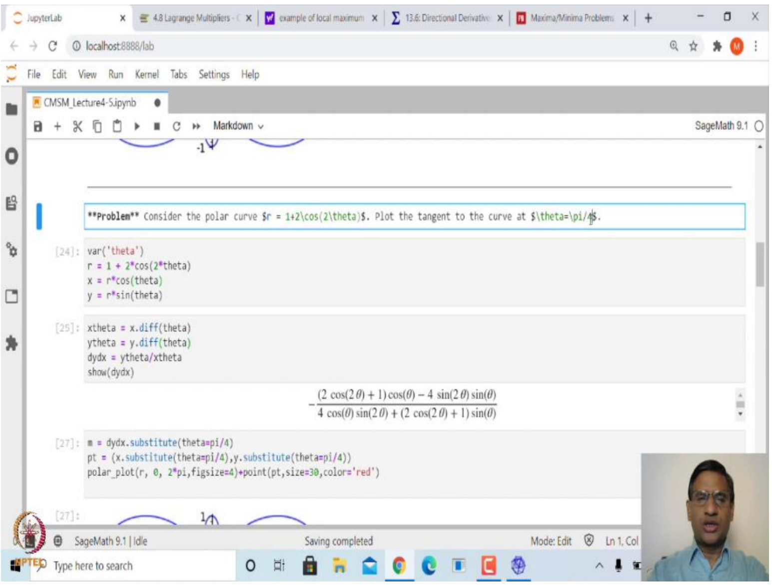

Let me change this problem instead of θequal to $\frac{\pi}{3}$ let me make it $\frac{\pi}{4}$.

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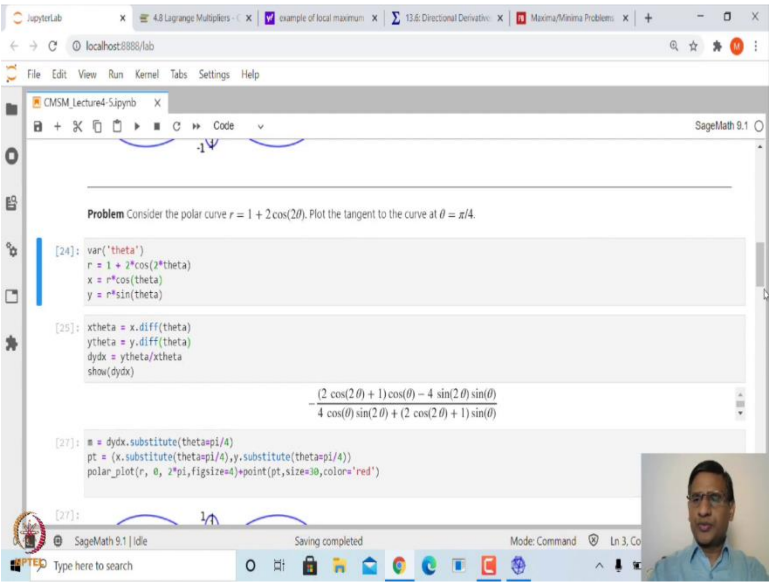

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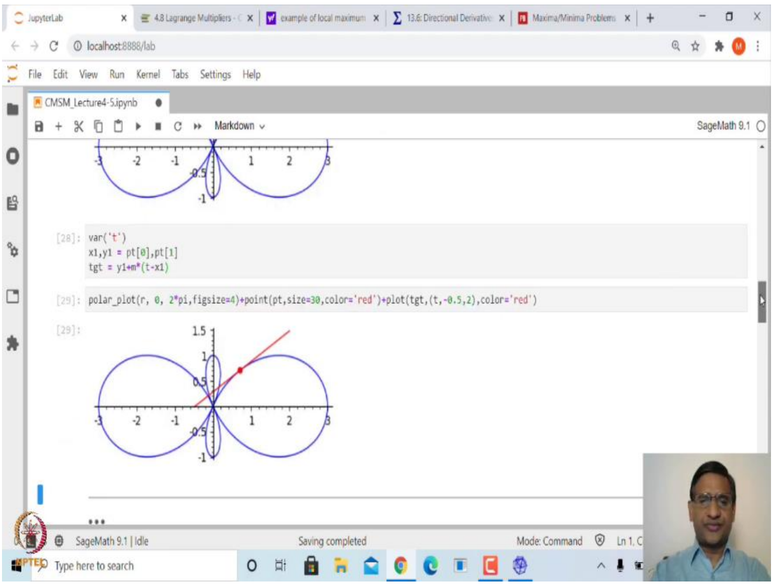

Here is the graph of this curve along with the point. Now, at this point, you can draw the tangent and for that, you need to define the equation of the tangent which is in slope-intercept form.

You have the equation of the tangent where the slope is m and it passes through (x1, y1). Here (x1, y1) are the x coordinate and y coordinate of this point. That is the tangent and lets me plot the curve along with the tangent and this is what you get. That is the tangent at this point. That is fairly simple.

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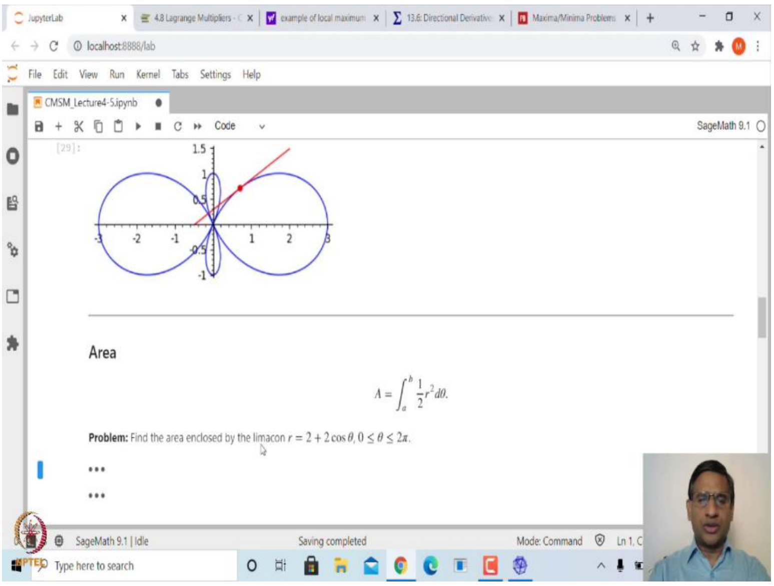

Similarly, suppose you want to find the area of a curve in polar coordinate, area of a curve defined in polar coordinate is given by integral of $\frac{1}{2}r^2 d\theta$, θvarying between a and b. And, this formula is quite simple to obtain one can use a change of variable and obtain this formula from the Cartesian coordinate.

Or you can look at what is the small area element in a polar form that will suppose if you draw a small area element here and θ varies this angle is dθ then and suppose this is the r, then this area will be $\frac{1}{2} * base * height$; height you can consider as r.

And, the base will be for a small arc this height will be $rsin(\theta)$, but $sin(\theta)$ for θsmall is very close to θ, therefore, you will get area element as $\frac{1}{2}r^2 d\theta$. you integrate this you will get the area. Let us suppose we want to find the area of this polar curve $r = 2 + 2cos(\theta)$ going from 0 to2πthis is what is called limacon.

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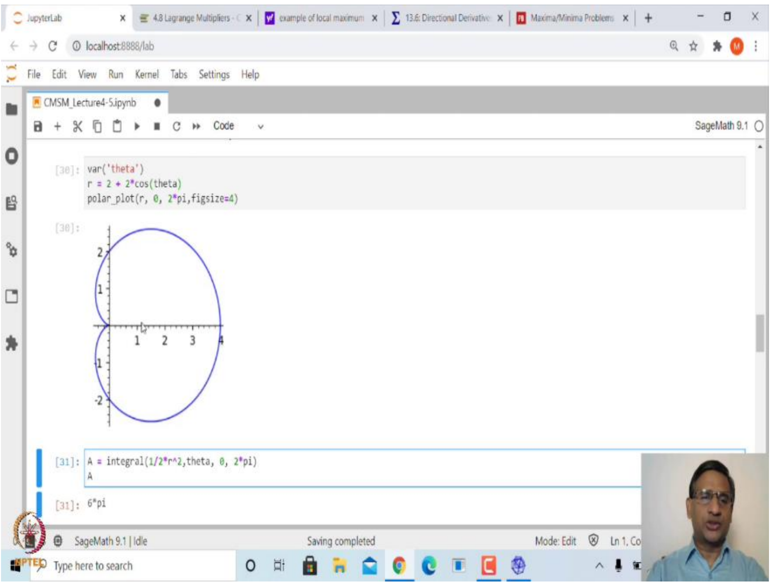

In this case, this is the graph and its area is given by integral of half r square with respect to θ, θ varying between 0 to 2π. That is the 6π. This 6π is the area enclosed by this particular polar curve.

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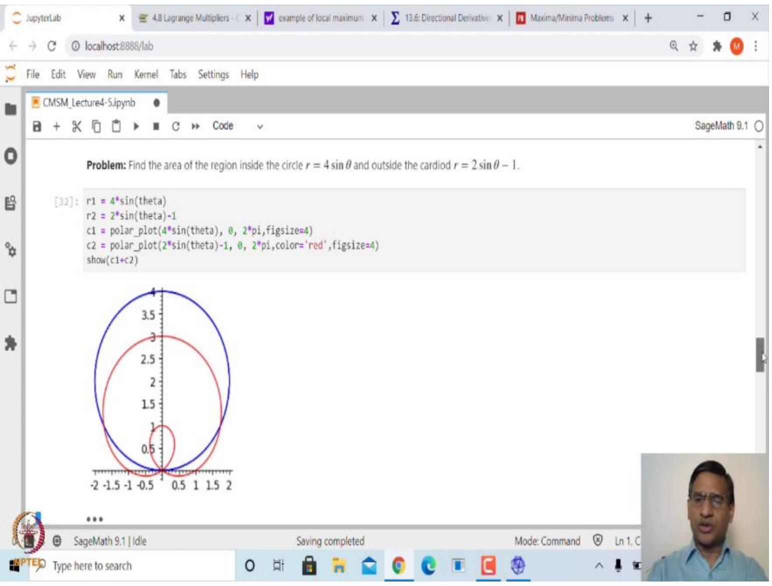

Suppose you want to find the area of the region inside this circle $4sin(\theta)$ and outside this cardioid $r = 2sin(\theta) - 1$. How do we find out? Let me define the two curves as $r_1$and $r_2$and those curves let us plot those curves.

When we plot these two polar curves the blue one is the circle which is 4 sin theta and the red one is the cardioid which is $2sin(\theta) - 1$ and we want to find an area inside the circle outside this cardioid this is the portion whose area we want to find out.

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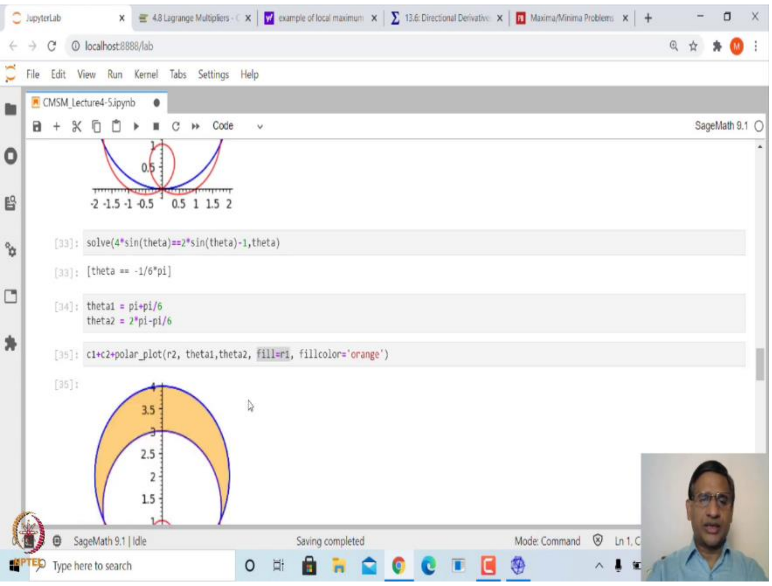
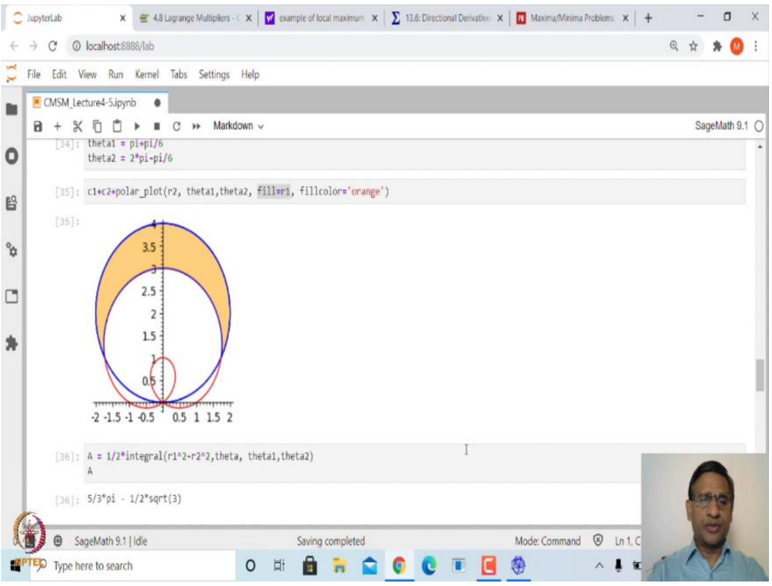


Let us plot this in shading. First, we need to find out what is the point of intersection of these two curves so that we can solve the command. Solverr_1 is equal tor_2 for θ. You will see that the $\theta$is in this case -$\frac{1}{6\pi}$, but we want this theta to lie between 0 to 2π.

This will be $2\pi - \theta$ that is $2\pi - \theta$ which is $\frac{\pi}{6}$and another one will be here. Let us write that our first one is $\pi + \frac{\pi}{6}$ and the second one is $2\pi + \frac{\pi}{6}$. These θ_1 is going to be $\frac{7\pi}{6}$ and θ_2 will be $\frac{11\pi}{6}$. Now let us plot the area inside this by using the option called to fill and give the fill colour. (Refer Slide Time: 08:49)



That is the portion whose area we want to find out. This is then in orange colour. Now, let us find out the area. If you want to find the area under this curve. This is your r_1, this isr_2, in this case, r_1 is bigger than r_2. We can find the area is half-integral of ${r_1}^2 - {r_2}^2$ and θ vary between θ_1 and θ_2. When you compute this area is $\frac{5}{3}\pi$ - $\frac{1}{2\sqrt{3}}$.

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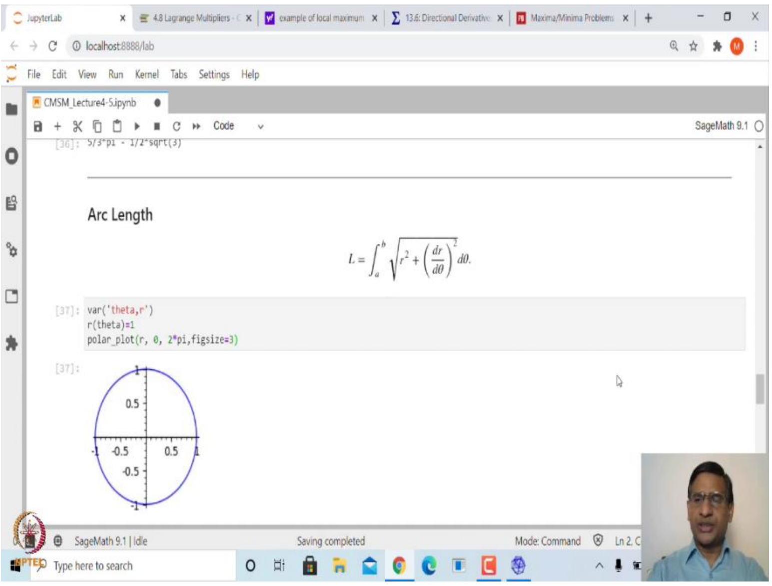

This is how you can find the area under the polar curve. Similarly, you can find arc length. If you look at Cartesian coordinates you can use a change of variable formula and an obtained arc length formula in polar coordinates like this. Let us look at one example. Suppose we look at the circle. The circle can be written in polar coordinate as r is equal to constant 1, circle of radius 1.

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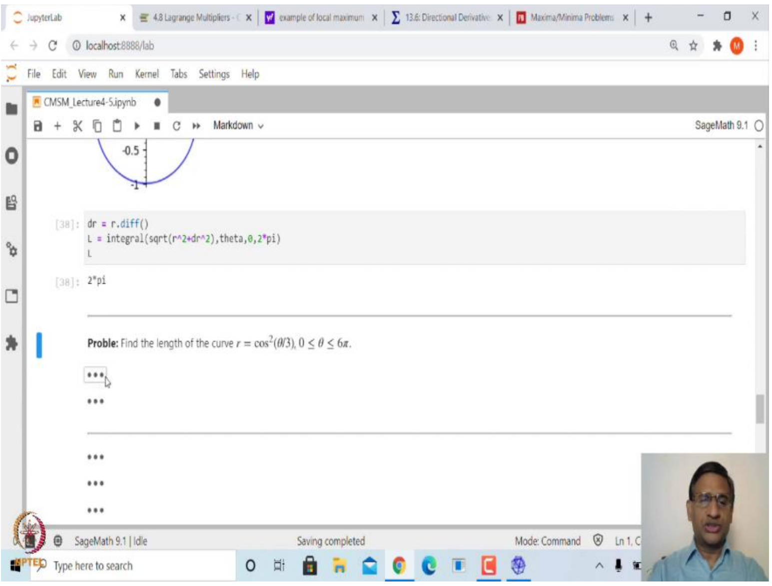

If I want to find the length of the circle already, we know it should be 2π. Let us compute this and see the answer is 2π. This formula is correct and supposes you want to find the length of the arc $r = \frac{cos^2(\theta)}{3}\ \theta$ varies between 0 to 6 π. Let us first plot a graph of this polar curve.

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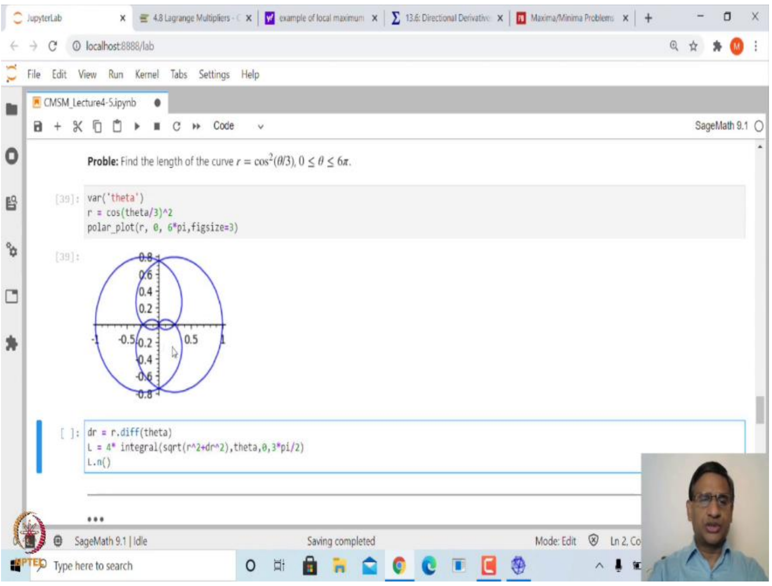

This is how the graph looks like.

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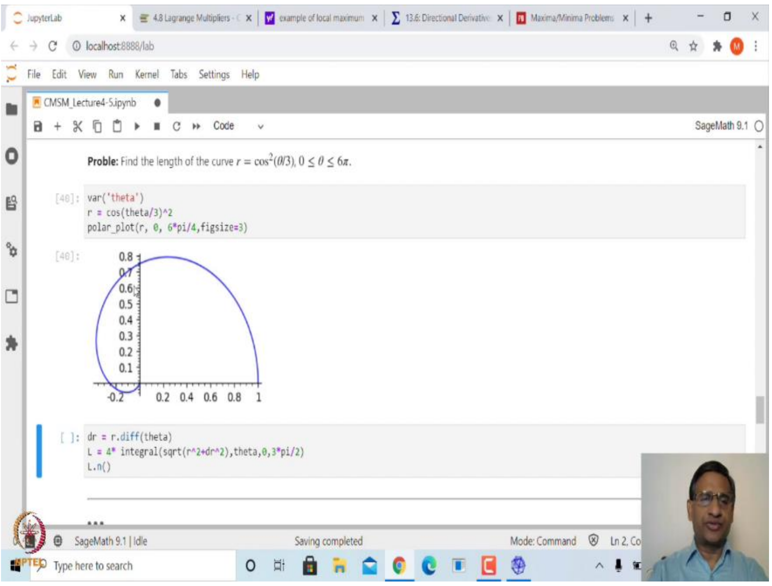

When you find the length of this curve so, in this case, what you can do is, suppose we plot this curve between this and let us say $\frac{6\pi}{4}$, this is only one fourth and the length will be 4 times this.

That is what we are doing. We will look at the length between 0 to $\frac{3\pi}{2}$ which is $\frac{6\pi}{4}$ and multiply this by 4 and when you compute the integral you can look at the numerical value of this.

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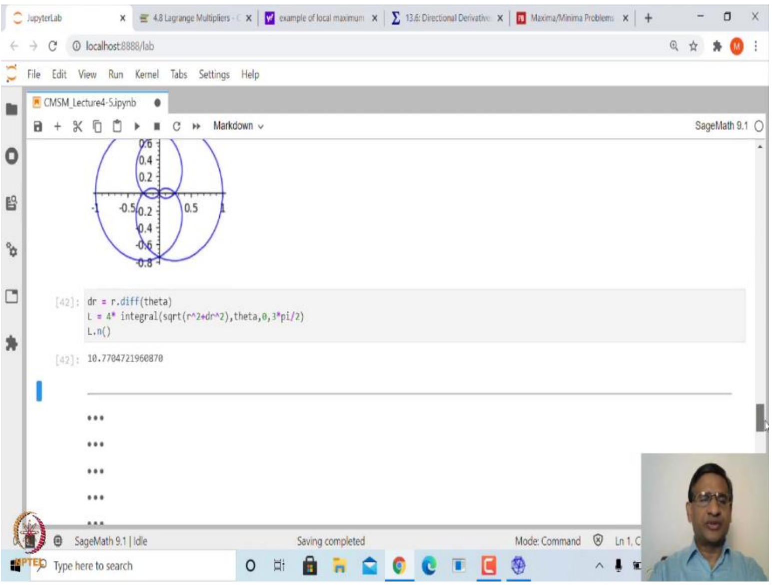

It is 10.7704. This is how you can find the area length of the polar curve and similarly, you can also find you can define what is the surface of revolution of a curve defined by polar coordinates and try to find out surface area and other things etcetera.

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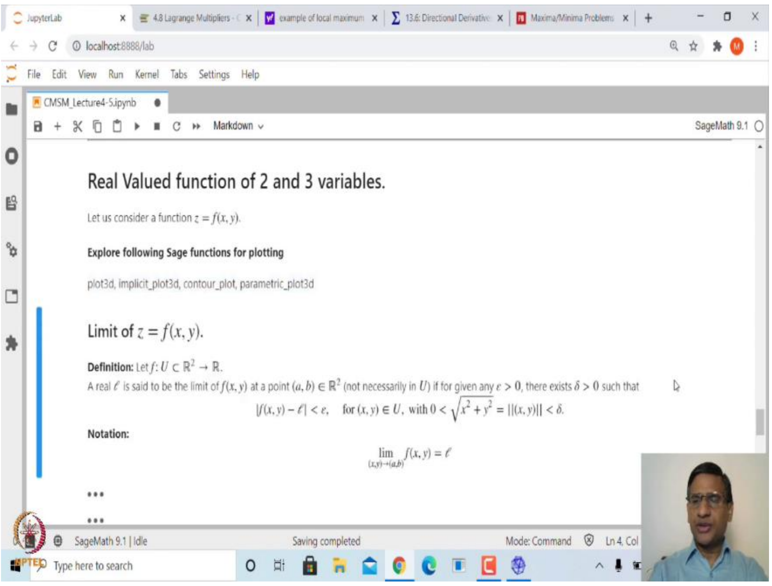

Next, let us look at if we have a function of 2 variables and its value is real, we call that as a real-valued function of 2 variables and in case it is a function of 3 variables real-valued function of 3 variables.

We will look at how to do various calculus concepts for this kind of function. First, we have already seen how to plot a graph of such functions in SageMath. If you want to plot that surface z is equal to f(x, y) you can use plot 3d.

If you want to plot implicitly define a function in three variables, you can use implicit plot 3d; if you want to plot contours contour underscore plot and in case the surface is defined using parametric coordinates you can use parametric plot 3d. We have already done it, but I request all of you to explore these functions and take some examples.

Now, when you have function z is equal to f(x, y) you can define the limit of this function at x equal to a, b and suppose that limit is a real number let say l when do we say this limit of f(x, y) at a, b exists and it is equal to l? If this says that the limit is equal to l if you take any epsilon positive, then you can find a delta positive such that all take any x, y in the disk of radius delta within this domain and look at the difference between f(x, y) and l in mod this should be less than epsilon.

For every x, y inside this u with this condition that x square plus y square in the square root which is the length of the vector x, y this should be less than delta then the difference between f(x, y) and l should be less than epsilon and in case the limit exists we denote that limit as f limit f(x, y); (x, y) going to (a, b).

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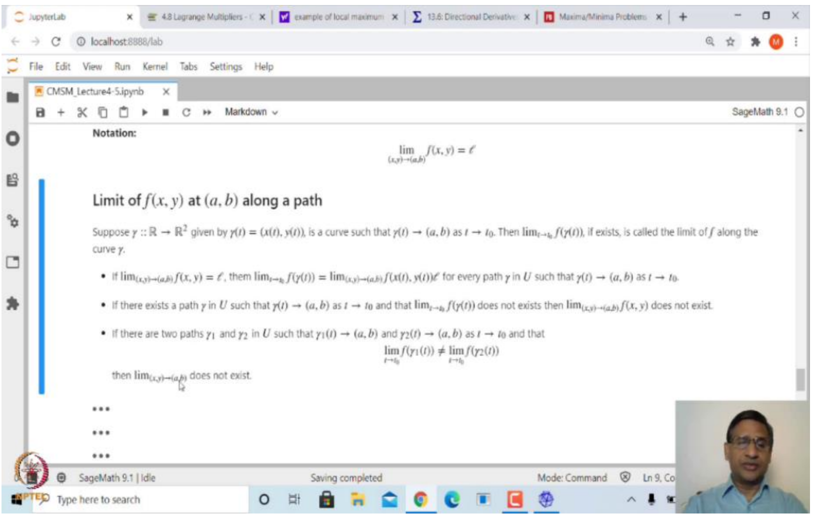

In case you are looking at a function in R2 in the case of one variable you had the notion of left-hand limit and right-hand limit, but in the case of two variables you do not have left-hand direction right-hand direction rather you can talk about limit along any curve. any curve inside U.

In case you have a curve gamma which is gamma t is equal to (x(t), y(t)), where x(t) and y(t) are x-coordinate and y-coordinate of this curve, then and this curve suppose gamma t goes to (a, b) as t goes to t not, then limit of f of gamma t as t goes to t not if that exists we say that this limit of this function along this curve gamma exists at t, not that is at the point a comma b.

And you can show that in case the limit exists of the function the limit along any curve will exist, but in case the limit does not exist what does it mean? That means, that you can find some curve gamma t in U such that gamma t goes to (a, b) as t goes to t not, but the limit of f of gamma t that is limit of f along the curve does not exist.

Or you can find two curves gamma 1 and gamma 2 both converging to (a, b) and such that limit of gamma 1 limit of f along with gamma 1 and limit of f along with gamma 2 if they exist but they are different then also the limit of the function does not exist.

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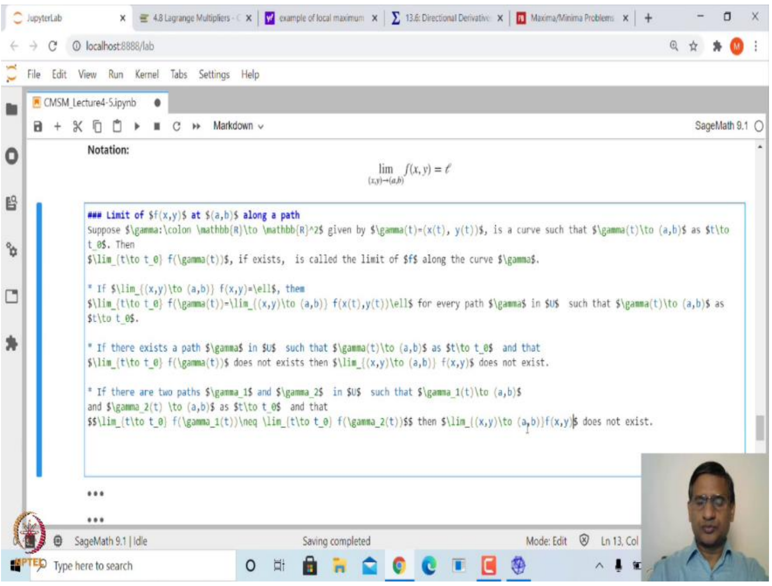

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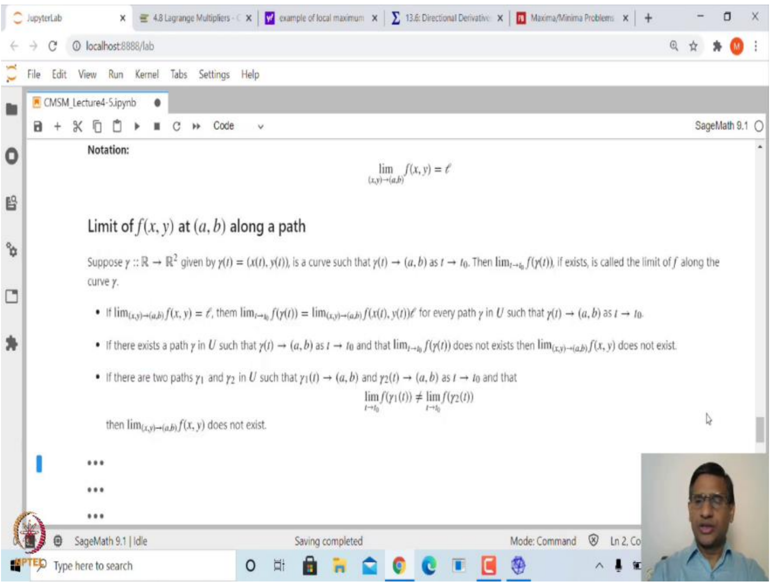

These are the things if you want to define the limit you can also define a limit along a path.

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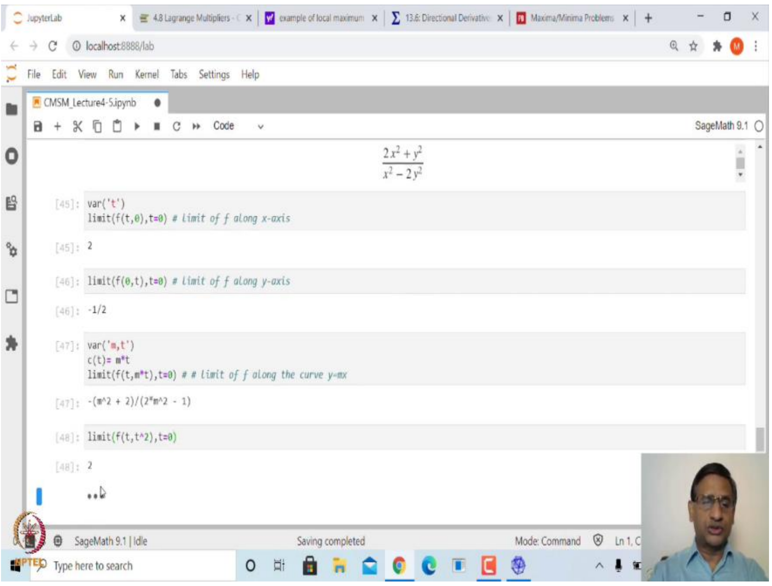

Let us take an example to suppose $f(x,y) = \frac{2x^2+y^2}{x^2 - 2y^2}$. Now, if you look at this function; let me just show you what is this function f(x,y) is the function and suppose we put $y = 0$; that means, and the limit of (f(t), 0) at t goes to 0. That means, we are taking the $y = 0$ in this is nothing, but the x-axis. We are saying we are finding the limit of this function along the x-axis at (0, 0); this is equal to 2, whereas if I want to find out the limit of this about x equal to 0 that is along the y-axis this limit is minus a half.

For this function, the limit along the x-axis is 2, the limit along the y-axis is minus half, therefore, this function does not have a limit at (0, 0). Any other place will have a limit other than the origin.

If you look at let us, say curve $y = mt$ as t goes to 0, this mt will go to 0. Then if you look at $y = mt$, this curve is a line along with $y = mx$ the limit of f along the line $y = mx$, this limit depends upon m.

If you take various values of m, you will get different limits. That is another way of saying that the limit of this function at origin does not exist. If you take, for example, the limit along y equal to t square this limit is 2.

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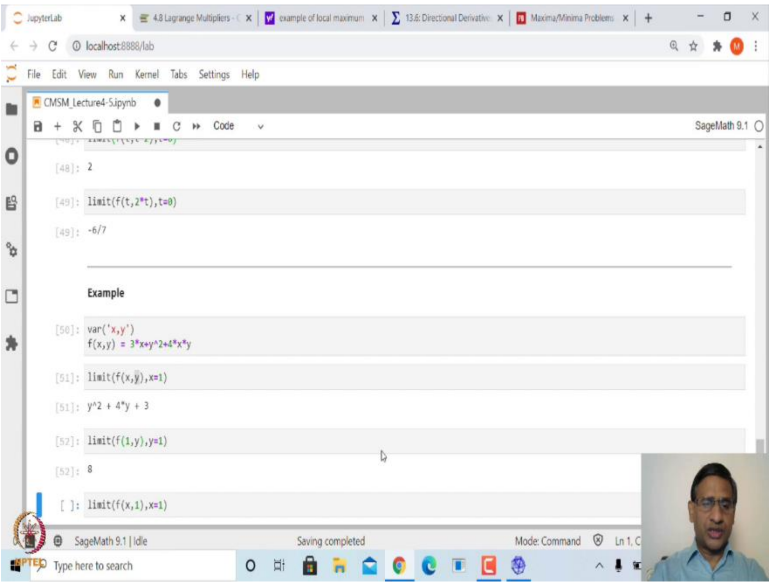

Whereas, if you take the limit around $y = 2t$, this limit is $\frac{-6}{7}$. Again, you can see that along various curves this limit of the function is different and hence the limit does not exist.

Similarly, you can take another example: let us say $f(x) = 3x + y^2 + 4xy$. In this case, first, we can find the limit at x equal to 1 and for any value y. That is the function of y and again we can find the limit of this at y equal to 1.

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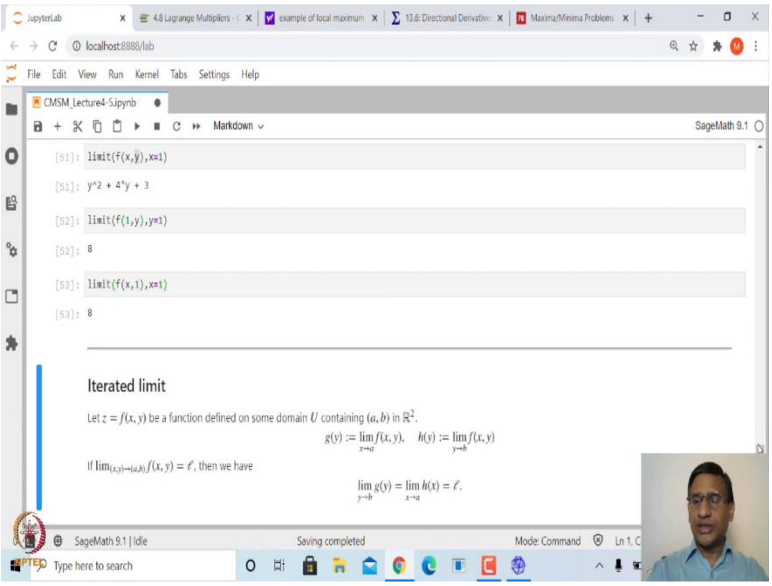

In particular, we are finding the limit first with respect to x and then with respect to y. You could do first with respect to y and then with respect to x and then c you will get the same thing. For example, here this is the same thing if I want to first find with respect to x and then with respect to y is same as the first find with respect to y and then with respect to x.

This is what is called the iterative limit. how does one define it? We just now saw that in case we find a limit along x at any y that is the function of y. We can denote that by g(y). Similarly, the limit when we find along y when y goes to b at any x that will be a function of x.

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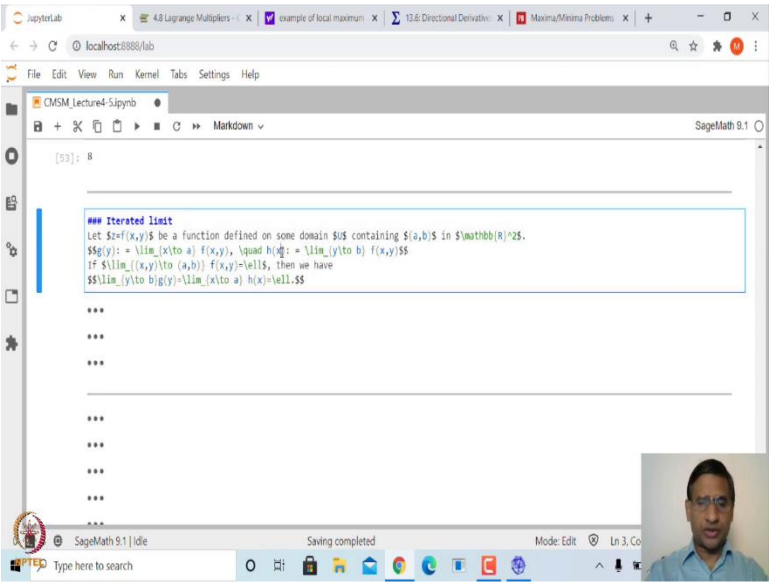

Then suppose the limit of f(x) at $x = a$, $y = b$ that is (x, y) going to (a, b) if that exists and is equal to l, then you can show that the limit of g(y) as y goes to b will be l and also limit of h(x) as x goes to a will be l.

But, this limit of g(y) as y goes to b is nothing, but the limit of f(x, y) as x goes to a and followed by the limit of f as y goes to b. These two are what is called an iterated limit and in case the limit exists, it does not matter you can find this limit in any order.

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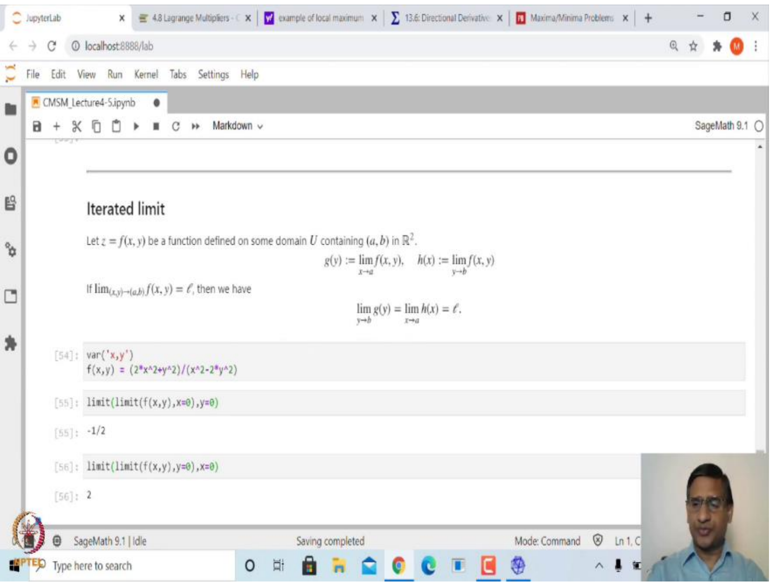

For example, if you look at this function the function which we looked at earlier $f(x, y) = \frac{2x^2+y^2}{x^2-2y^2}$. In this case, if you try to find the iterated limit at x equal to 0, y equal to 0, that is the origin.

When we take the limit first with respect to x and then with respect to y in this case, this limit is minus half whereas if I take first with respect to y and then with respect to x this limit is 2.

Again, you can see here these two iterated limits are not the same. That again proves that the limit of this function at 0, does not exist because if the limit exists then both these iterated limits should have been the same. Since this iterated limit is not the same, the limit cannot exist.

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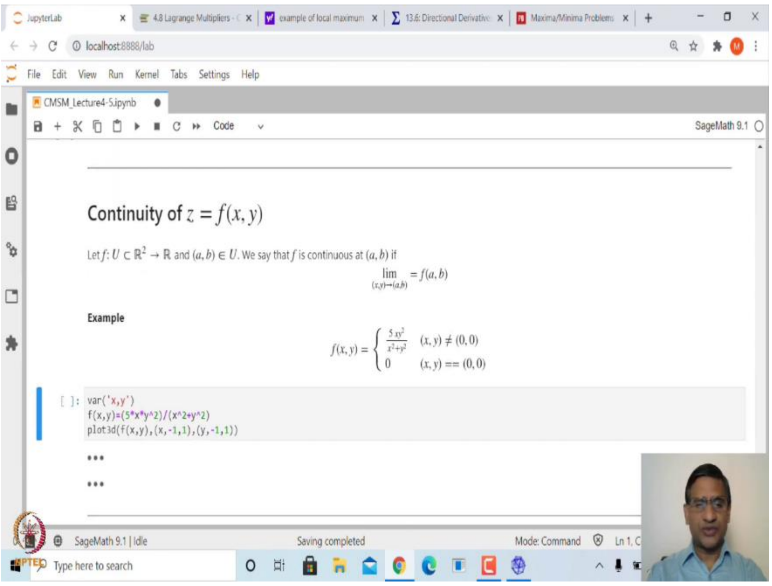

Next, let us look at if you have a function z is equal to f(x, y) how do we define continuity of this function at any point (a, b)? for that suppose f is a function defined on let us say U which is a subset of $R_2$to R and then you need this point at which you want to talk about the continuity inside the domain and function should be defined at that in that domain.

Then we say that the f is continuous at (a, b) if the limit of f(x, y) at a, b exists and the limit is equal to the value of the function, that is the standard deviation. Of course, one can also define limit and continuity using a sequential approach. You can talk about the sequence in R_2 and general in R_3, in R_n etcetera and its convergence.

We say that function f(x, y) at (a, b) is continuous. If you take any sequence x and y and converge to (a, b) then f (x, y) should converge to f(a, b). That can also be taken as a definition and solve all the problems using that. Let us take an example. suppose you have f(x, y) is equal to let us say $\frac{(5xy)^2}{x^2+y^2}$.

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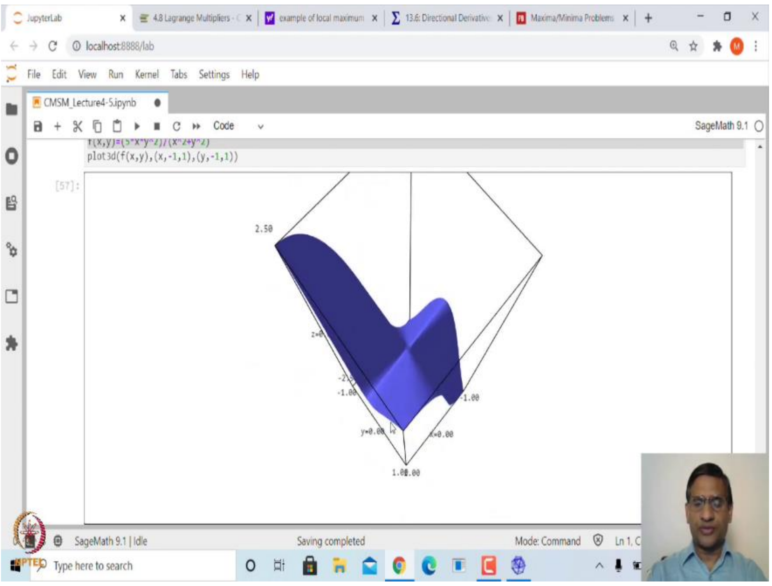

Now, if you look at this graph of this function suppose we plot a graph of this function then this graph looks like this. at the origin you can see here there is hardly any there is no break actually though the function has some strange behaviour at the origin, this function has to be continuous at (0, 0).

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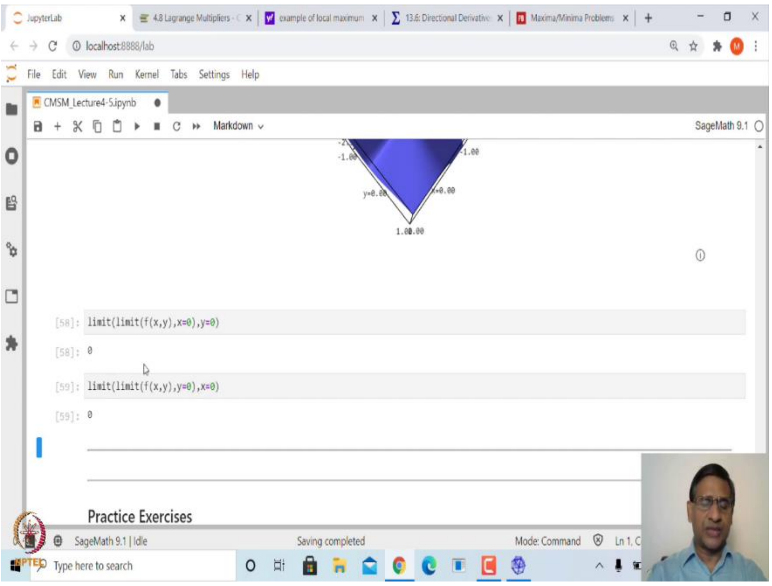

Let us look at suppose we want to find the iterated limit this limit is 0, similar to the other one this is 0. First, it does not matter whether you find a limit with respect to x equal to 0 first and then y equal to 0 or you first find with respect to y equal to 0 and then find x equal to 0 both are the same.

When you want to prove the existence of the limit the best thing is to use the $\epsilon - \delta$ definition. This and $\epsilon - \delta$ working with an $\epsilon - \delta$ definition or given an ϵ finding delta using any software may not be all that convenient. Here what we are trying to do is we are just trying to find out the limit and then say whether they are equal.

In this case, you can see here that these two limits are the same and the limit is 0 which is the value of the function at the origin therefore, this function is continuous. You can look at other functions and all these definitions in 2 variables can be extended to 3 variables and n variables.

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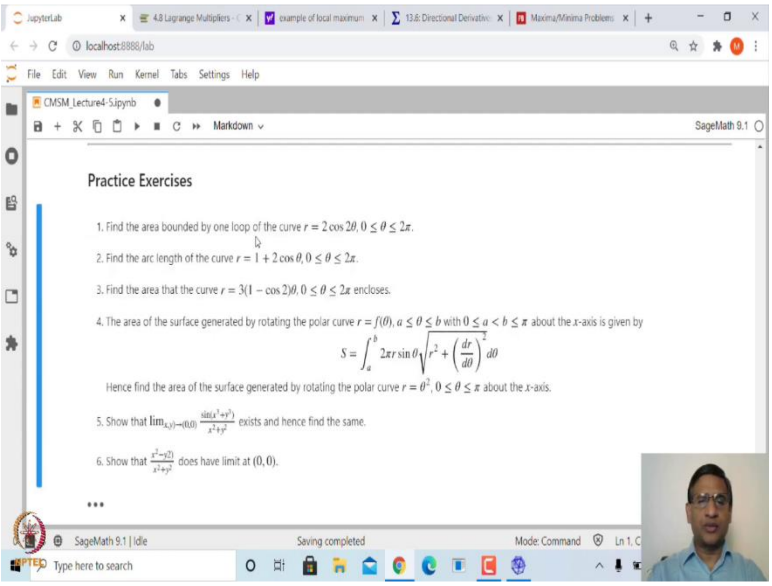

Let me leave you with some simple exercises that you can try. These are the few exercises. First, find the area bounded by one loop of this polar curve. Similarly, find the arc length of this particular curve between 0 to 2π.

Find the area that curve this encloses and find the area of the surface generated by rotating a polar curve $r = f(\theta)$, theta varies between a and b and you need to assume that a is strictly less than b and both lies between 0 and π and in that case, this area is given by integral of $2\pi r sin(\theta)\sqrt{r^2 + (\frac{dr}{d\theta})^2}$, $d\theta$and these limits are between a and b.

Again you can obtain this formula of area of the surface from the Cartesian coordinate; you already know the Cartesian coordinate formula. you just use a change of variable, you will get the answer and the next two problems are on limits. Try to show that the limit of this function exists, but the limit of this function does not exist.

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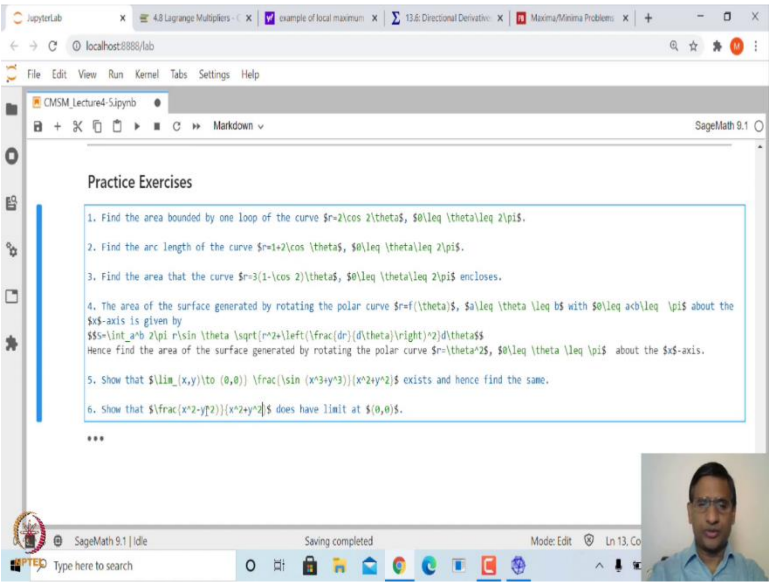

These are some exercises.

Thank you very much. I will see you in the next class and we will look at partial derivatives and various concepts related to partial derivatives.