Computational Mathematics with SageMath
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Lecture – 22
Improper Integral using SageMath

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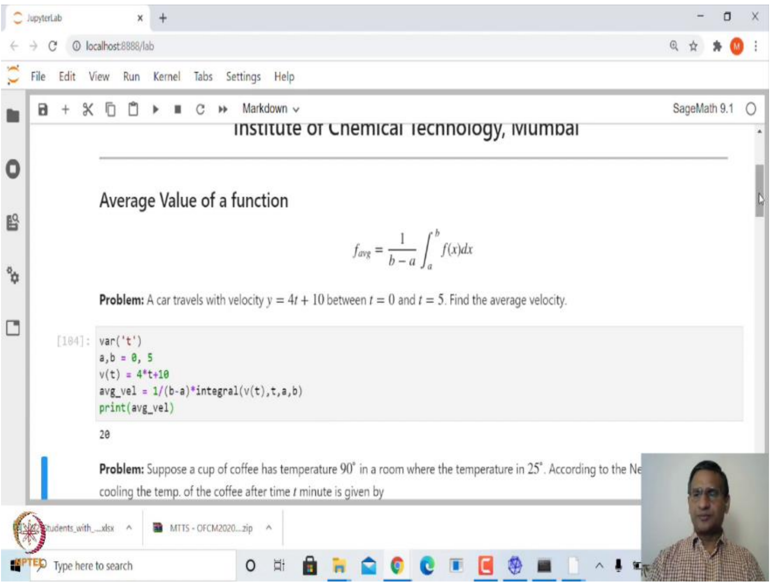

Welcome to the 22nd lecture on Computational Mathematics with SageMath. In the last lecture, we looked at the computing integral of various functions using SageMath, including some numerical integral wherever it was necessary. In this lecture, we will look at two things; one we will look at what is the average value of a function, and second, we will look at how to find improper integrals.

Let us get started. If you look at the average value of the function if you are given a function f(x) is equal to some function y equal to f(x) some function in the interval [a, b] then how to find the average value? If you recall the average of a set of values or set of numbers, one finds using adding all of them and dividing by the total number of the points.

The idea of finding the average value of the integral is also very similar. Instead of adding these values in the case of individual points, you integrate the function; that means, find the area in that interval and divide by the length of the interval what we called the average value of the function. So, this is how we calculate. Of course, we need a function to be Riemann integrable, then only this integral will make sense, we can find the area.

For example, if I look at a simple problem, a car traveling with velocity $y = 4t + 10$ between t equal to 0 and t equal to 5. If you have to find the average velocity, then what will you do? You will just integrate this function between 0 to 5 and divide by the length of the time interval which is 5.

Let us calculate this using Sage. It is quite simple. Define t as a variable a and b as 0 to 5. This is the velocity. And the average velocity is 1 upon b minus a into the integral of v(t) with respect to t from a to b. And if you print, this is what you get. 20 is the average velocity when the car is traveling using this velocity.

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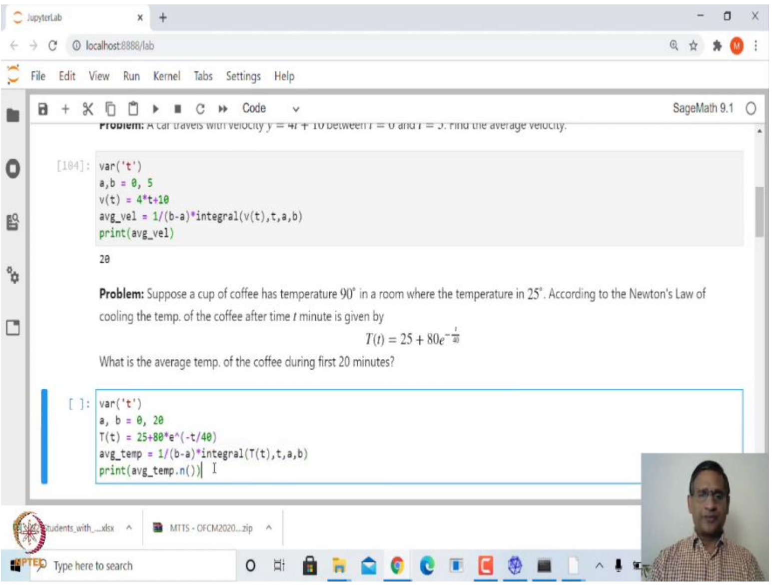

Similarly, let us look at another problem. Suppose you have a room in which the temperature is 25 degrees Celsius. And keep a cup of coffee at 90 degrees Celsius. Then, according to Newton's law of cooling, suppose at t minute, this temperature of the coffee is given by 25 plus 80 into e to the power minus t by 40.

Then what is the average temperature of the coffee during the first 20 minutes? How do we calculate? Again, it is quite simple. If you look at this cooling function as a time defined as capital T and then calculate the average value of this capital T between 0 to 20.

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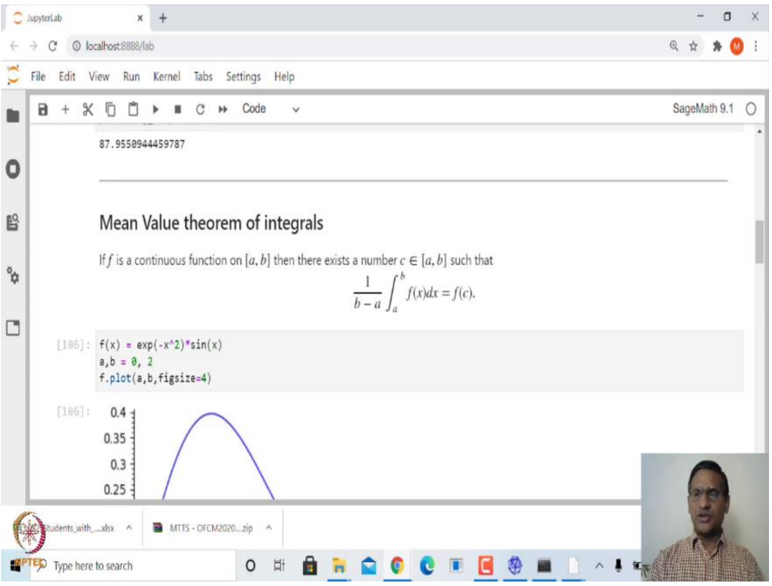

That is 87.955 is the average temperature which is quite high. So, it simply means that within 20 minutes if you drink this coffee, it will be hot. Similarly, there is another important result on the average value of the function.

It says that if you look at the average value of the function in the interval a to b, then this is equal to f(c) that is it attained at some value c between a and b. This is called the mean value theorem of integral. Let us look at geometrically what it means. Suppose you have a function $f(x) = sin(x)e^{-x^2}$ in the interval 0 to 2.

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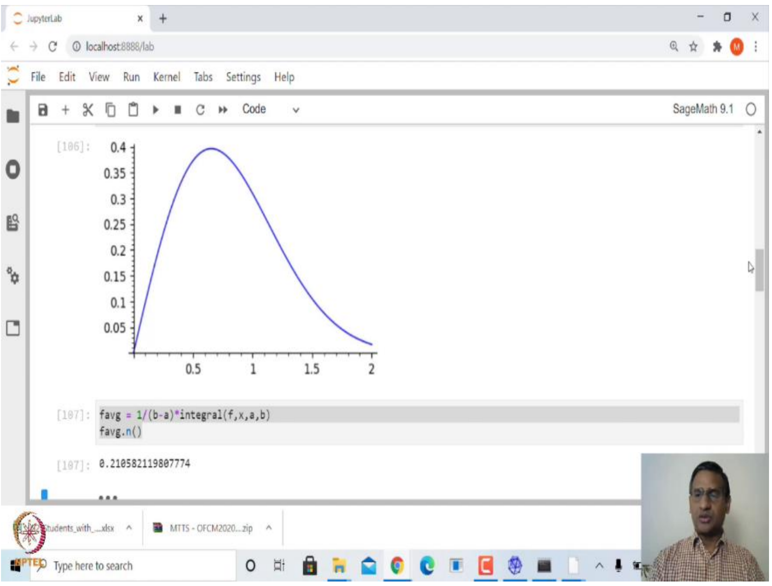

Let us plot the graph of this function. That is the graph of this function $sin(x)e^{-x^2}$. And if you calculate the average value, this is equal to 0.210. If you draw a horizontal line at 0.210, then it will intersect this function at two points in this case, so that is the meaning of this mean value theorem of integral.

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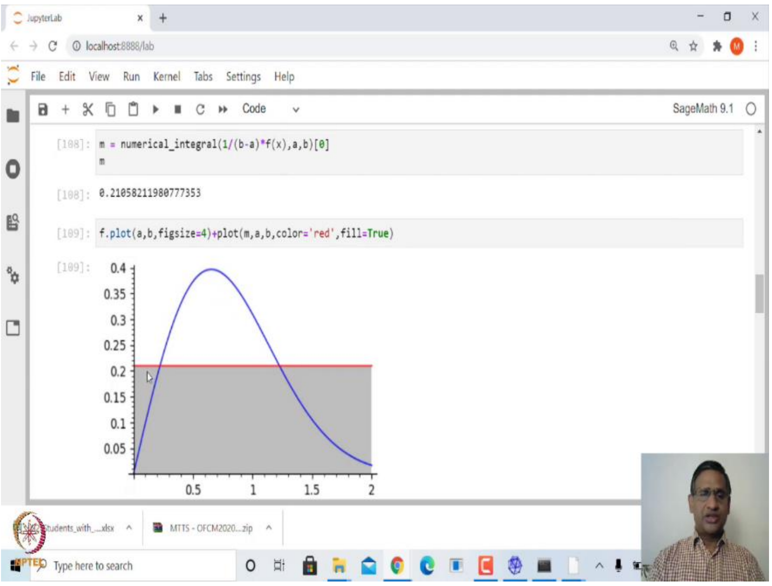

Let me store that value in m. Though, I have written here a numerical integral. In this example, you do not need a numerical integral. But if you have a complicated function, you need to calculate the numerical integral.

Now, let us look at what will be the intersection of the function along with this. First, let us plot the graph of the function along with the horizontal line y is equal to m with red. You can see here, this is the rectangle of height m in this interval, that will intersect this horizontal line and will intersect at two points. And we can find out this using the find root command.

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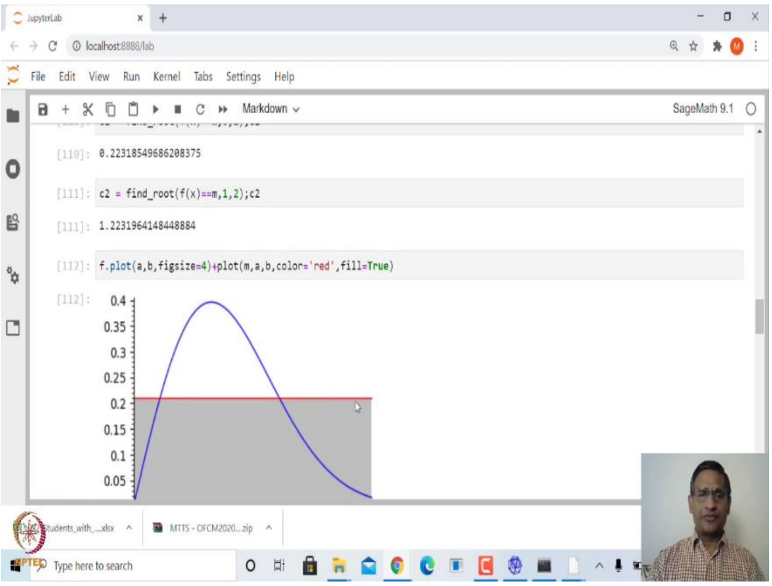

This find root of f(x) is equal to m between 0 to 1. You can see that there is 1 root between 0 to 1. There is another root between 1 and 2. The first root I am calling at c1 which is 0.2231. And the second root is c2 which is 1.2231. And if you try to plot both together, then this we have already seen. This is how it looks. That is the geometric meaning of the average value of the integral, which is attained at some point of the function. This is called the mean value theorem of integral.

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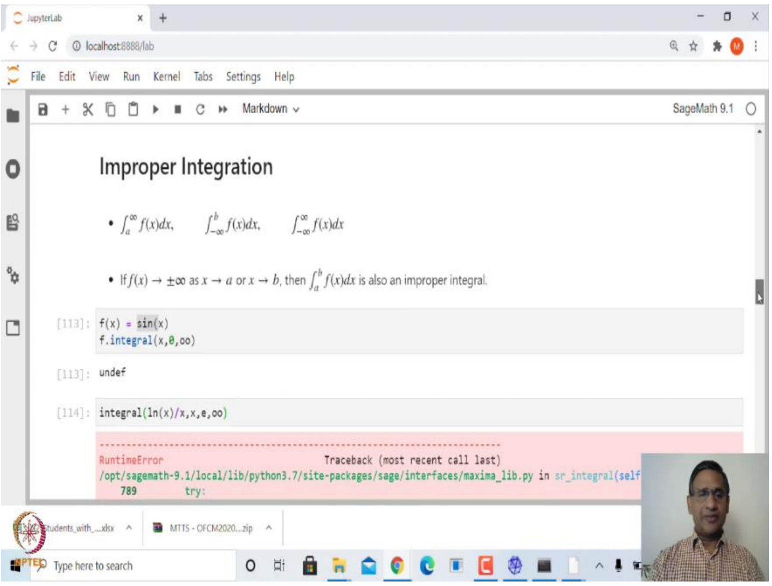

Now, let us look at improper integrals. If you have an integral f(x) from a to ∞, or -∞ to b, a and b are a finite number or -∞ to ∞, so these kinds of integrals are called improper integrals. Improper integrals in this case are ones in which either one or both the limits are ∞ or - ∞. You can also have a situation where the function at some intermediate point a, or at the endpoint may shoot to plus-minus infinity, such integrals are also known as improper integral.

One can define when this kind of integral exists, one can define this integral using limit. For example, if I want to define the integral of f(x) from a to ∞ one finds an integral from a to t f(x) dx and takes the limit of that as t goes to ∞. If that limit exists, we say that this integral is convergent, and the value of that limit is the value of this integral.

Sage can find improper integrals, but not everything, of course, has limitations. For example, if you look at f(x) = sin(x), and if you try to find an indefinite integral from 0 to ∞, then it says it is undefined. For example, in this case, because this integral of sin from 0 to ∞ will go to ∞.

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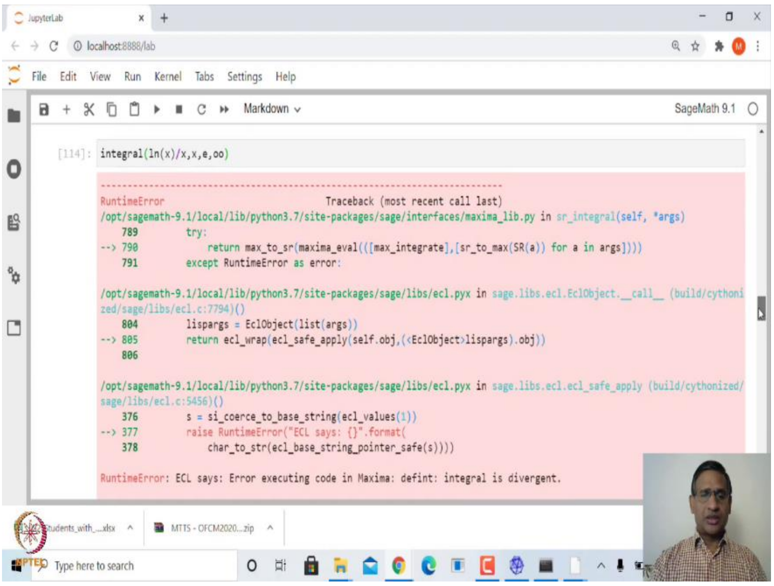

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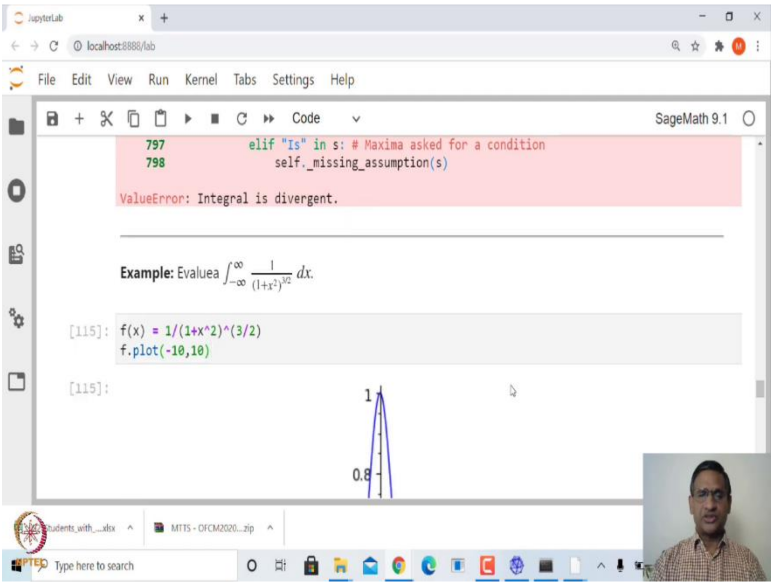

Whereas, if you look at the integral of log(x) upon x from e to ∞. Then again, it says that if you come down with this error; they say that the integral is divergent. This in case the integral does not exist it may give you integral is divergent or it will say undefined. Let us look at one example. So, if you want to find this indefinite integral from -∞ to ∞ of $\frac{1}{(1+x^2)^{\frac{3}{2}}}$, then what do we get?

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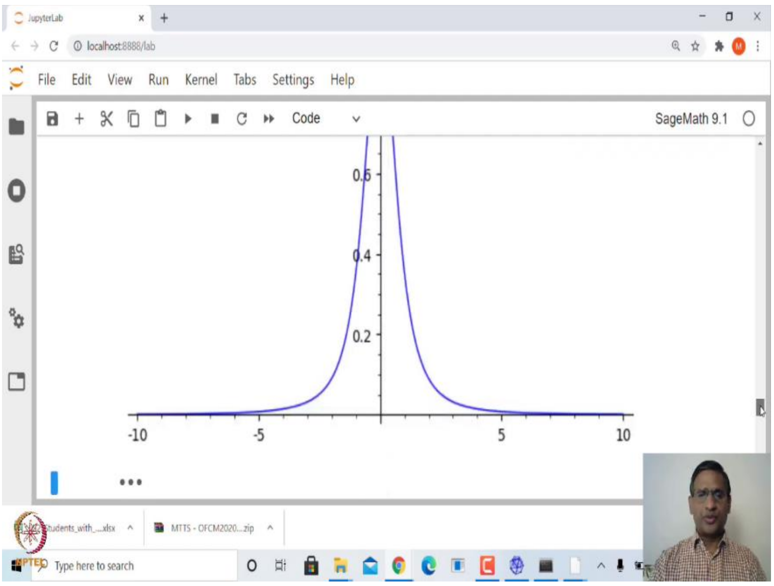

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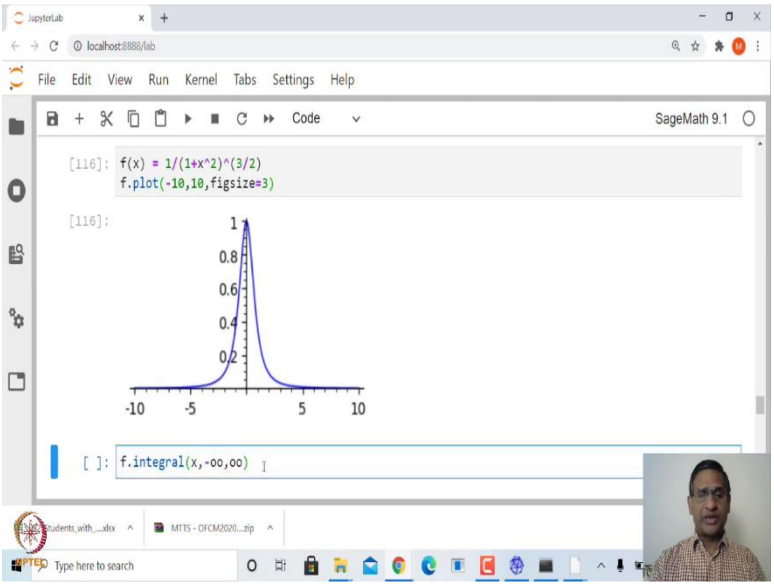

First, let us plot a graph of this function between - 10 and 10. If you plot this graph, this is how the function looks like. Let me make the figure size small so that it will be visible properly fig size is equal to let us say 3. You can see here after a certain stage; this function is almost very close to 0.

What matters is the area in this finite domain rest can be made as small as possible. So that gives you an idea that this indefinite improper integral exists. And if you try to find out, this improper integral the answer is 2. So, this is convergent.

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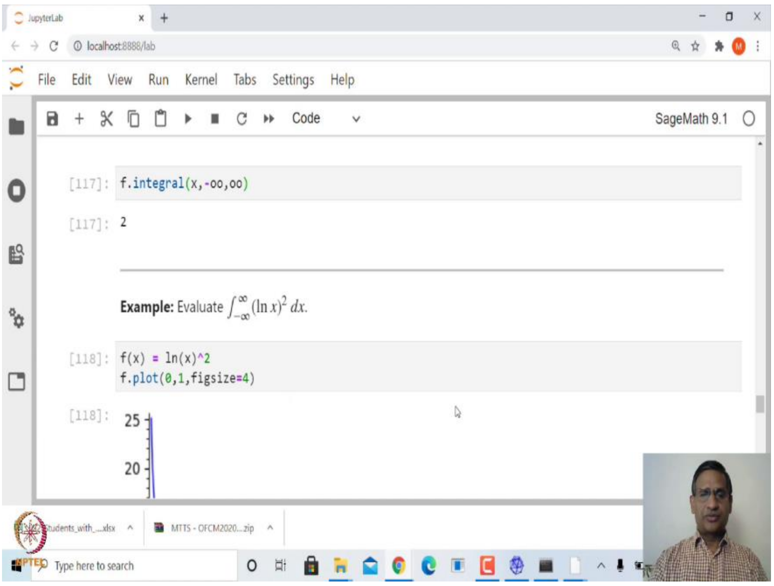

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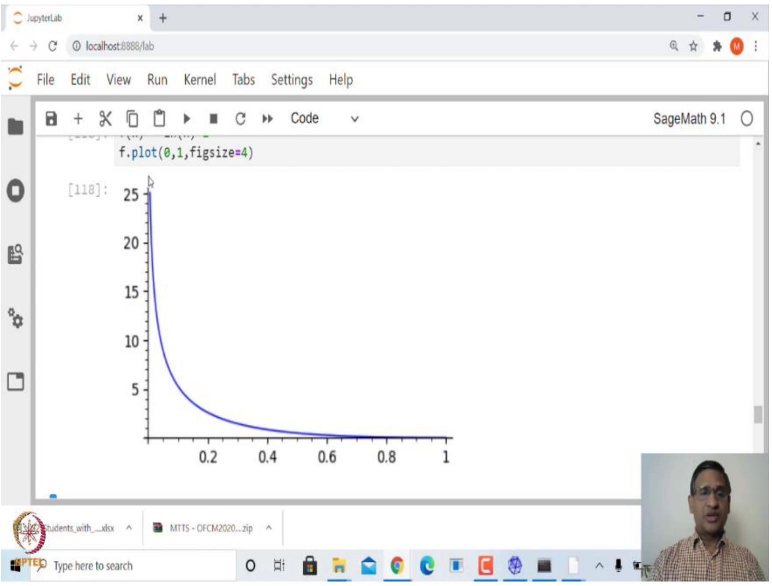

Similarly, if you try to look at the integral of log(x) ^{2} from -∞ to ∞, first try to plot a graph of log(x) ^{2}. If you take log(x) ^{2} from 0 to 1, it looks like it. At 0 it goes to ∞, but that log very close to 0 will be quite high in negative, very small quantity in ∞. And then it is we who take the square of that will be a very large positive quantity.

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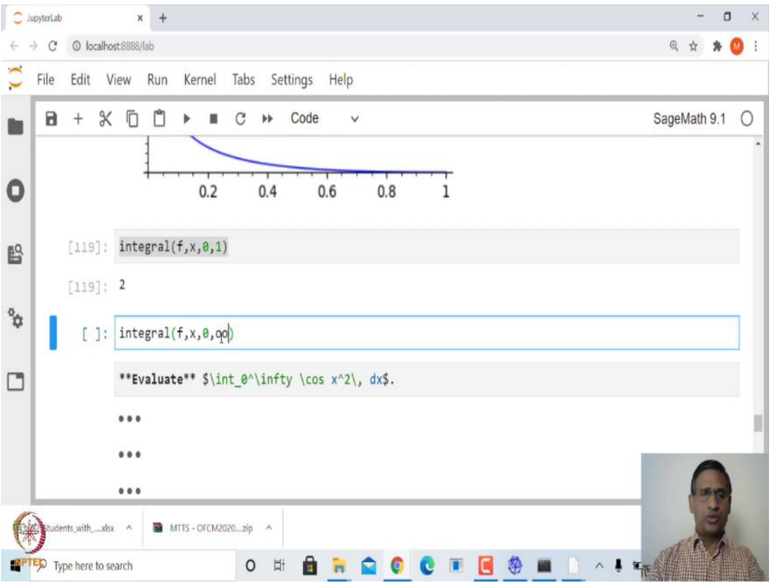

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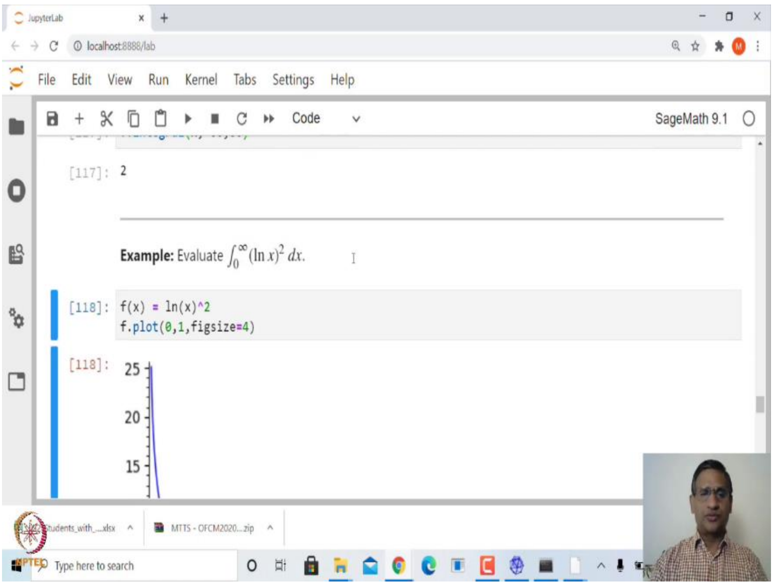

If you try to find the integral of f between 0 and 1, this is equal to 2. Whereas let us find between 0 to ∞, not from - ∞ to ∞, from 0 to ∞, and then see what you get?

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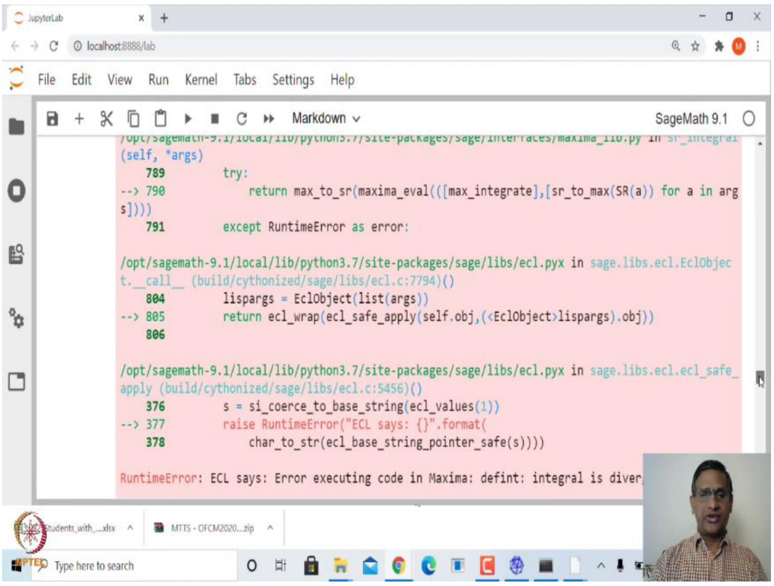

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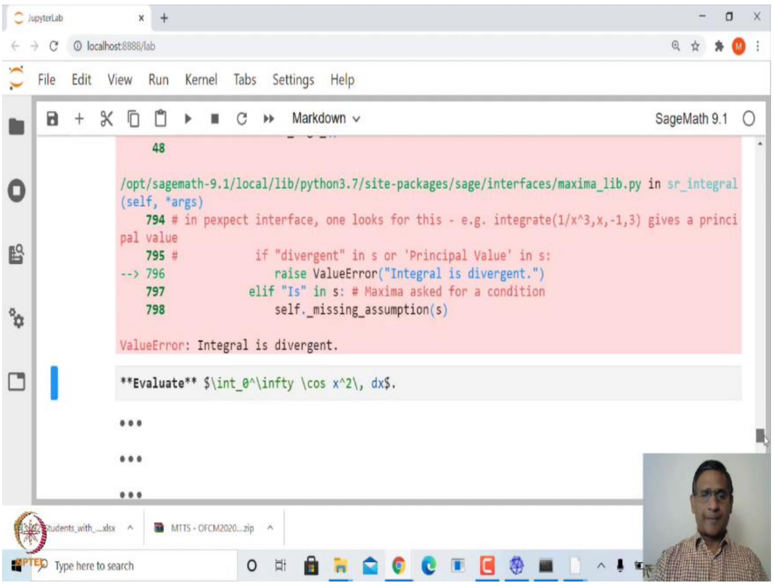

Let us calculate from 0 to ∞. Again, this says it is divergent. Whereas, in 0 to 1, it is finite.

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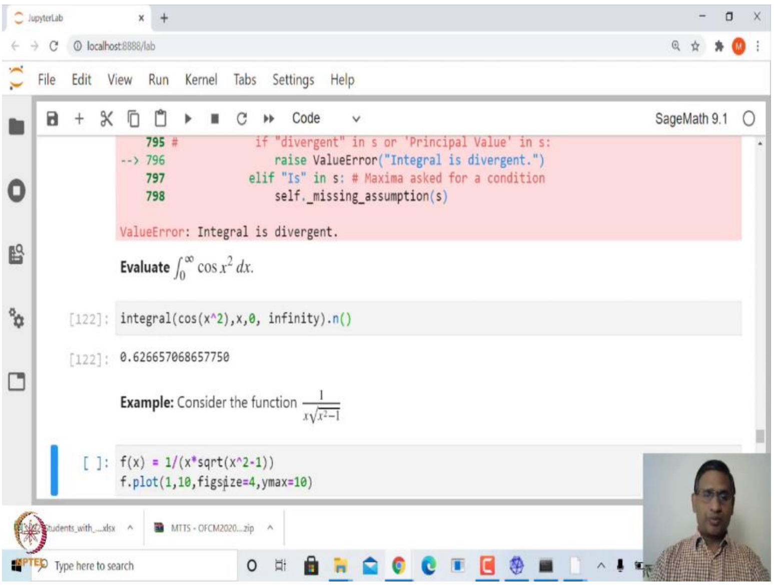

Let us take another example, this integral of cos(x) square from 0 to ∞. Again, this value is equal to $\frac{\sqrt{2}*\sqrt{\pi}}{4}$. And if you try to find the numerical value of this, then dot n, this will give you 0.6266, so that is the numerical value. This is convergent, and its value is 0.62.

Similarly, if I look at this function, for example, it is not defined at one. And it is also not defined as - 0 and - 1.

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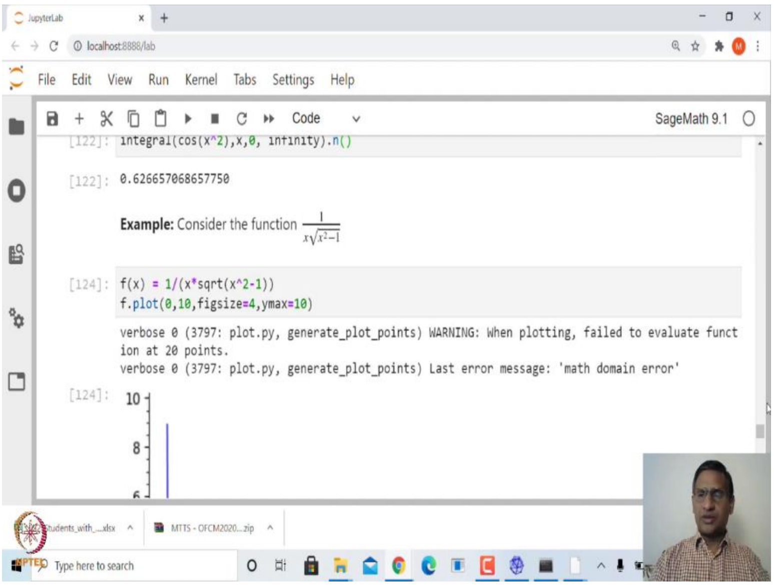

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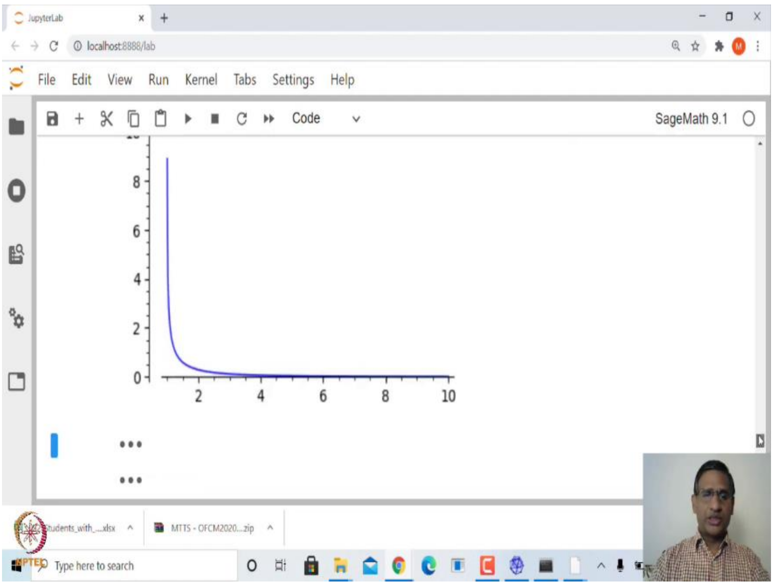

If you try to plot the graph of this function first let us try to plot between 0 to 10, then you can see here it gives you some error that there is domain error at a point very close to 0 and 1 as a problem. So, this is how the graph looks like.

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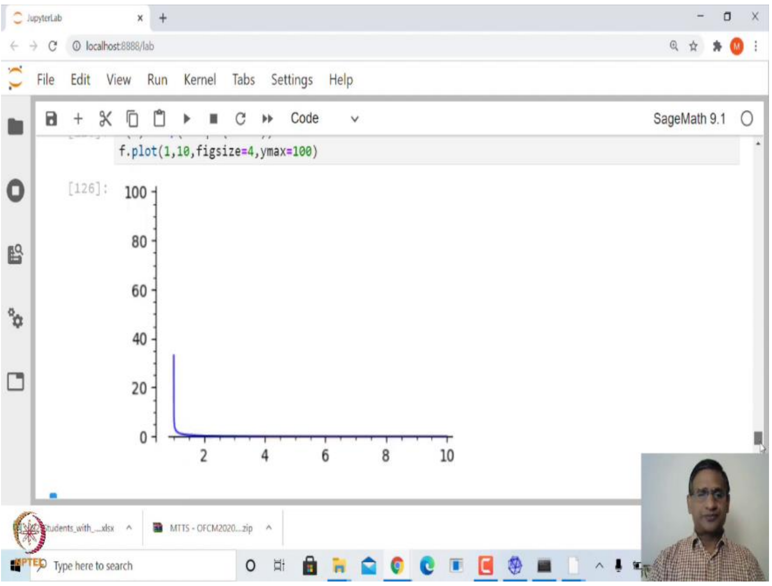

For example, if you change the domain, let us say 1 to 10, then at 1 again it goes to infinity. I have restricted the y range to 10, you can make it let us say 100. In that case, again this is how you can see.

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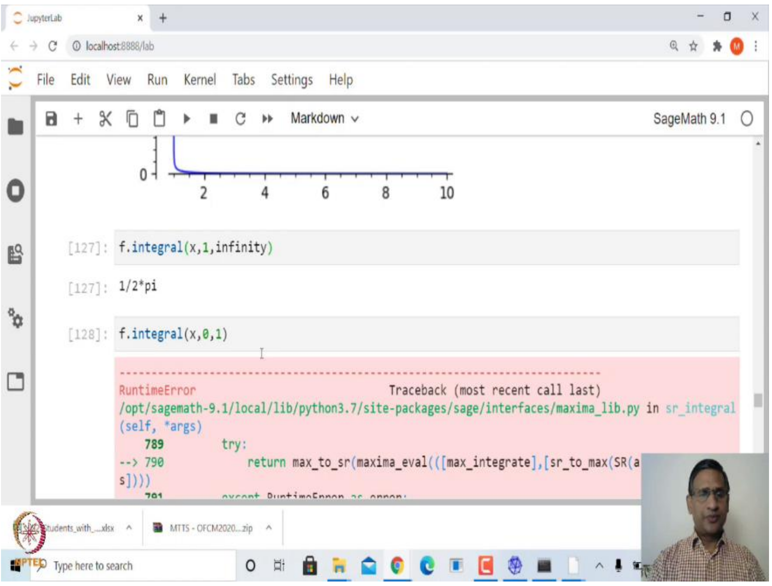

If you try to find the integral of this, what is the function? The function is 1 upon x into square root x squared minus 1. The integral of this between 1 to infinity this value is pi by 2. And the same integral between 0 to 1 does not exist.

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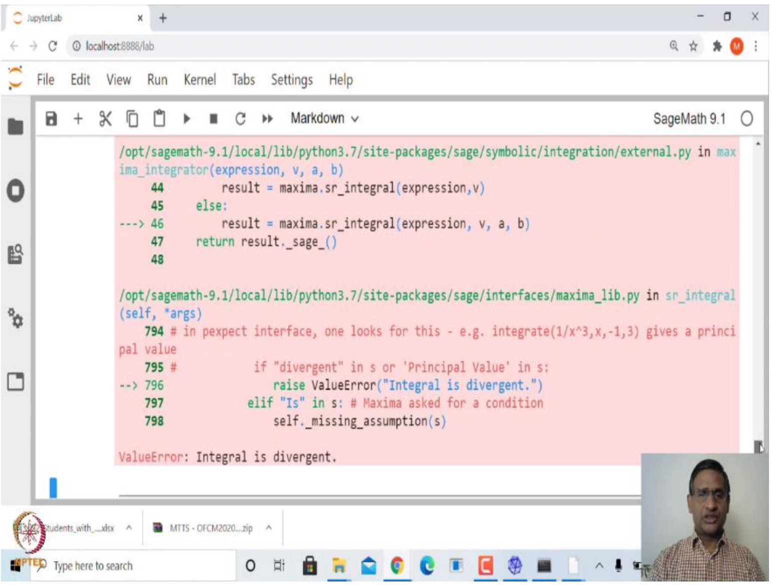

You can see here again this integral is an improper integral because it will be hot t will be hot at 1, it has the function goes to infinity and at 0; the function goes to infinity. It is not defined there. These are called improper integrals. And many times, if the improper integral exists, then Sage will be able to find it.

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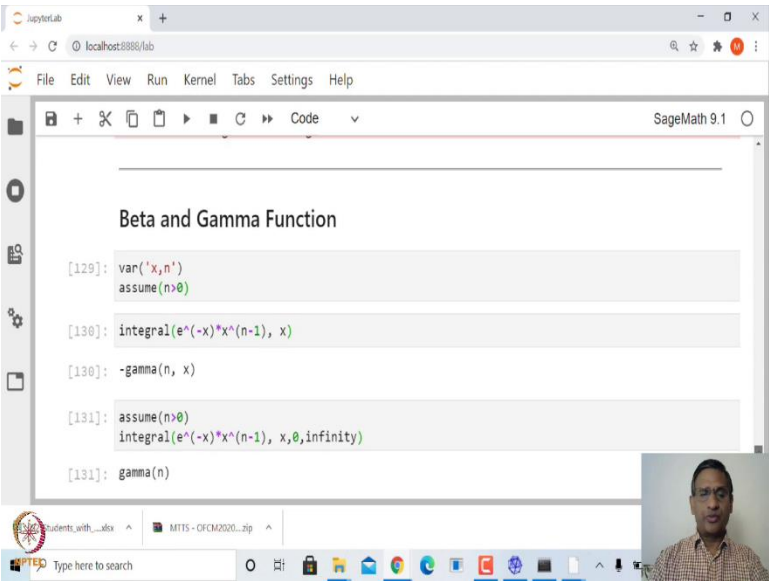

One application of improper integral you must have seen is called beta-gamma function. How does one define gamma function? So, suppose you define x and n as variables, and we will think of an integer or it can be any positive number. If you take the integral of $e^{-x}x^{(n-1)}$ x, this is minus gamma n, x. Whereas, if you assume n to be positive and calculate this integral, this is called the gamma of n.

The gamma of n is defined as integral from 0 to ∞ of the function $e^{-x}x^{(n-1)}$. This is called the gamma function, gamma of n. And this is actually this one can think of as a generalization of the factorial function.

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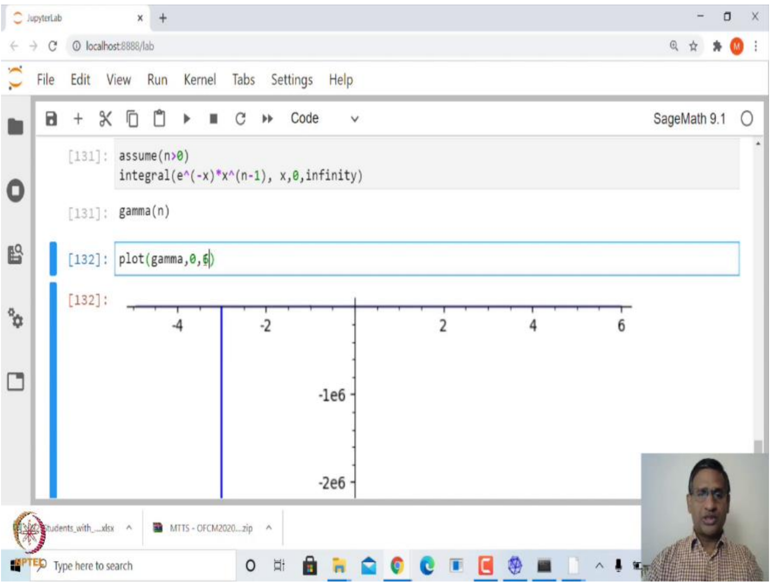

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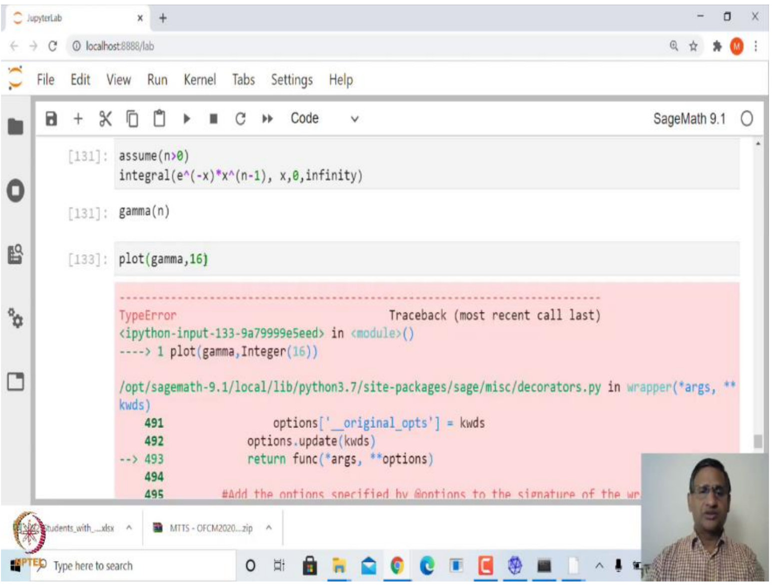

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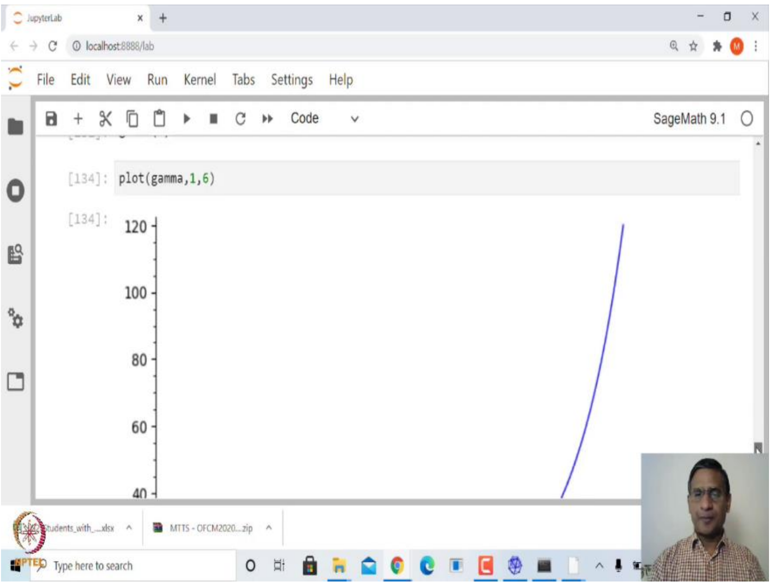

For example, if you look at what it is if you try to plot the gamma function, this is how it looks like. Let us plot in the smaller domain. Let us say 1 to 6, the gamma function is inbuilt in SageMath though we defined.

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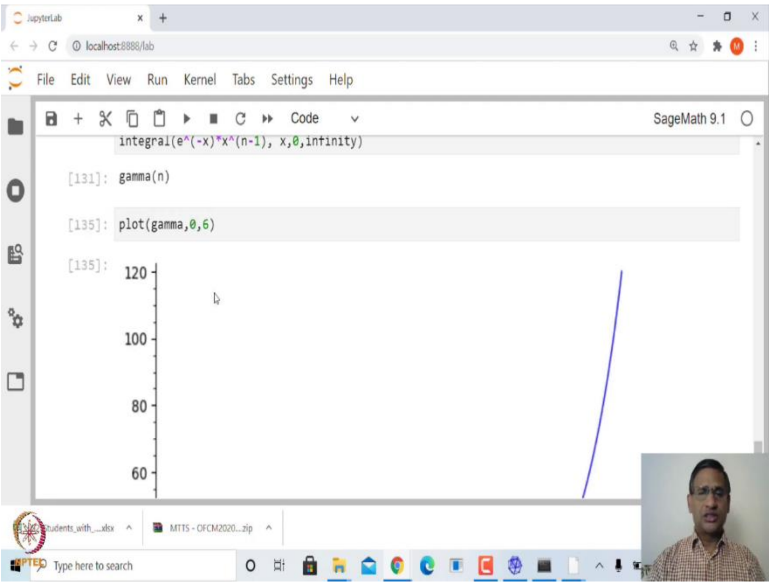

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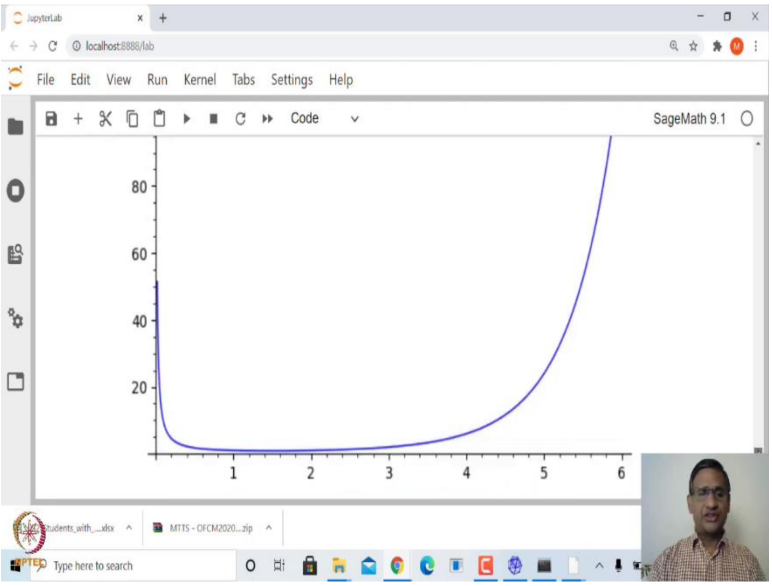

Here it is. There is no problem. Whereas, if you try to define between 0 to 6, then you see that at 0, it will go to somewhat infinity.

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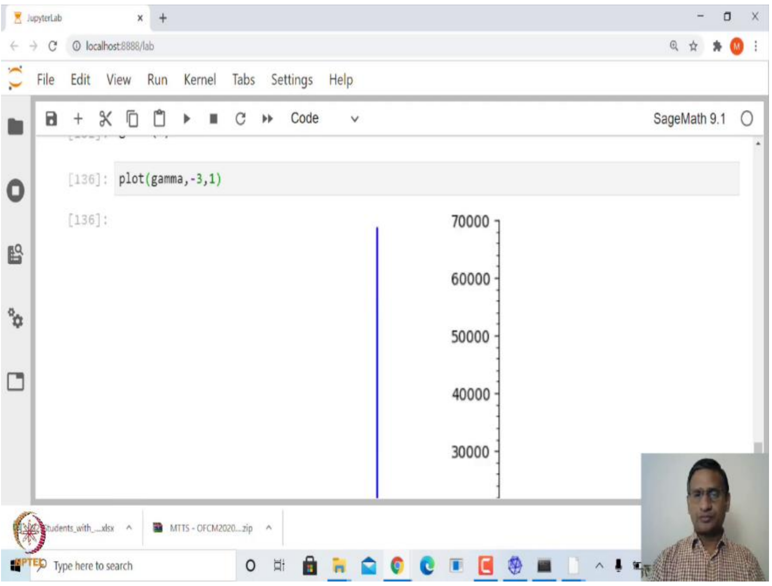

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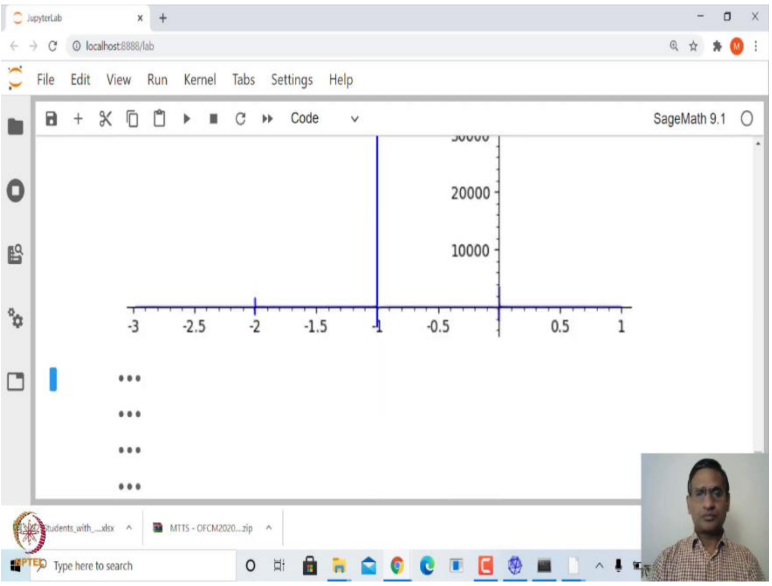

If you took a look at from minus 3 to let us, say 1, then you will see that at negative integers, this gamma function is not defined.

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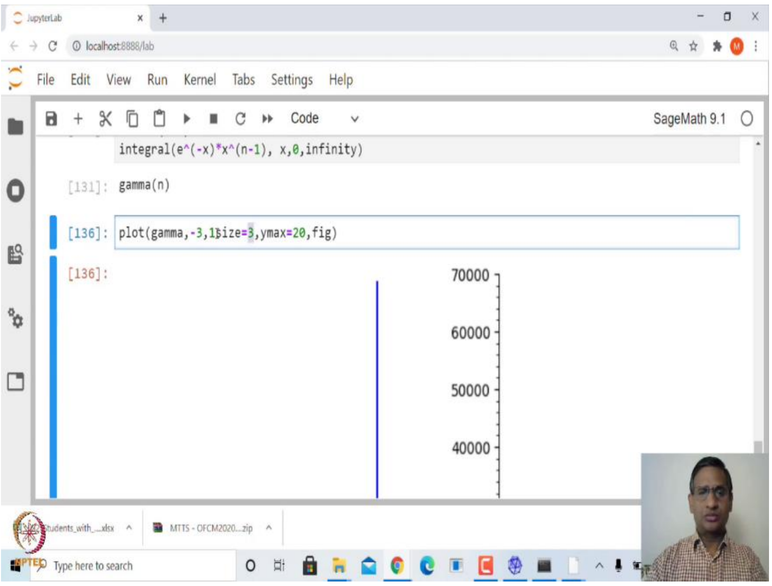

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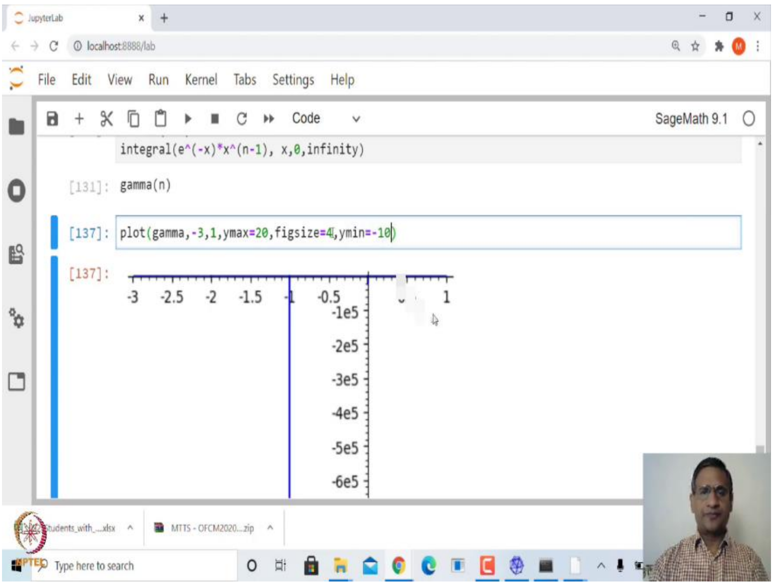

If you try to restrict the y range, let me say y max is equal to say 20, and let me also say fig size is equal to 4. y min is equal to let us say -10.

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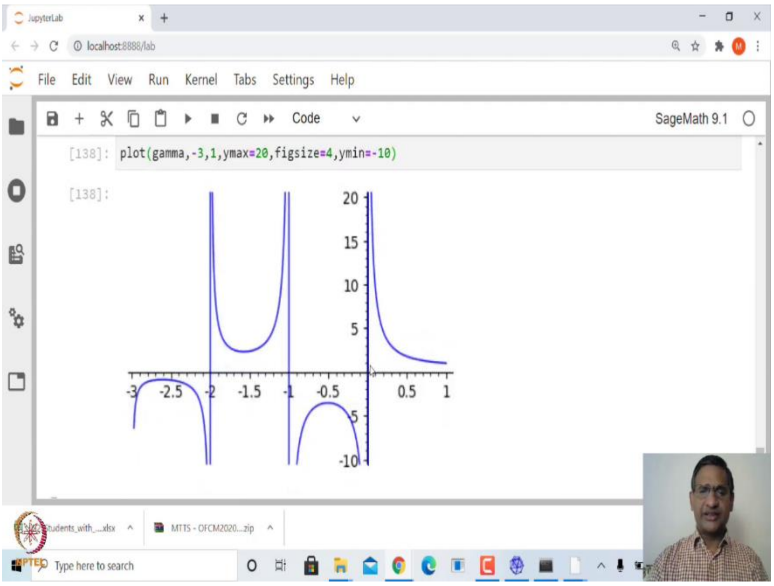

Now we can see here except at negative integers and 0, the gamma function is defined.

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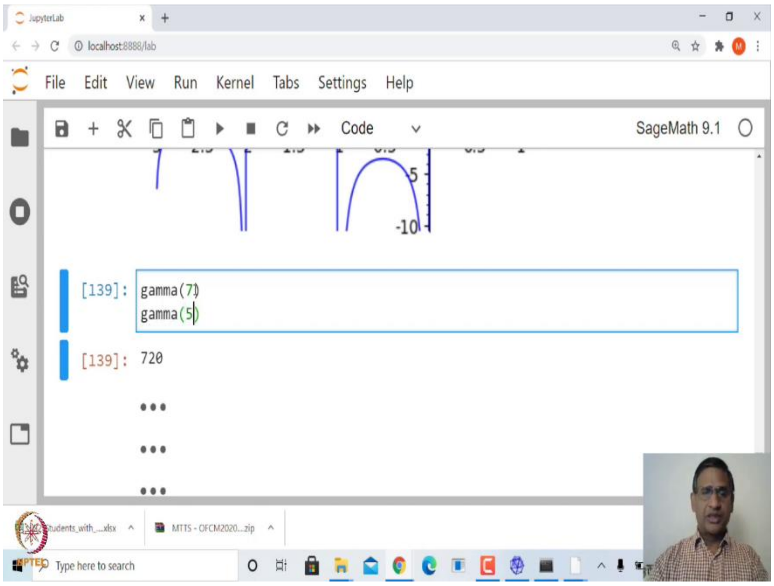

If you look at what is the value of gamma 7, what is the gamma value of gamma 7 this is 720 which is a factorial of 6. If I look at what is the value of gamma, let us say gamma 5 this should be 24 which is 4 factorial. So, if n is an integer, then gamma n will be n minus 1 factorial. You can verify this.

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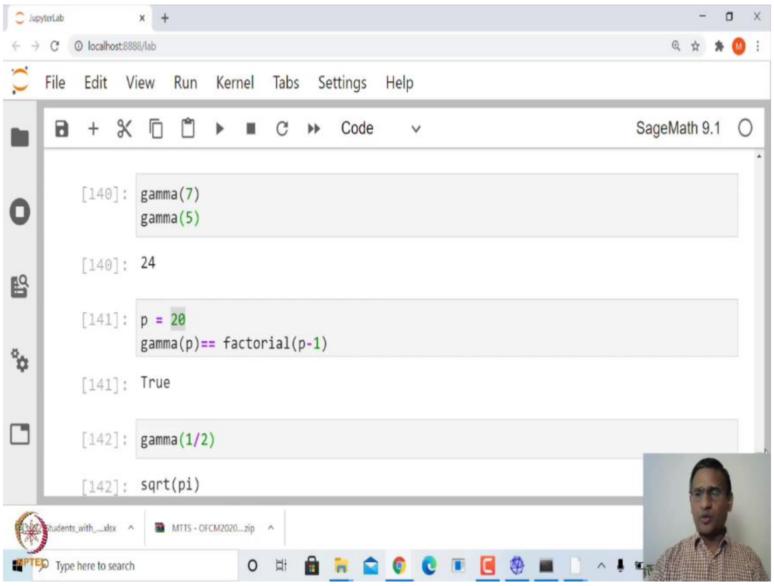

Gamma of p is equal to factorial of p minus 1 for any value of any integer positive integer p. That is why we say that the gamma function is a generalization of the factorial function.

If you look at what is the gamma of the half, this is square root pi. What will be the square root π? it will be n is equal to half.

$e^{x}x^{-\frac{1}{2}}$ that is 1 upon square root of π. $\frac{e^{-x}}{\sqrt{x}}$ if you try to integrate which means 0 and ∞, this you will get as square root π that is a very important approximation.

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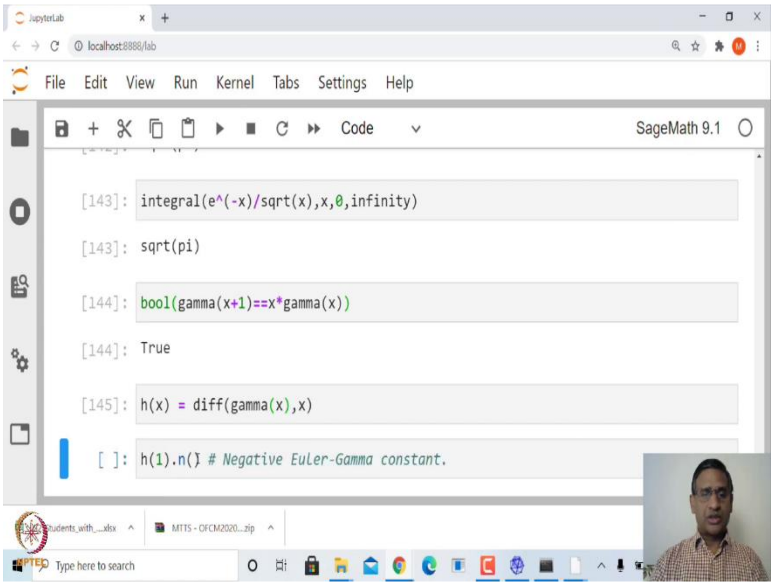

$\frac{e^{-x}}{\sqrt{x}}$ will be square root π which is the gamma of the half. And gamma of x + 1 if you try to simplify, you can use a change of variable, then that is nothing but x times gamma x. So, you can verify this. Boolean means it will give you true or false value without Boolean, it will not give, it will simply return this is equal to double equal to this.

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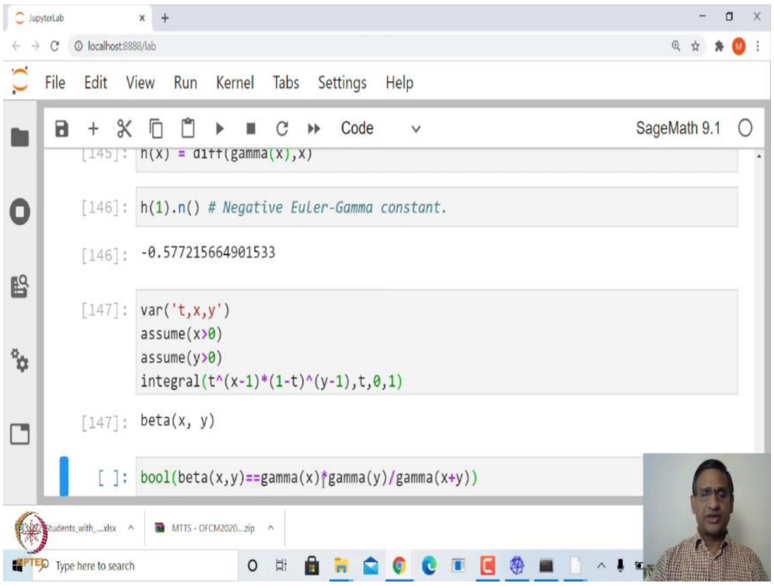

If you take the derivative of the gamma function and try to calculate what is the derivative of the gamma function at 1, this gives you negative 0.577 which is negative, or the Euler gamma constant. This is called the Euler gamma constant, again a very important constant.

Next, if you look at the integral of $t^{(x-1)}(1-t)^{(y-1)}$, where x and y are positive and t between 0 and 1. This is called the beta function. So, this is the beta of x, y. And beta x, y is a function of the gamma function. So, a beta of x, y is equal to gamma x upon gamma y divided by a gamma of x plus y.

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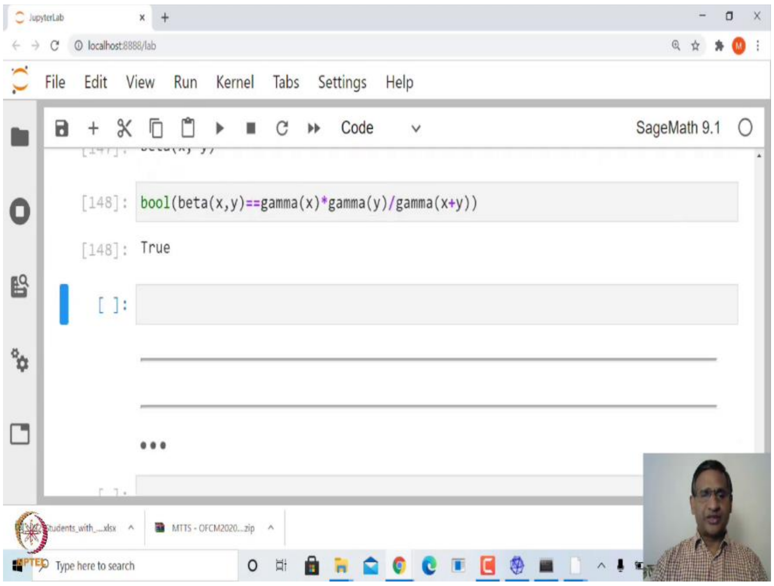

This is verified here. The beta of x, y is equal to gamma x into gamma y divided by a gamma of x plus y. So, and this beta-gamma function has lots of applications in statistics. Similarly, if you try to look at what are some properties of the beta-gamma function, you can easily try to verify in SageMath.

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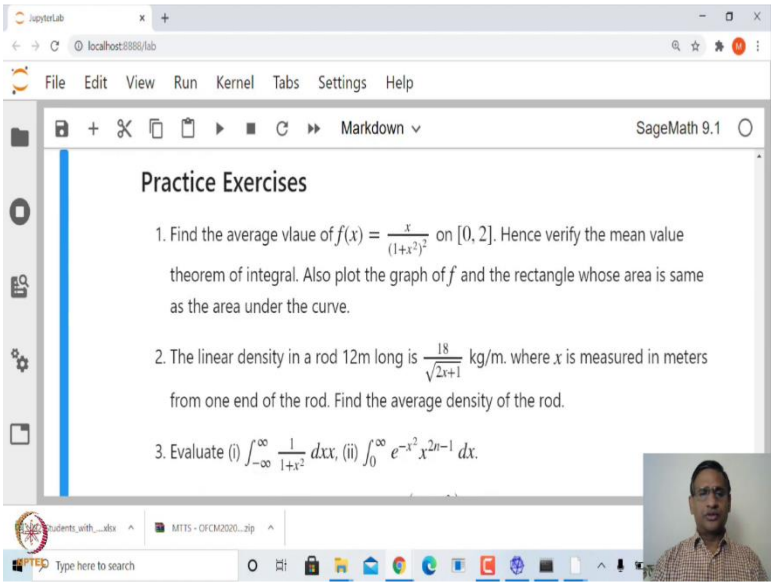

At the end, let me leave you with some simple exercises. The first two exercises are to verify the relationship between the average value and the value of the function or mean value theorem for integrals. Take the function $f(x) = \frac{x}{(1+x^2)^2}$ in the interval 0 to 2, and verify this mean value theorem for integrals. Now, you can also plot the rectangle as we did above.

Similarly, you take the second problem which is that the linear density of a rod 12-meter long is $\frac{18}{\sqrt{2x+1}}$ kilogram per meter, where x is measured in meters from one end of the rod to find the average density of the rod.

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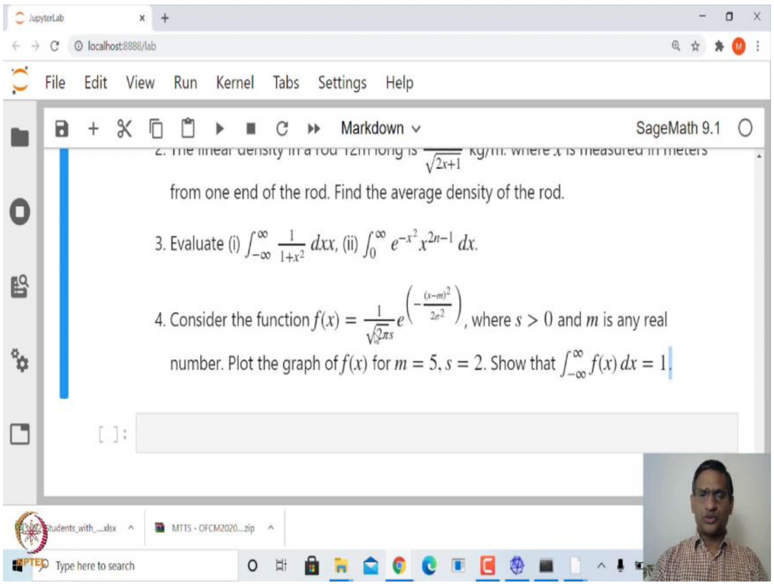

And then try to evaluate these improper integrals. And the 4th one is if you look at this function, for example, $f(x) = \frac{1}{\sqrt{2\pi s}} e^{\left(-\frac{(x-m)^2}{2s^2}\right)}$. You can take sigma as well. So, this is called the Gaussian function.

And try to plot the graph of this function, this will be a bell-shaped graph around m; m is the mean, is the standard deviation. And try to find this integral from minus infinity to infinity of f(x). This will be the total area from minus infinity to plus infinity, and this turns out to be 1, so that is why this is a probability distribution function because the total area under this curve is 1.

Thank you very much. Next time, we will look at some applications of these integrals.