Computational Mathematics with SageMath
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Lecture – 21
Calculus of one variable with SageMath Part 2

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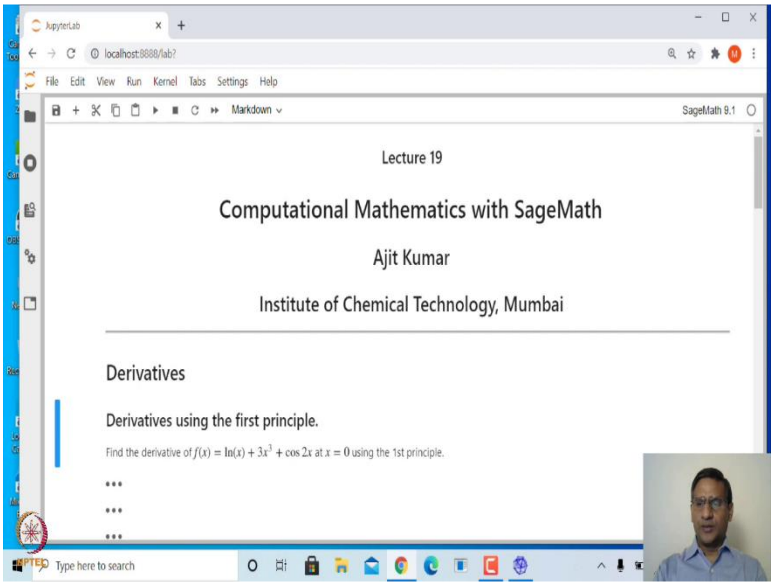

Welcome to the 21th lecture on Computational Mathematics with SageMath. In the last lecture we looked at how to find limit of a function and limit of sequences. Now in this lecture we will look at; finding derivative of a function and some of the concepts related to that. So let us get started. So, suppose you have a function $f(x) = \log(x) + 3x^3 + \cos(2x)$ and you want to find its derivative at x equal to 0. The way you begin with you find the derivative using first principle which is using the limit.

So, if you want to find the derivative of f. Then you find if the derivative exist you find the limit of $f(x+a) - f(a)$ divided by x and as x goes to 0. So, this is what we will try to do. Since we have already found the limit, we have explored how to find the limit we can make use of that.

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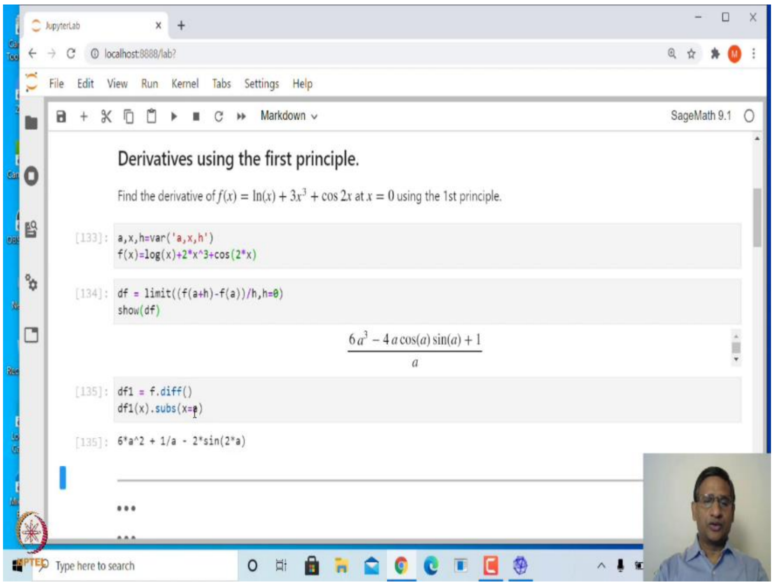

So, let us see so first let us define a, x and h as variables and declare the function $f(x) = \log(x) + 2x^3 + \cos(2x)$. And we want to find the limit the derivative of this function at x equal to a which is 0 right.

So, let us look at if I take generic a and look at what will be the limit of this difference quotient $f(a+h) - f(a)$ divided by h and if that limit exist then that will be the derivative. So, in this case the limit at x equal to a of this difference quotient $f(a+h) - f(a)$ divided by h. As h goes to 0 is $(6a^3 - 4a\cos(a)\sin(a) + 1)/a$.

So, if you can cancel a and then what you are left with is $6a^2 - 4\sin(a)\cos(a)$ this will be sin 2 a. So, this will be a into sin 2 a + 1 by a and that is the derivative and at x equal to a. So, let us verify that using inbuilt function diff. So, if I say f dot diff and substitute the value of x is equal to a then you will see that this is the derivative which is same as limit of this difference quotient ok.

So, generally we do not find derivative using this limit of difference quotient we learn certain rules of derivatives etcetera and then we apply that to the some of the derivative of standard functions ok. So, I am not going to get into verifying all these rules etcetera that is quite easy you can verify.

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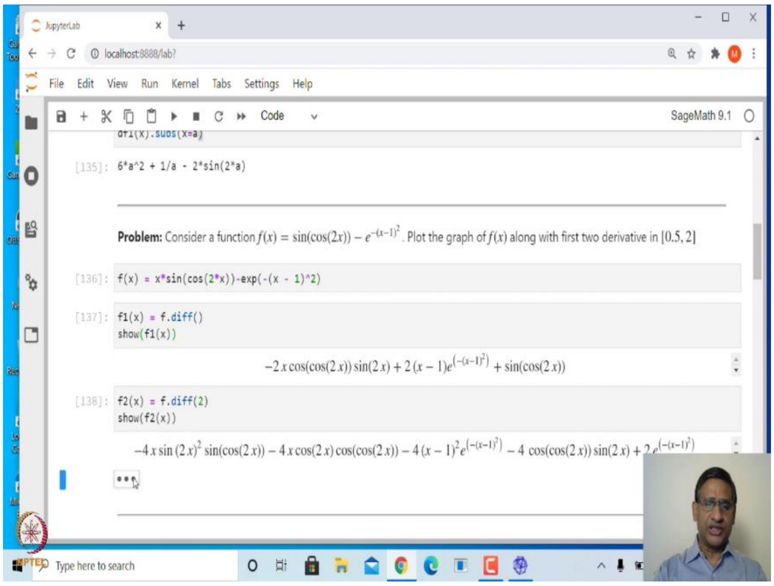

And so let us look at a function $f(x) = \sin(\cos(2x)) - e^{-(x-1)^2}$. Suppose we want to plot graph of this function along with first two derivative first derivative and second derivative. We already know how to find any kth order derivative using diff function.

So, let us define this function f(x) and let us define f1 as derivative of f which is equal to this and let us define f2 as derivative of f second derivative of f with respect to x that is the f2 it looks very complicated, but it is able to find quite easily.

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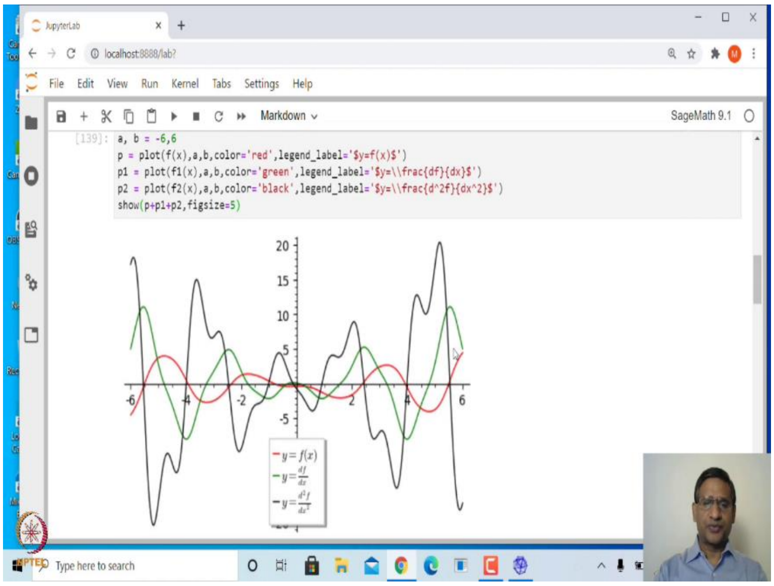

Now, let us plot graph of the both the functions namely graph of f(x) between -6 and 6 so this is the plot graph of f(x) and we are giving the legend label to be f(x) then p1 is the graph of f1 and the p2 is graph of f 2 and then add p + p1 + p2 and let me show. So, that is the graph of f which is red in color derivative which is green in color and second derivative which is black in color.

Now, you can even relate the behavior of derivative in terms of the various concept that you would have learned about the function similarly the second derivative right. So, for example, if you look at this portion of the red curve it is increasing. So, in that case in this domain the second derivative will be somewhat, second derivative will be negative right.

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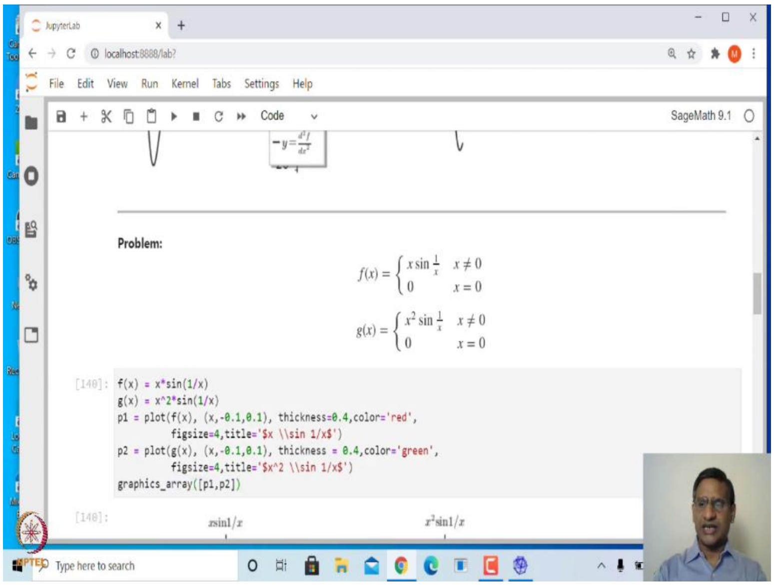

So, let us look at suppose you have a function $f(x) = x \sin(1/x)$ when x is nonzero and is equal to 0 when x is 0 and second one is $x^2 \sin(1/x)$ when x is nonzero and equal to 0 when x equal to 0. So, if you look at these two functions you must have seen that you can find the limit of these two functions and the limit at x equal to 0 exist; but how about the derivative.

So, if I try to plot graph of both these functions side by side. So, that is where we have used graphics array. If I plot graph of this functions side by side; then this is how the graphs look like.

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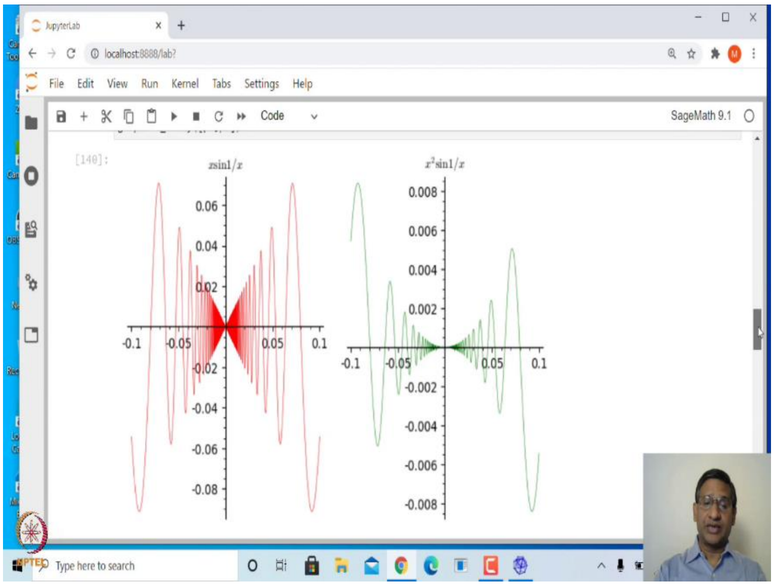

This is a graph of x sin $(1/x)$ and this is graph of x^2 sin $(1/x)$. So, in this case if you look at this is somewhat very sharp turn here sharp edge here in corner I should say. In fact, you can draw a line $y = |x|$ and you will see that this near 0 the value is very close to that mod x function which is not differentiable at x equal to 0.

So, this function you expect this is not to be differentiable at x equal to 0. Whereas, in this case this is somewhat actually close to $y = x^2$ from above and $y = -x^2$ from below. So, this at x equal to 0 that and y equal to x^2 is differentiable function.

So, you can expect that this x^2 sin $(1/x)$ to be differentiable at 0 whereas, x sin $(1/x)$ not differentiable at x equal to 0. Now let us find out if I want to check whether the limit or whether the derivative exist. I have to find out limit of the difference quotient, so we want at x equal to 0. So, we have to say $f(x) - f(0)$ divided by x limit as x goes to 0.

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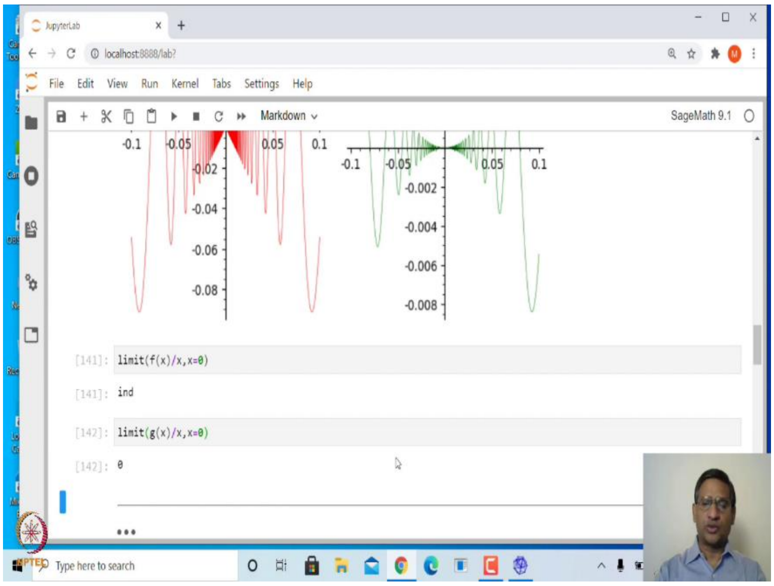

So, that and f of 0 is 0 therefore, it is same as limit of f(x) divided by x and x equal to 0. In this case it says that it is undefined that is it is not defined or indefinite right. Whereas, if I look at the second 1 the second $g(x)$ limit of g(x) by x at x equal to 0 this limit is 0. So, the derivative of g exist and it is equal to 0 at x equal to 0. So, graphically we are able to at least get some idea about whether the derivatives should exist or not and which we have verified ok.

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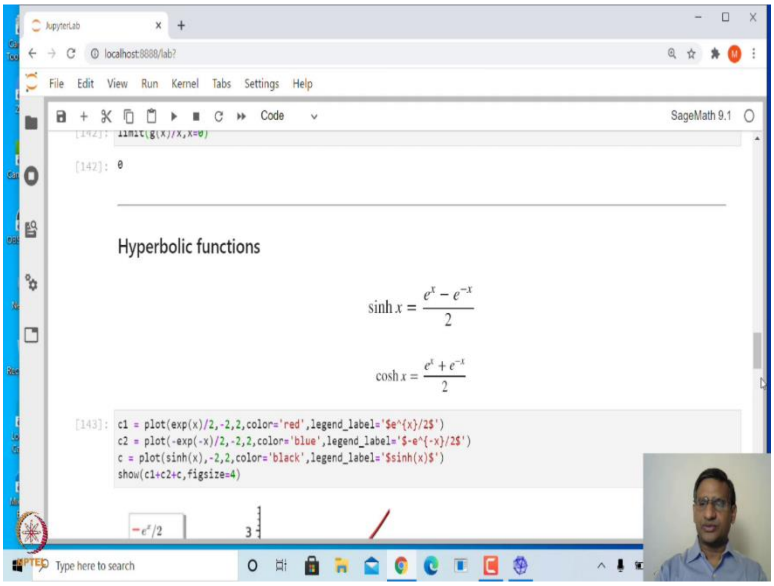

Now let us look at so you must have seen how this hyperbolic sine and cosine are defined. So, hyperbolic sin is defined by $(e^x - e^{-x})/2$ whereas, $cosh(x)$ is defined as $(e^x + e^{-x})/2$. So, if you look if you want to plot graph of sin hyperbolic and cos hyperbolic one way to visualize this is we can plot graph of e to the power x and -e to the power -x by 2 and then add these two together and then you will get graph of sin hyperbolic.

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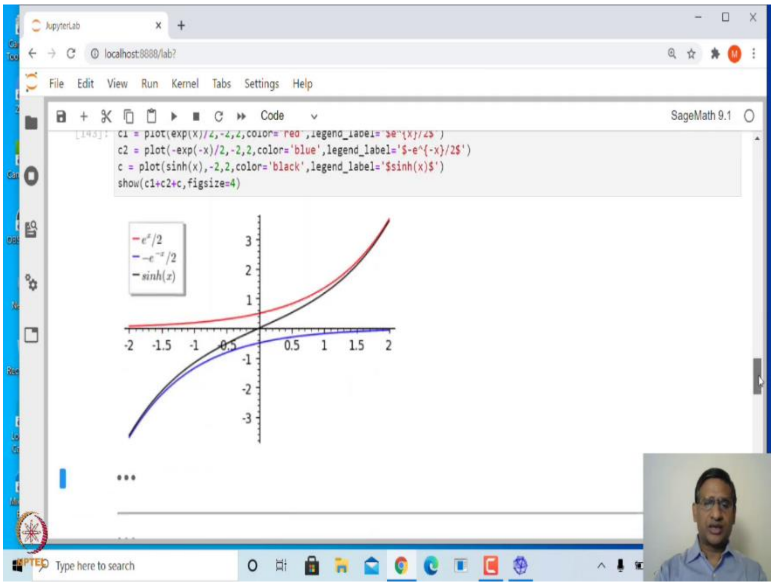

So, if you look at this graph this red is graph of e to the power x by 2 blue one is -e to the power -x by 2 and when you add these to the graph which you get is the graph of sin hyperbolic. So, that is how the graph of sin hyperbolic between -2 and 2 looks like.

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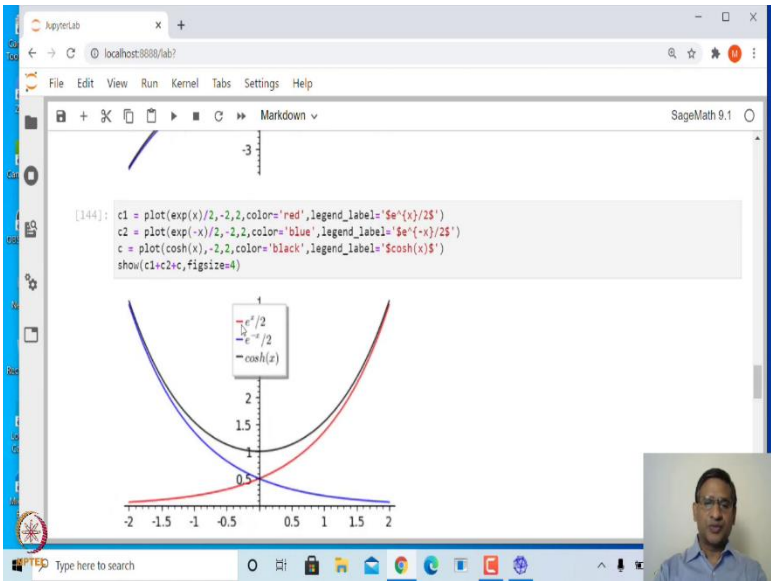

Similarly, you can do the same thing for $cosh(x)$. So, let me plot this cos hyperbolic x. So, cos hyperbolic x how do we do that? Plot the graph of e to the power x by 2 which is red in color and graph of e to the power -x by 2 which is blue in color and when you add these 2 you get graph of cos hyperbolic x which is black in color in this case. So, that is how you can visualize cos hyperbolic and sin hyperbolic right. You can try with other trigonometrical hyperbolic functions.

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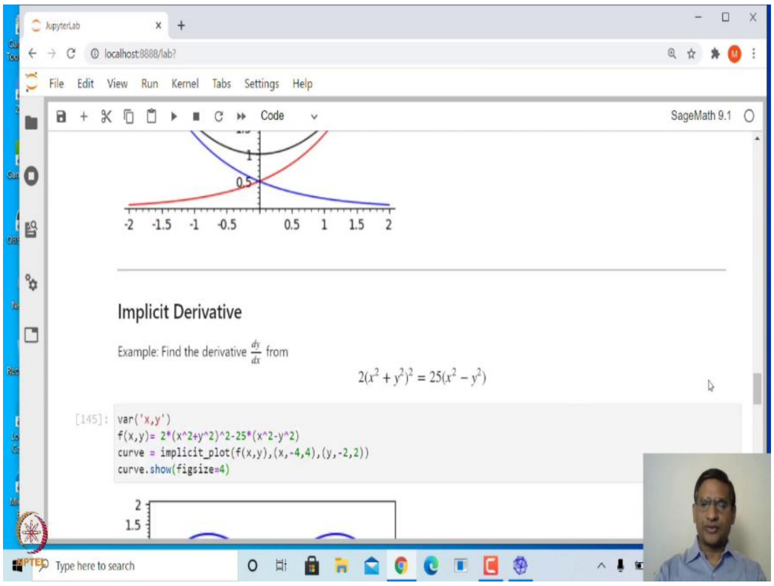

Now, many times you your function is not defined explicitly it is defined implicitly. So, that means, y and x is satisfied by some equation whereas, y is a function of x. So, in that case you want to find derivative of y with respect to x and how do we do that?

So, let us take an example, suppose you have this x and y defined by this equation this implicit equation that is $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ and here y is a function of x therefore, you would like to find derivative y with respect to x.

The way you would have done is find derivative of this entire expression on the left hand side right hand side and wherever you have y you write that differentiate with respect to x using chain rule. So, you will get an expression in dy dx and you solve for dy by dx.

But Sage has inbuilt function to find the implicit derivative, but first let us plot graph of this curve which is represented by this equation and we can use implicit plot we have already seen how to use this. So, if we plot graph of this function this is how it looks like.

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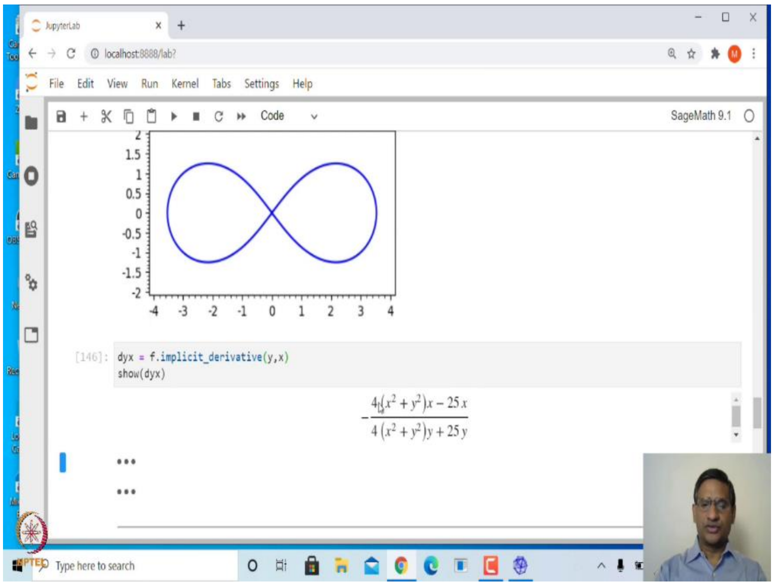

It is here; it is here a kind of infinity symbols we also call this as figure 8. Now, suppose at some point so for example, at x equal to 3 you can check that y will be + 1 and -1. So, (3, 1) and (3, -1) both are lies on this curve similarly (-3, -1), (-3, 1) also lies in on this curve. So, suppose at that point we want to find the derivative and look at what is the geometric meaning of this.

So, let me find the derivative implicit derivative using a function called implicit underscore derivative of y with respect to x. So, if I say you have to say f dot implicit derivative y comma x. So, that is how the derivative of y with respect to x looks like. So, this you should try to.. I am sure you can find out by hand also right.

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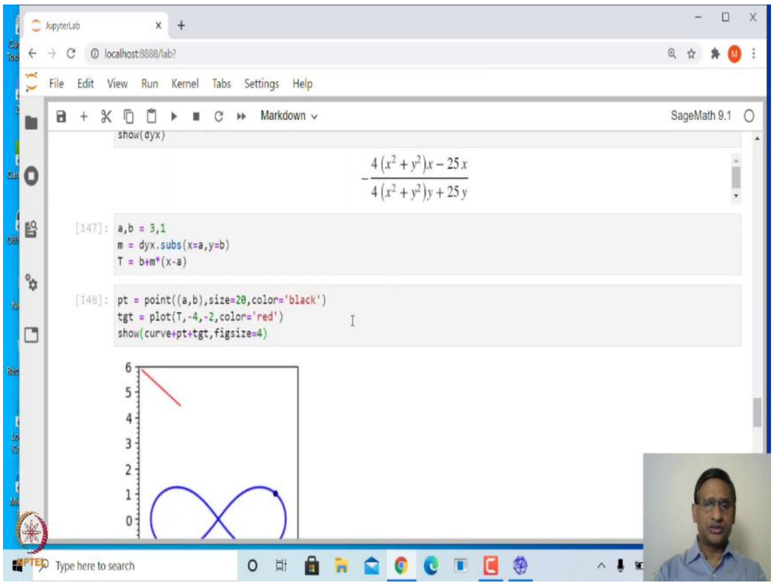

Now let us look at this point let me define the derivative at x equal to let us say 3 and 1 is equal to m. So, that is the in this you put x = 3 and y = 1. And then let us store that in m and let us define a tangent passing through (a, b) with slope m.

So, that is the tangent line or line passing through (a, b) with slope m is what we are plotting. Now suppose we want to plot the curve along with this point and the line passing through (3, 1) with slope m and then see what it means. So, therefore, now you can see here this I should say 2 comma 4.

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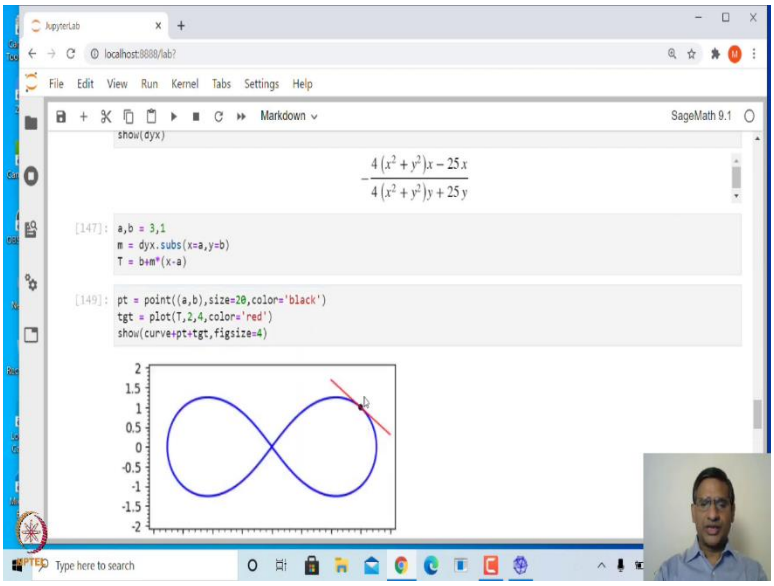

Then you can see here this is the point 3 comma 1 and this line which we plotted which is the line with slope m which is the derivative of the function at this point is tangent to this curve. So, therefore, that is what the geometric meaning of the implicit derivative is when you find it. So, it is actually the slope of the derivative dy by dx at this point represents the slope of the tangent to this curve ok. If I if you try to change this instead of 3 1 if you say -3 and 1 and in this case.

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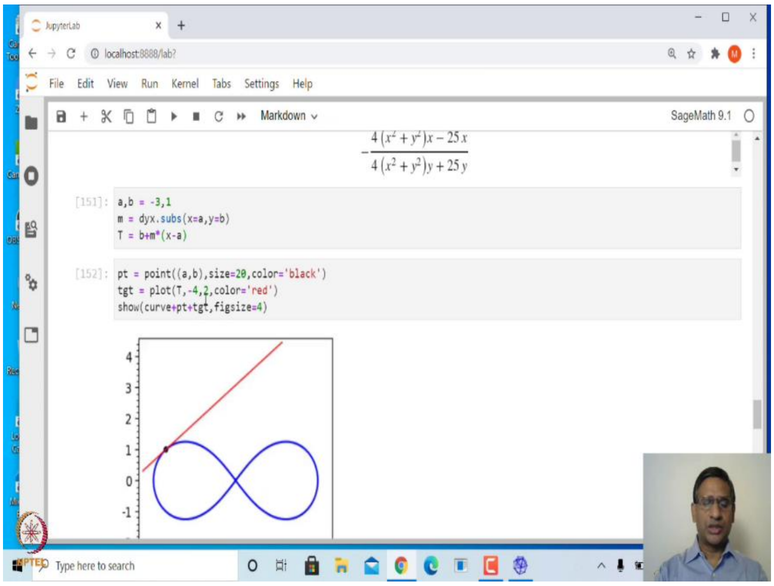

Now, let me say it should be -4 comma 2 in that case you will get, this is the tangent at this point -4 comma -2 let me put right.

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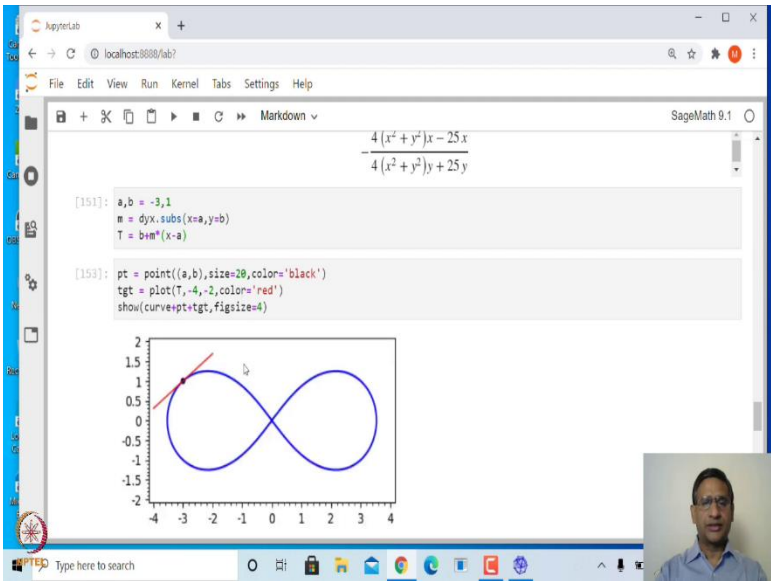

Similarly, you can explore this derivative at other points.

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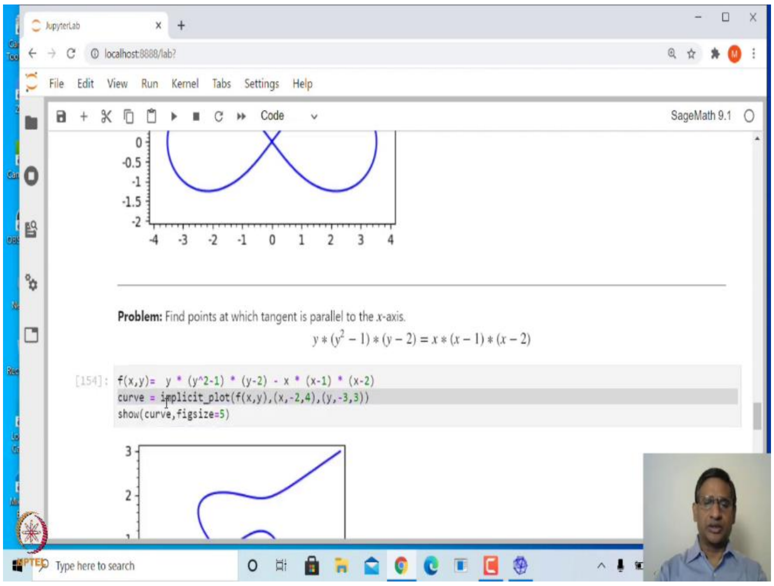

Let us do one more example in this implicit derivative and see what is how we can explore Sage Math. So, you have a function implicitly defined function this and you want to find points at which the tangent is parallel to x axis. So, first let us declare this function and plot graph of this curve.

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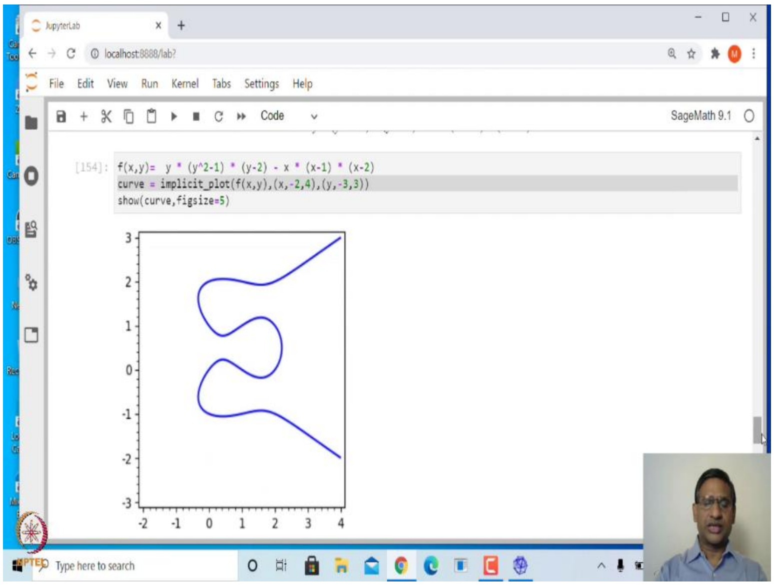

Graph of this curve looks like this is quite complicated one. So, if I look at for example, in this domain some points where the tangent is parallel to x axis there will be several points. So, what we will do is first of all.

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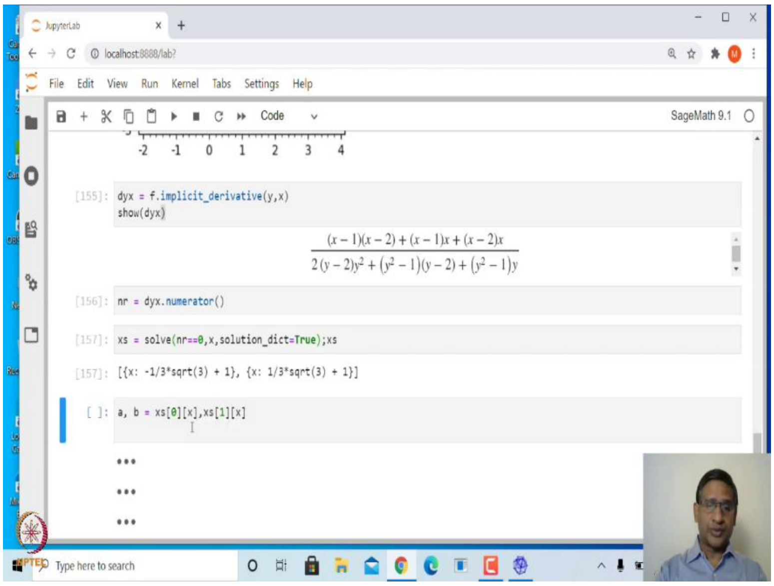

Let us find the derivative of y with respect to x using implicit derivative so that is the implicit derivative. Now the slope will be parallel to x axis the tangent will be parallel to x axis if slope is 0; that means numerator this these functions should be 0. So, in particular numerator is 0.

So, you extract the numerator in nr by using dot numerator function and then you find the 0s of this is a polynomial. So, you will be able to find 0's of this. So, it has actually three two 0's 1 is $-1/3\sqrt{3}+1$ which is same as $1-\frac{1}{3}\sqrt{3}$ and the other one is $1+1/3\sqrt{3}$ these are the two points.

So, and if I look at the curve at $1+1/3\sqrt{3}$. So, this is some kind of vertical line and it will intersect around 4 points similarly this side. So, it looks like there will be 8 points where the tangent is parallel to x axis. So, now, what do we how do we find out?

So, we have found out the points x at which the derivative is going to be parallel to x axis at these points you need to find out what is the value of y right. So, let us extract these points which we have found out the points where x that slope is parallel to x axis. Let us store this in a and b and let me say what are a and b.

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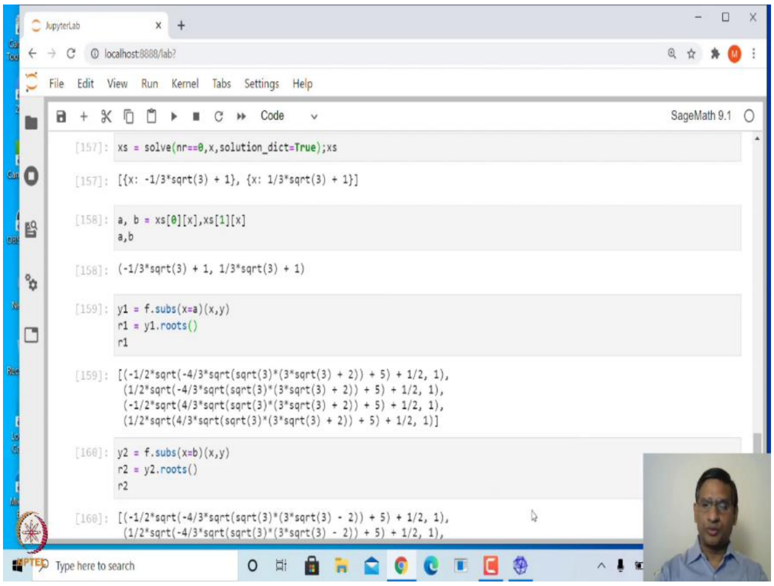

So, these are a and b now we want to find substitute x equal to a in this expression so that I am calling as y and then find roots of that y. So, that will give you there are four points in this case you can see here there are four points and these four points. So, x equal to a comma these four points will be these points. Similarly, you can do by substituting x equal to b. So, that you get another four points.

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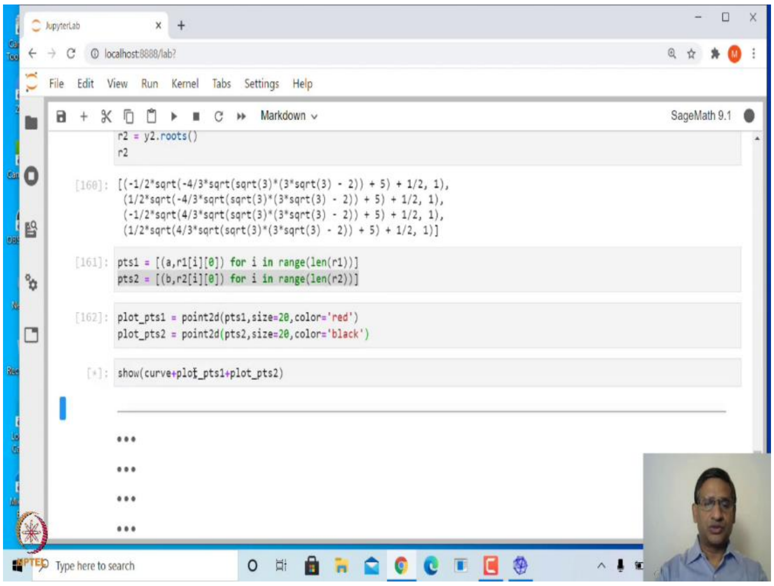

Now, all these 8 points so first let us make a list of these first four points for x equal to a and the second one as points 2 the second for x equal to b. And then let us plot points of these first four points in red color and second points using point 3d in black color and then let us add these two points to the curve. So, when we add to the curve.

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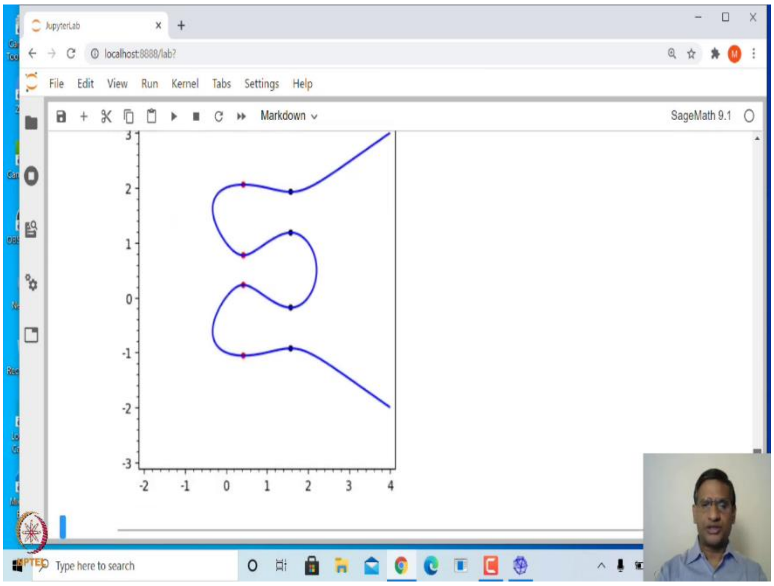

These points you will see here these are the 8 points at which the tangent is parallel to x axis. Of course, you could have also plotted the graph of this the tangent at these points fine.

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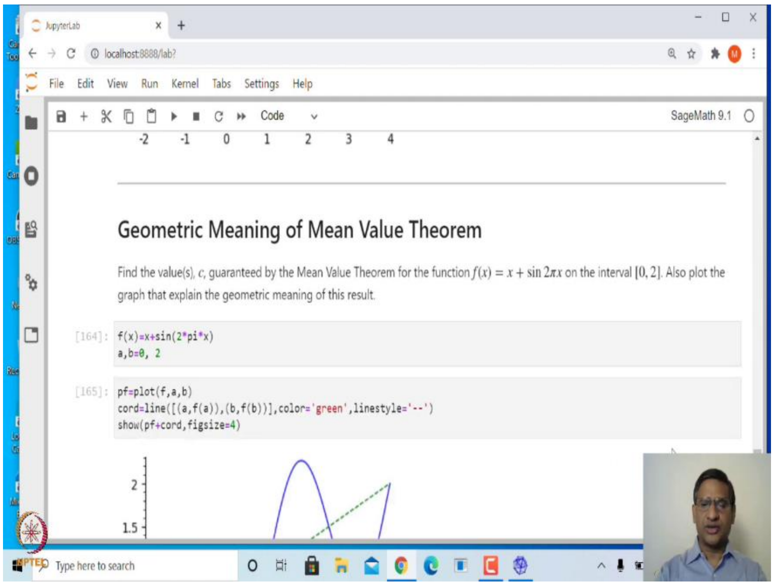

Let us look at all of you must have studied this mean value theorem which says that if you have a function f(x) defined on closed interval a b which is continuous enclosure interval a b and differentiable in open interval a b. Then if you look there is a point c strictly between a and b that is in open interval a b such that f dash at c is equal to f b -f a upon b -a.

So, let us try to verify that for this function $f(x) = x + \sin(2\,pi\,x)$. So, how do we do that? So, let us define this function f(x) and a is equal to 0 and 2, a b is 0 and 2. And let us first plot graph of this function.

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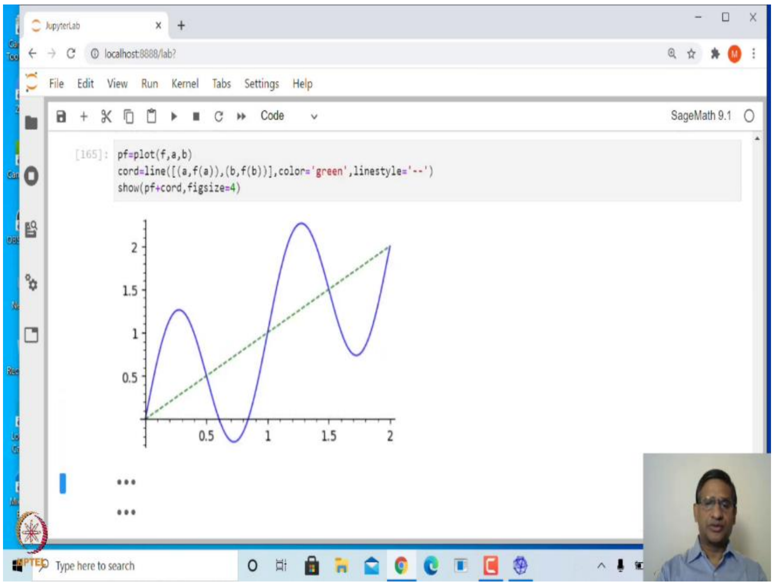

So, this is the graph of this function and endpoints are joined by this chord. So, the geometric meaning of mean value theorem says; this is called Lagrange mean value theorem it says that if you look at this chord then there are points on this curve at which the tangent is parallel to this chord.

So, in this case you can say that there will be some point here, there will be some point here, there will be some point here, there will be some point here. So, there will be four points in the strictly between a and b at which the tangent is parallel to this chord.

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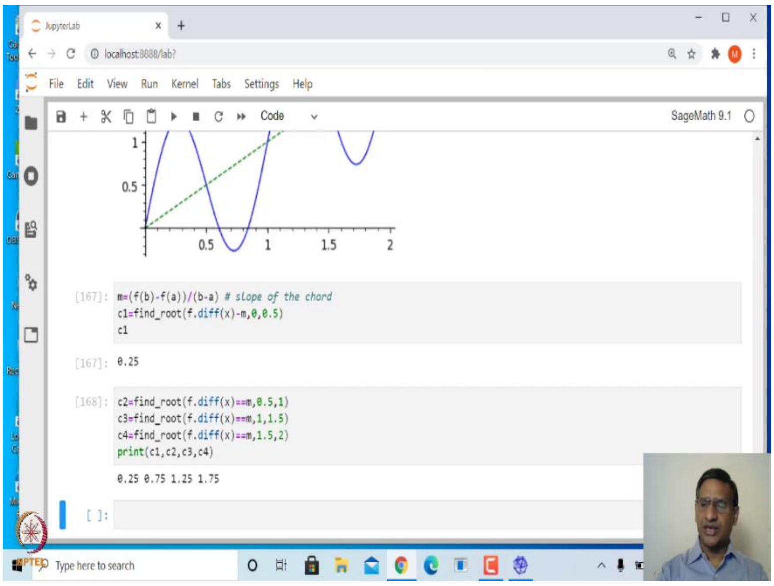

So, that is the geometric meaning of, so how do we do that? Let me again store m as the slope of this chord and find the points. So, if you look at this curve this is the slope and then you want to find out f dash x equal to m. m is the slope of this chord.

So, that means you want to solve or find 0's off f dash x -m. So that is what is done. So, first root in this case you can see here there is one root has to be between 0 and point 5 that is actually 0.25 the second root third root and fourth root they are at 0.75 1.25 and 1.75.

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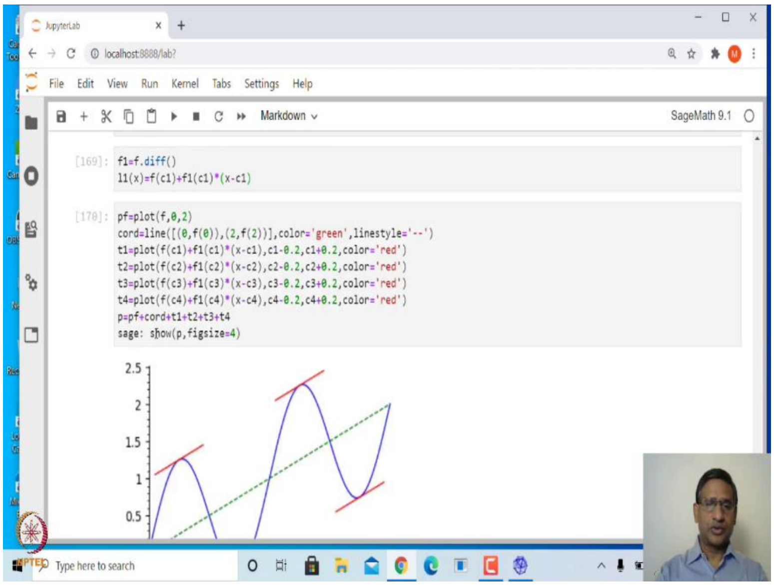

And at these points let us plot the tangent. So, what will be the tangent? The tangent l1(x) at c1 is equal to $f(c1) + f1(c1)$ that is the value of the function at c1 into $(x - c1)$. And then try to plot tangent to this curve let me show you this is quite easy.

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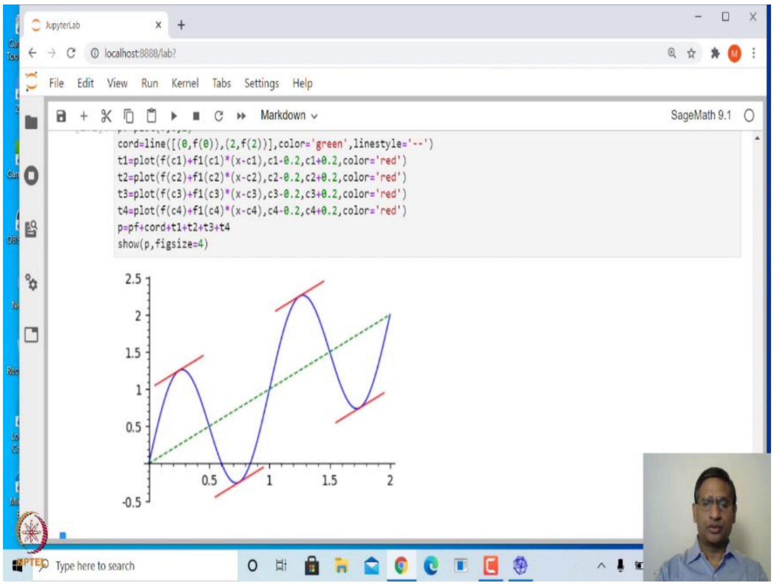

This I do not require Sage here, but that is fine. So, that is the; that is the geometric meaning of Lagrange mean value theorem; you could also apply to Rolle’s theorem and then explore it is geometric meaning.

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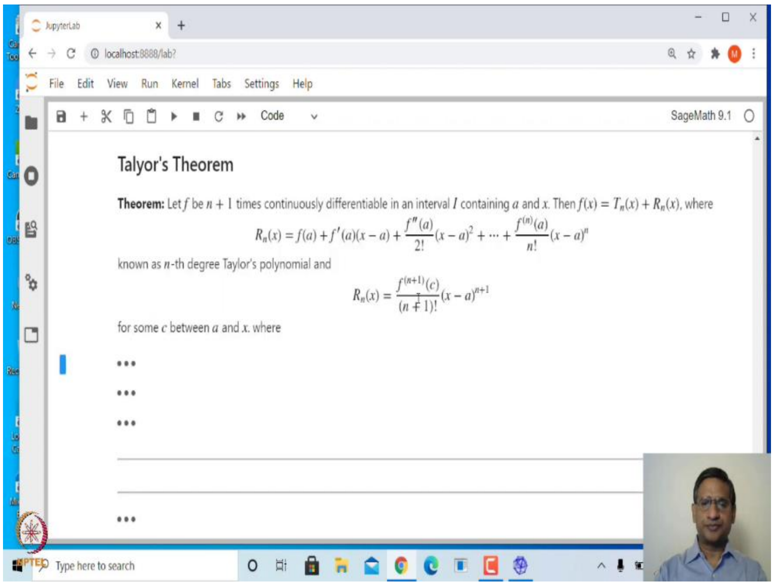

You could also do .. there is another mean value theorem known as Cauchy mean value theorem that I left leave this as an exercise try to explore geometric meaning of Cauchy mean value theorem. Let us look at what is Taylor's theorem.

So, I am sure most of you would remember, but just in case you do not remember I am just stating that result. So, if you have a function $f(x)$ defined on some interval I in R which is n + 1 times continuously differentiable. And then if you have a any point inside this interval and look at any x in this interval. Then you can write $f(x) = T_n(x) + R_n(x)$ where $T_n(x)$ is actually nth degree polynomial which is known as Taylor's polynomial nth degree Taylor's polynomial.

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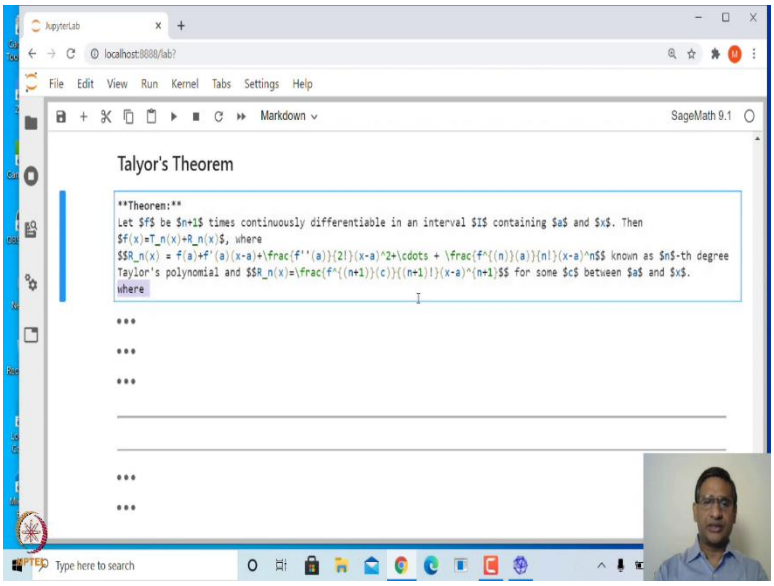

And R_n we call as a remainder term which is given by $(n+1)th$ derivative of f at some point c between strictly between a and b divided by n + 1 factorial times $(x-a)^{n+1}$. And how do we write this Taylor's nth degree polynomial this is actually this should be $T_n(x)$.

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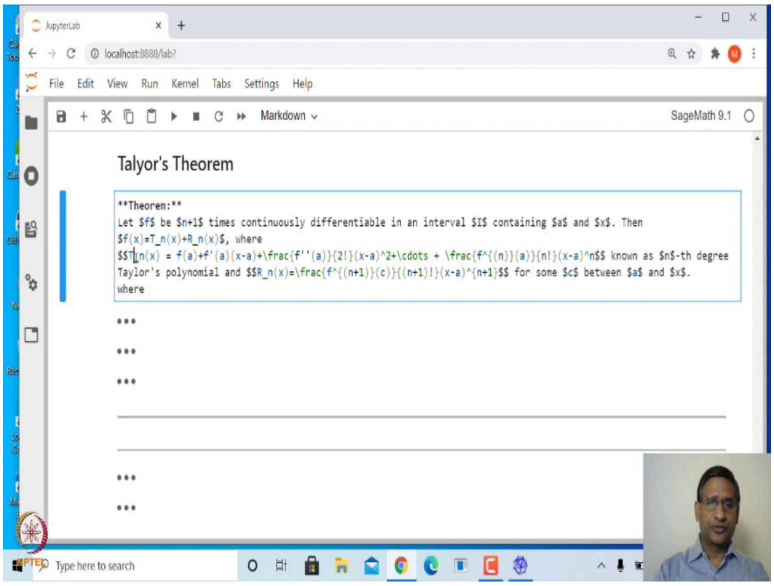

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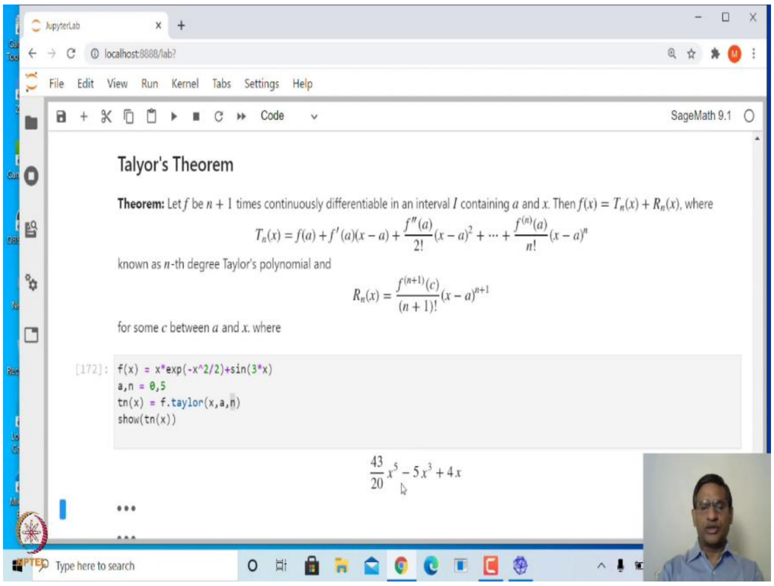

So, $T\_n\ (x)\ = f(a)\ + f'(a)(x-a) + \left(\frac{f''(a)}{2!}\right)(x-a)^2 + \ldots + \left(\frac{f^{(n)}(a)}{n!}\right)(x-a)^n$ that is called nth degree e Taylor's polynomial. And the function is actually approximated by nth degree polynomial with remainder. And in case you have higher derivative one can show that this remainder goes to 0 as n increases right.

So, this is actually a basis for all these numerical evaluations of function. For example, when you want to find out sin of something cos of something exponential of something the actually it uses this expansion depending upon the accuracy and gives you the value. So, that is the very important concept which is kind of used for approximating the value of the function.

Now, let us take an example suppose I have function $f(x) = xe^{-\frac{x^2}{2}} + \sin(3x)$. And suppose we want to find out 5th degree Taylor's polynomial of f about x equal to 0.

So, here a is equal to 0, n is equal to 5 and then I am defining that Taylor's polynomial is $tn(x)$ = f dot Taylor with respect to variable x and at the point a and the degree is n. So, let us ask it to show 5th order Taylor's polynomial. If I say instead of 5th if I say 10 this will give me 10th degree Taylor's polynomial right.

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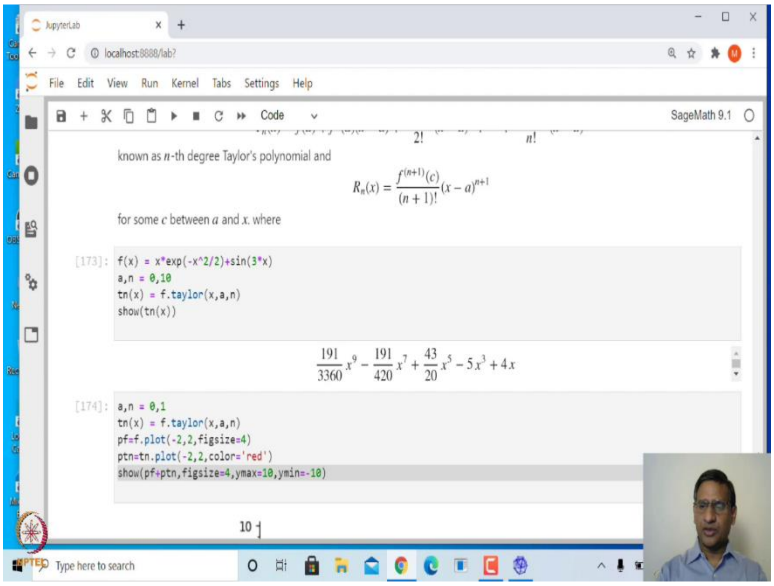

So, that is how you can find Taylor's polynomial or Taylor's approximation of any function. Now let us try to plot graph of this function along with the Taylor polynomial. Let us say we want to plot graph of first order Taylor's polynomial. First order Taylor's polynomial will be nothing, but the tangent. So, it is the linear approximation. So, let us try to plot it is graph.

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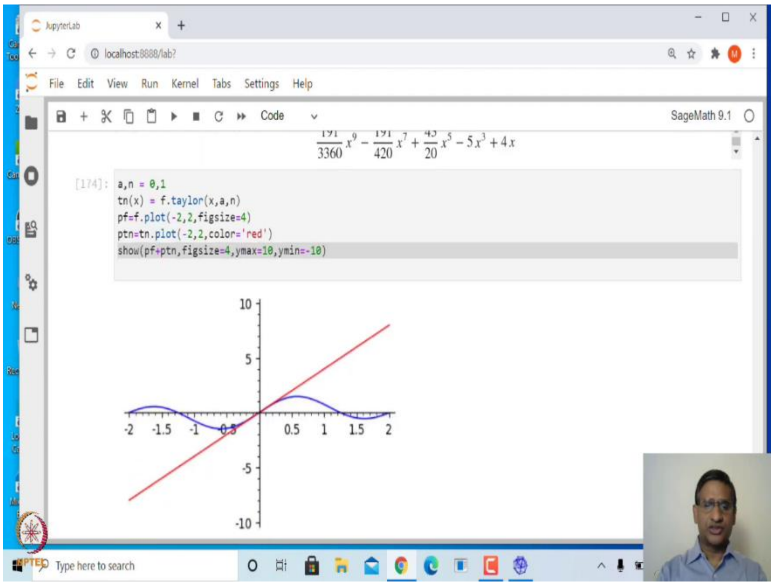

So, you have graph of this function and we are plotting Taylor's polynomial approximating the function about x equal to 0 and first order…

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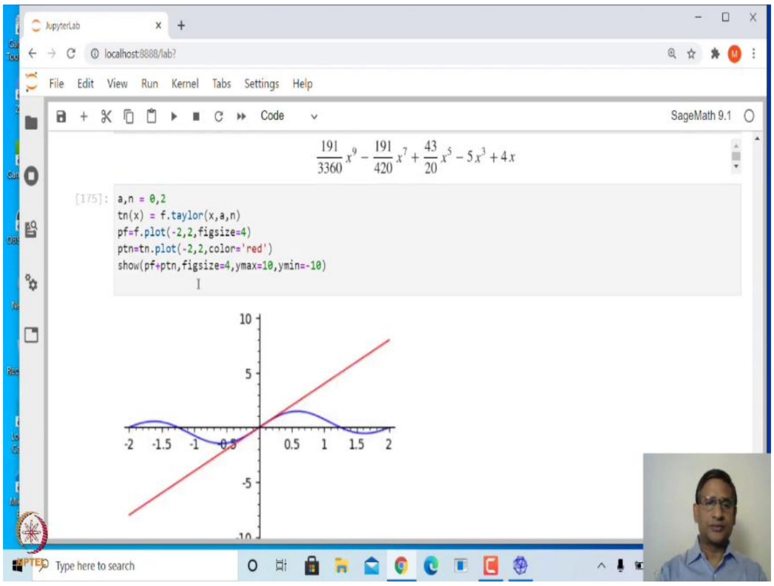

So, that is the tangent. If I increase n to 2 this will be quadratic Taylor's polynomial.

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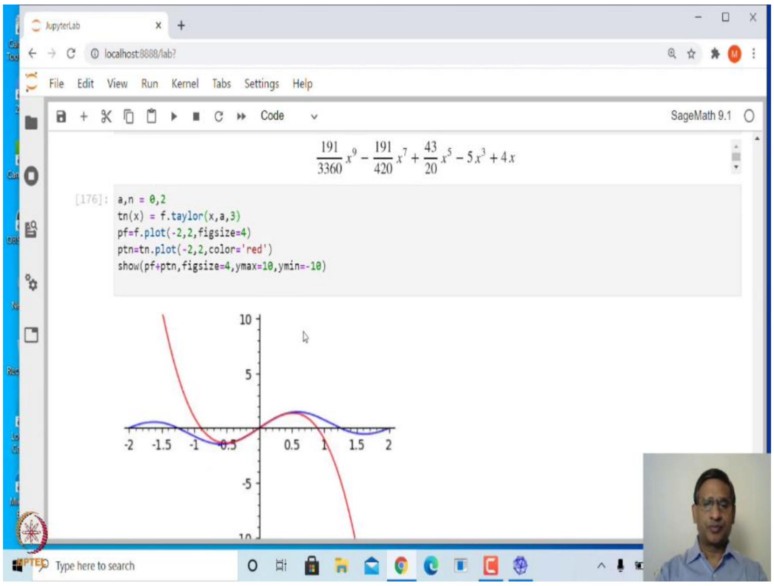

If I increase 2 to 3 then that is the actually in this case third quadratic polynomial is very quadratic. The 3rd degree term will be missing. So, this is I have to say here n.

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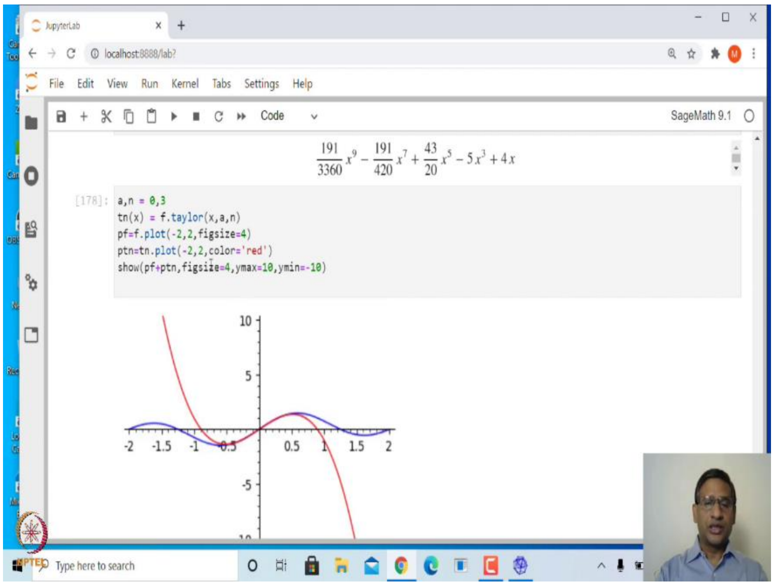

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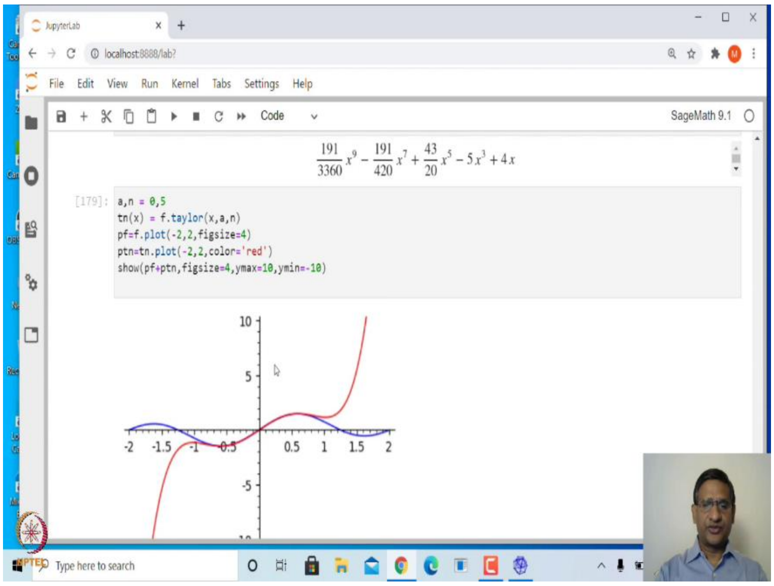

And increase this to 3 right. And if I increase it to let us say 5 this is the 5th and you can see here this red color which is the graph of the Taylor’s polynomial is very close to this actual function.

So, that is what I meant by saying as you increase the degree the function will be very-very close to the actual that polynomial Tn that is nth degree Taylor’s polynomial will be close to this the actual value of the function. So, you can even create animation for this let me just leave you …

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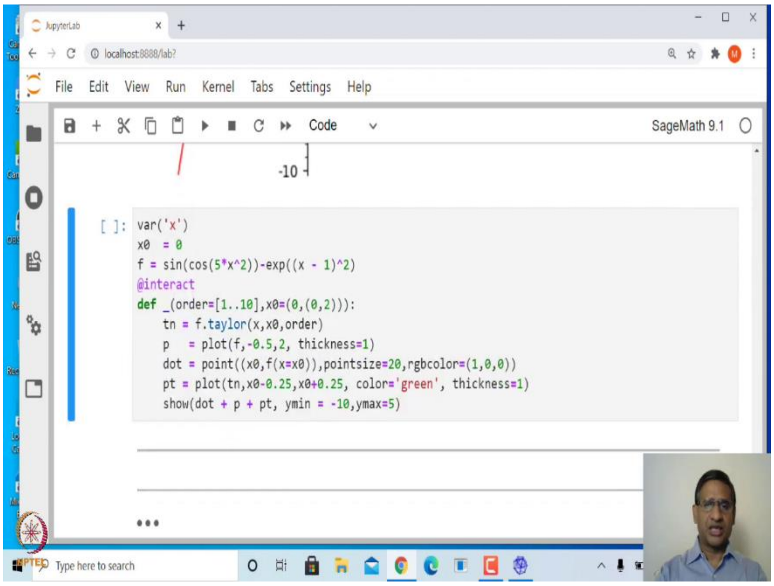

… this code using Sage interact which will kind of create an animation where you can vary the points and also plot the various Taylor's polynomial and other things. So, you can explore this you can learn this in Jupyter notebook.

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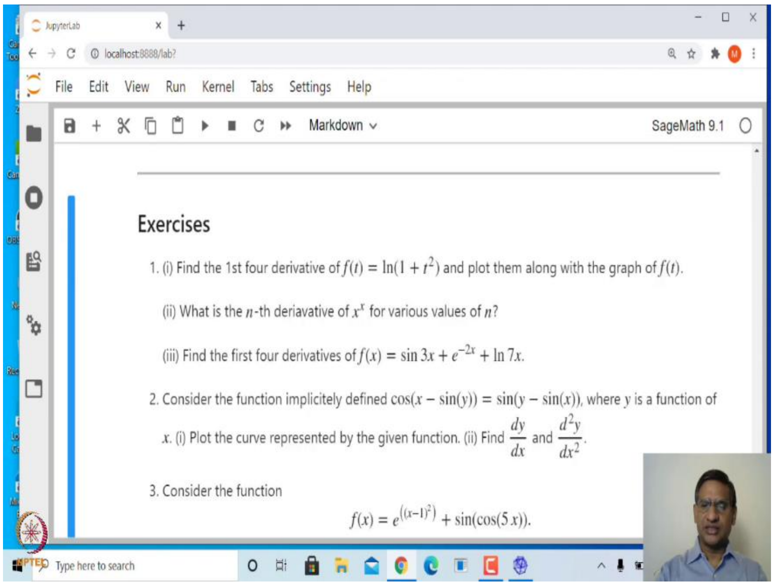

At the end let me leave you with few exercises. So, these are some of the exercises.

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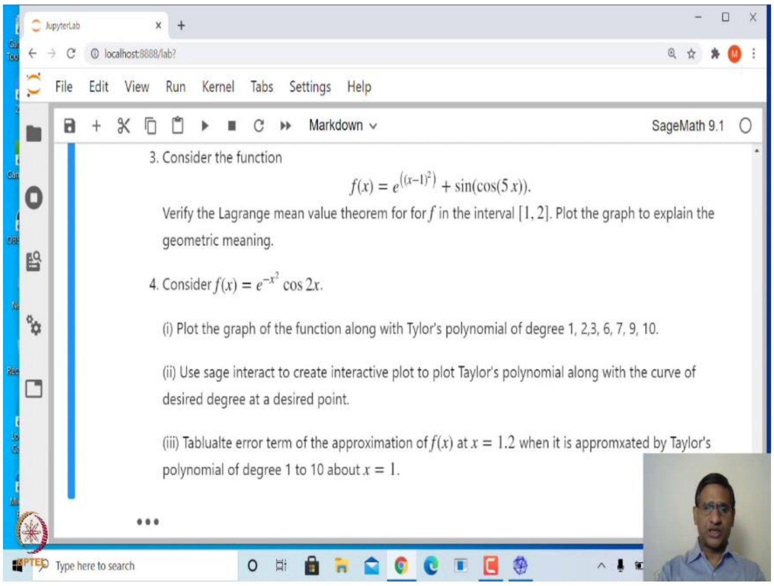

The first exercise is to find the first derivative of this function and plot it is graph along with the graph of the function. And for various values of n you find the derivative of x to the power x. Then the third one is look at this function find first four derivatives then you look at this implicitly defined function find first derivative and second derivative.

And then the third problem is to verify Lagrange mean value theorem for this function. And the fourth problem is find Taylor's approximation to e to the power -x square into cos 2 x and plot it is graph along with the Taylor's polynomial of degree 1, 2, 3, 6, 7, 9, 10.

So, just this is just to demonstrate the approximation of the function using Taylor's polynomial. You can even tabulate the error term and you can create a sage interact whose code I have already given you. So, these are the some exercises all these exercises are there. So, you can try to solve them they are quite simple. So, next time we will look at some more applications of derivative using Sage Math.

Thank you very much.