Computational Mathematics with SageMath
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Lecture – 19
3d Plotting with SageMath

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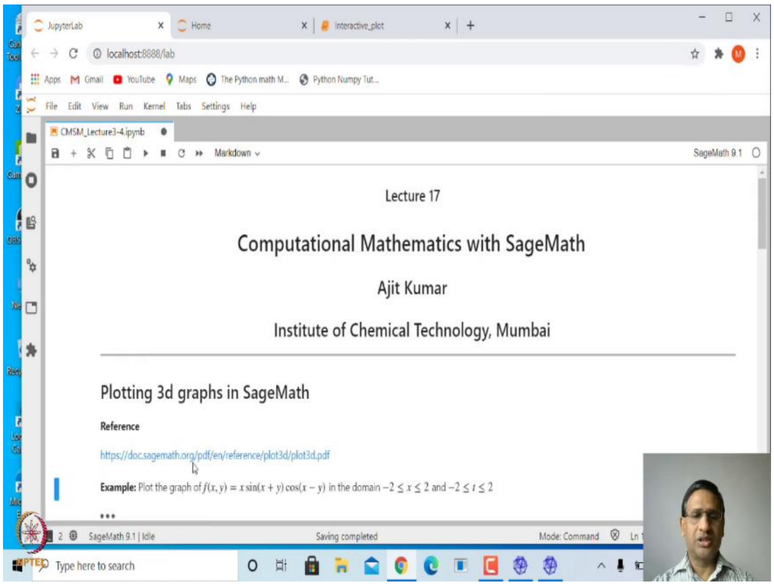

Welcome to the 19th lecture on Computational Mathematics with SageMath. In this lecture, we will explore 3d plotting in SageMath. In the last lecture, we looked at 2d plotting of various objects like functions defined explicitly, function defined in terms of parameters or parametric curves, polar curves, also functions defined implicitly.

Then we also looked at how to plot regions and how to plot inequalities. Let us look at how we can plot 3 dimensional objects using SageMath. So, first let us look at this is the reference plot3d, let me make it slightly bigger.

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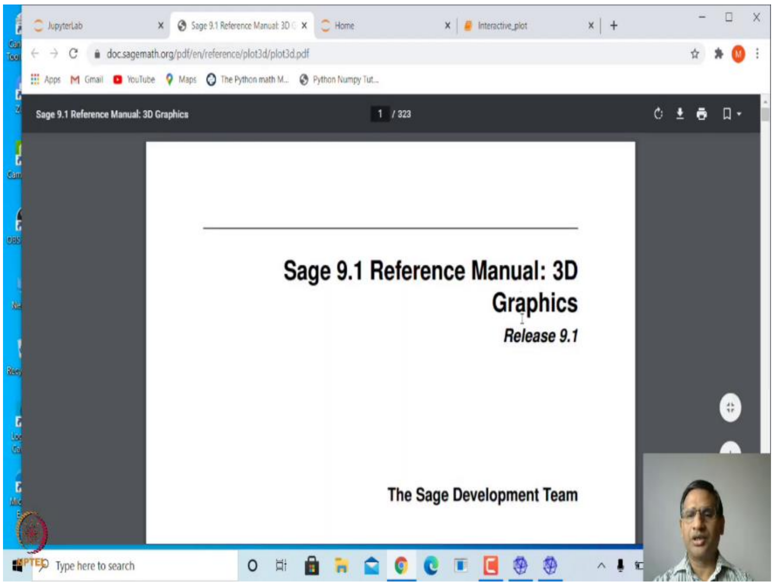

So, this is again if you click on this particular link, it will open a 3d graphics manual. This is about 323 pages. You can go through this manual which will tell you with examples regarding plotting of 3 dimensional objects in SageMath. So, let us look at first example. We want to plot graph of a function $f(x, y) = x\,\sin(x + y)\,cos(x - y)$ in the domain x between -2 and 2 and also y between -2 and 2. This should be y.

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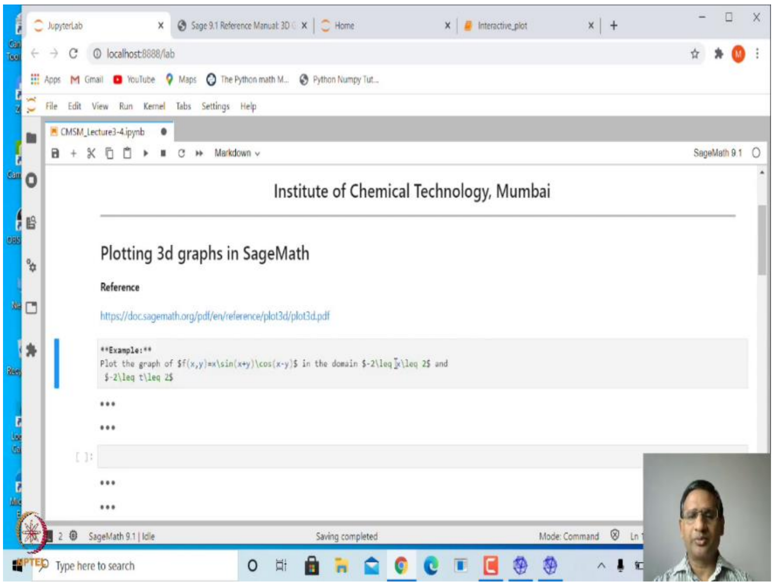

So, let me change this to y, x and y varying in the domain -2 to 2. So, first we need to define this function $f(x, y)$ as a function of two variables x and y.

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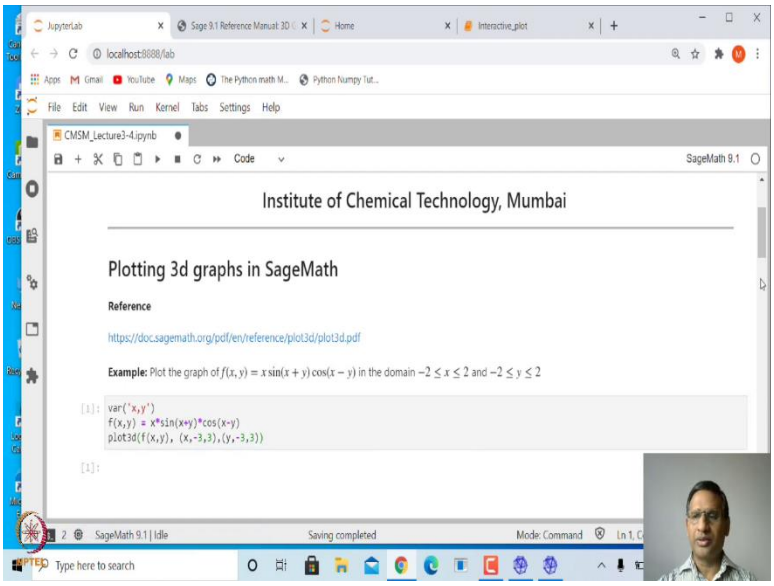

So, we need to first declare x and y as variables. So, we have declared x and y as variables. And then we define $f(x, y) = x\sin(x + y)\,cos(x - y)$, and then we can use the function plot3d. Instead of plot, we have now plot3d and then again you write $f(x, y)$, and then give the domain of x, domain of y.

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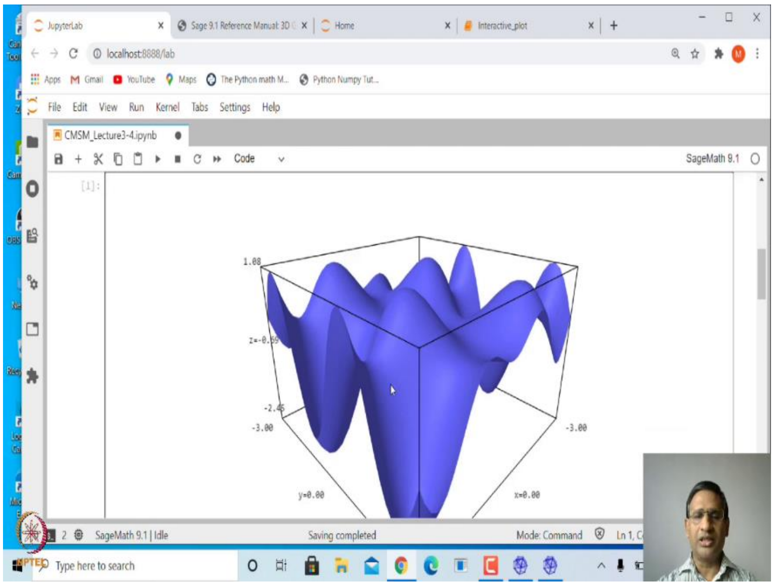

So, let us plot this. This may take little time depending upon the speed of the computer right. So, this is a beautiful graph of this function.

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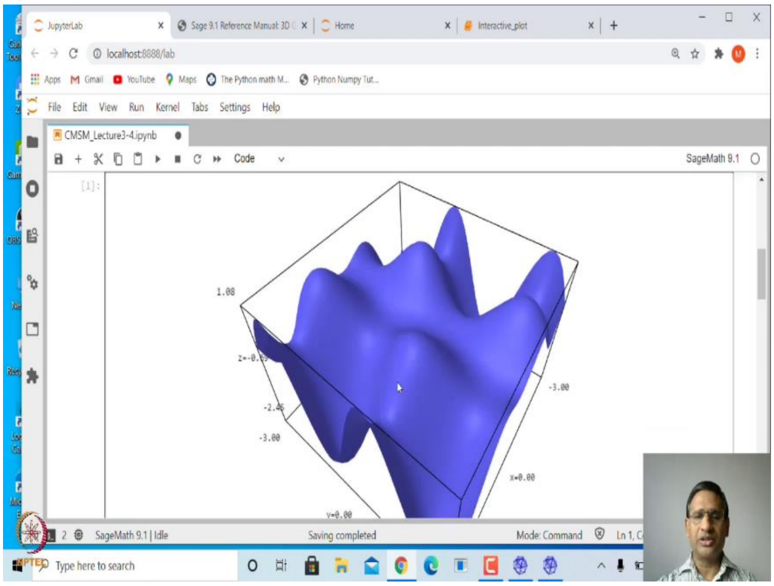

And you can rotate this. You can just click on this graph and use your mouse or touch pad to rotate this. And whatever is the best angle you can just keep it right. You of course, you can change color of this graph, you can change the point size. So, you can look at help on plot3d. plot3d question mark or double question mark that will tell you how to change various options inside this plot.

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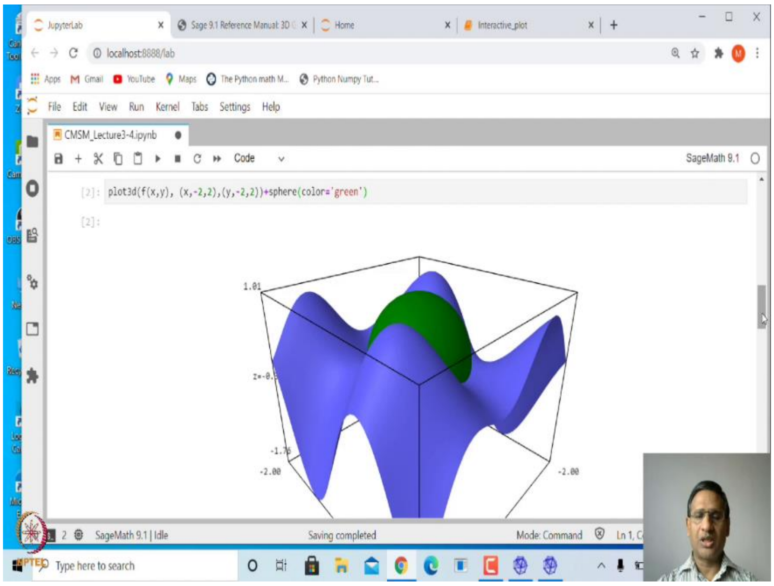

You can even add two 3 dimensional plots. So, for example, in this particular surface which is graph of z = f(x,y), suppose we want to add sphere of unit length. So, Sage has inbuilt function to plot sphere; this is called sphere and I am writing or I am giving the color of the sphere to be green.

So, by default, this will plot sphere centered at the origin (0,0,0) with radius 1. If you want with different radius and origin at the center, it you have an option. You can look at help on sphere; it will tell you all these things. So, I am just saying plot $f(x, y)$ from x from -2 to 2, y from -2 to 2, and plus sphere with color green.

So, let me plot these two. So, we will get, now you can see here the sphere has been added to this particular surface. So, you can see that the two these are two surfaces, and they intersect along a curve ok..

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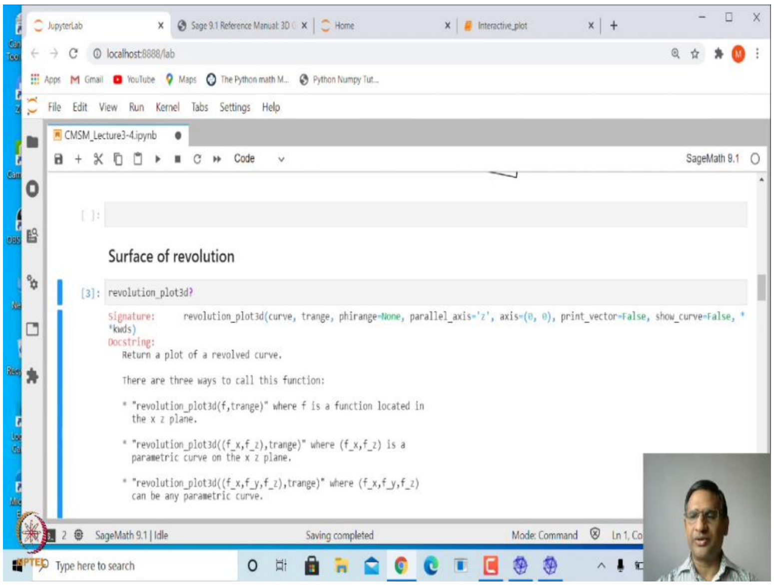

Next let us look at how we can plot surface of revolution. This concept is very important in calculus. So, Sage has inbuilt function to define surface of a revolution, this is called revolution underscore plot3d. So, if you take help on this revolution underscore plot3d, it will tell you how to use this function.

So, there are various options inside this revolution plot3d. You need to give the curve; you need to give the range in which this curve is defined. And this is $phirange$ is the rotation by which you want to rotate that curve, and about which axis that you can mention in parallel underscore axis.

And this axis is the point by which our point through which this axis of rotation passes, and this is you can even actually print the parametric curve when you rotate this curve this is the option. And then you can also show the curve along with the surface.

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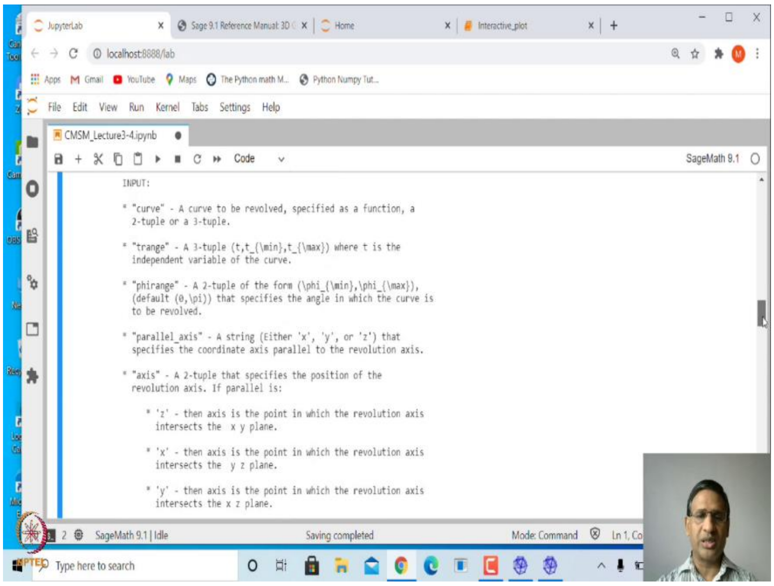

So, let us take an example. Of course, it this also gives you several examples, you can make use of this example itself.

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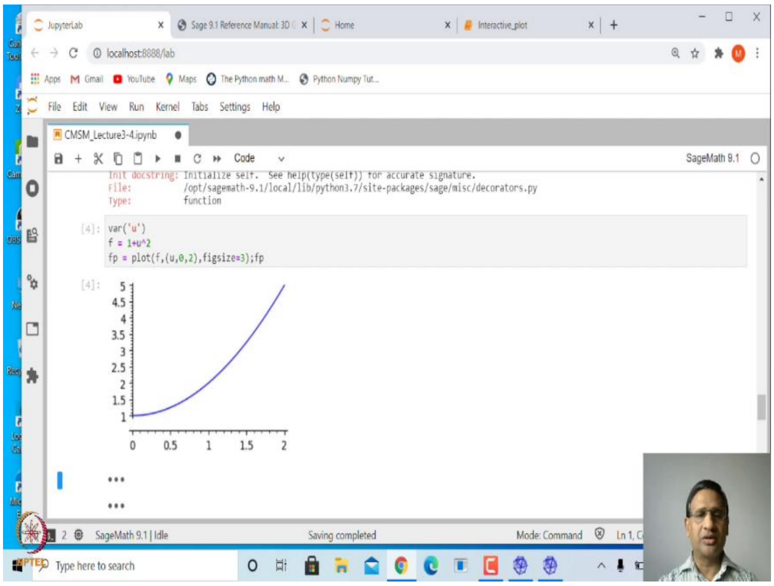

So, let us go to the first example. This is let us say we want to plot surface of revolution of the curve $y = 1 + u^2$. And u varying between 0 to 2, so that is the curve. Now, this curve suppose we rotate about x-axis, what will it generate? This will generate a surface which is called surface of revolution of this curve.

Similarly, you can rotate this about z-axis, you can also rotate about y-axis. So, let us look at one by one what kind of surface we are going to get when we rotate this curve about x-axis first, then about y-axis, and z-axis.

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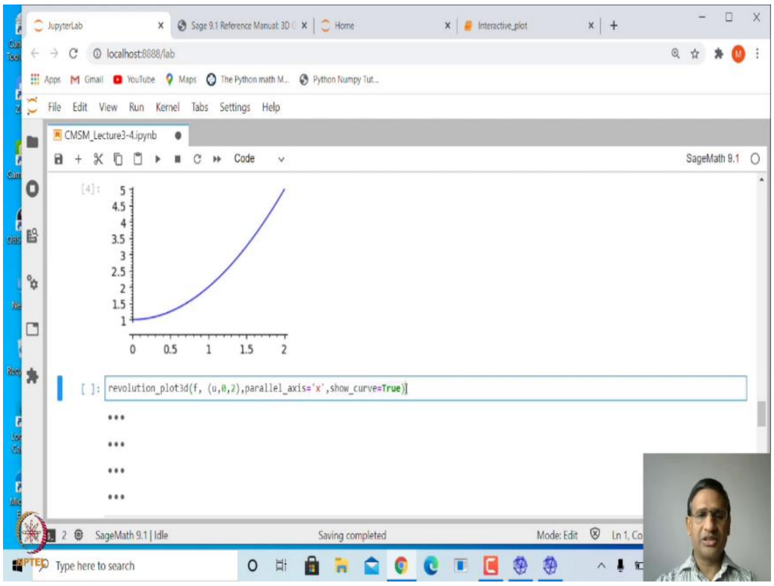

So, when and how do we use this? So, you simply say plot3d plot revolution underscore plot3d. The function is f which is $1 + u^2$, u varies between 0 to 2. And this is parallel axis is x, and you also show the curve along with the surface.

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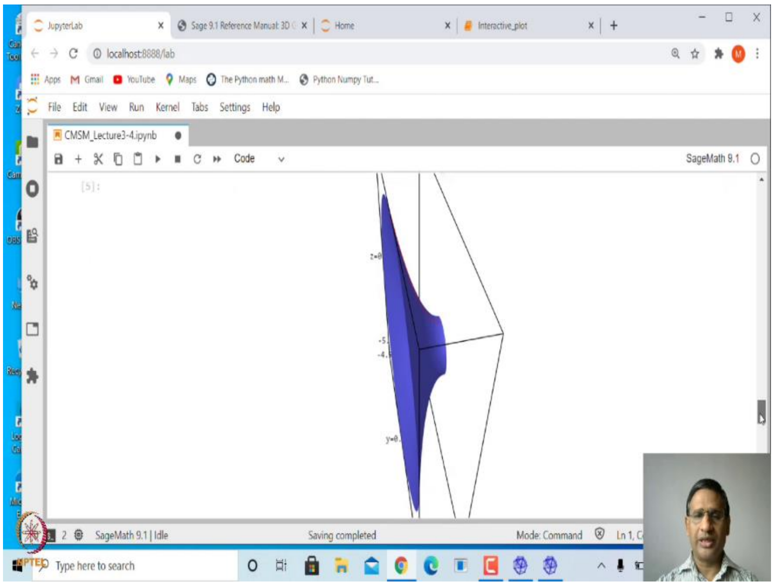

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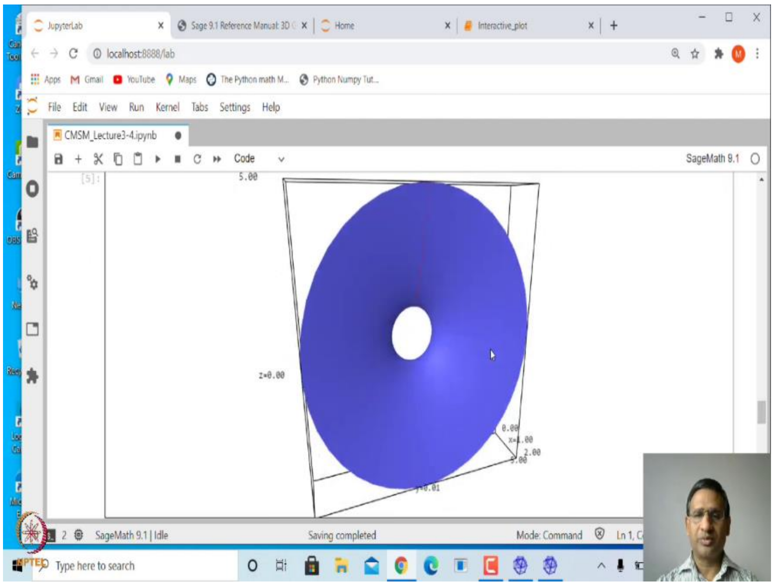

So, if I execute this, this is what you will get. This is a surface of revolution. So, let me rotate.

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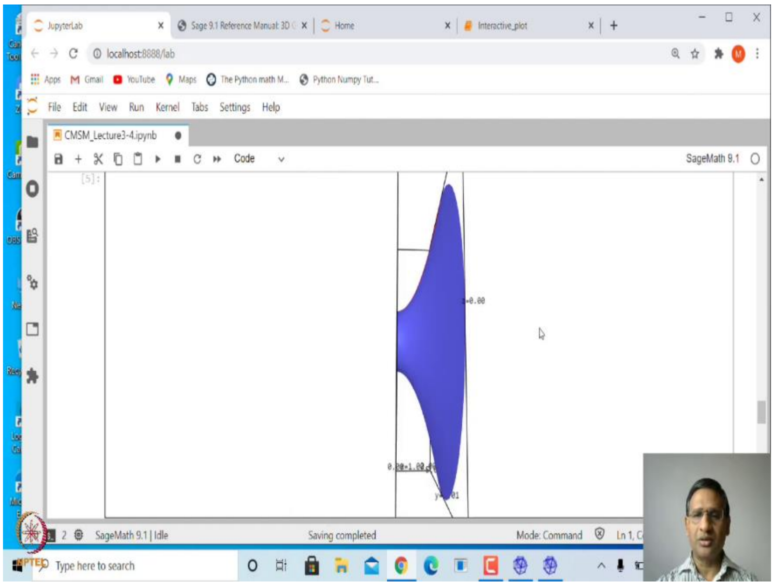

And then see this should be actually the correct way in which you can visualize this surface right, so that is the surface of revolution. If you want, you can make the figure size also small.

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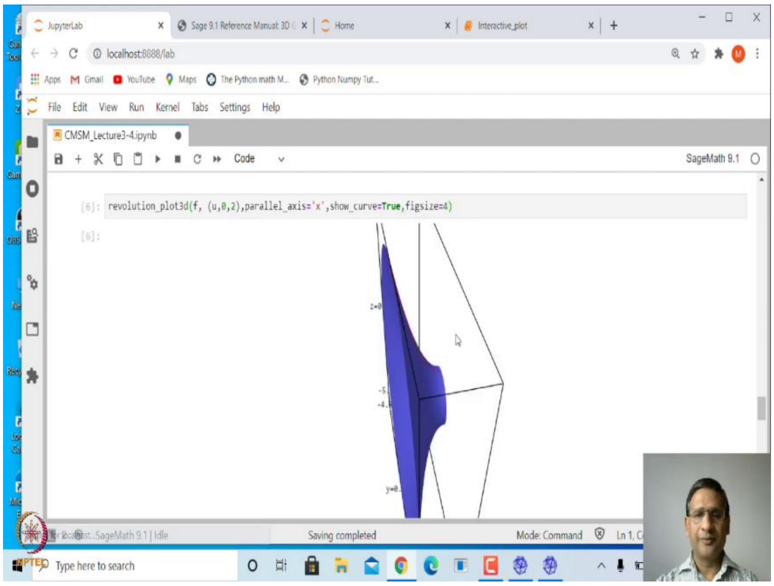

So, let me say that $figsize\ =\ 4$ words here, yeah. So, this figure size has become smaller right.

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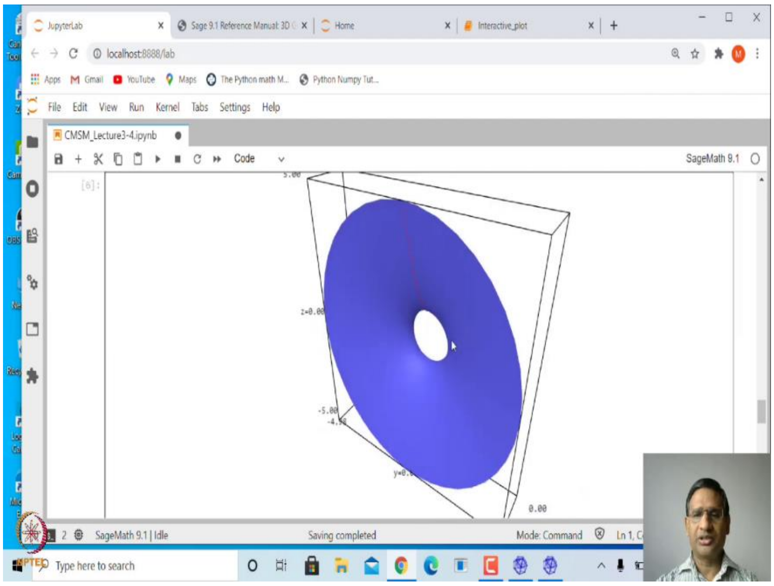

So, similarly, suppose if you plot this surface of revolution by rotating this curve about y-axis, so this is in x-y plane and if you rotate about y-axis, what you are going to get will be just a kind of disk right. So, let us check this.

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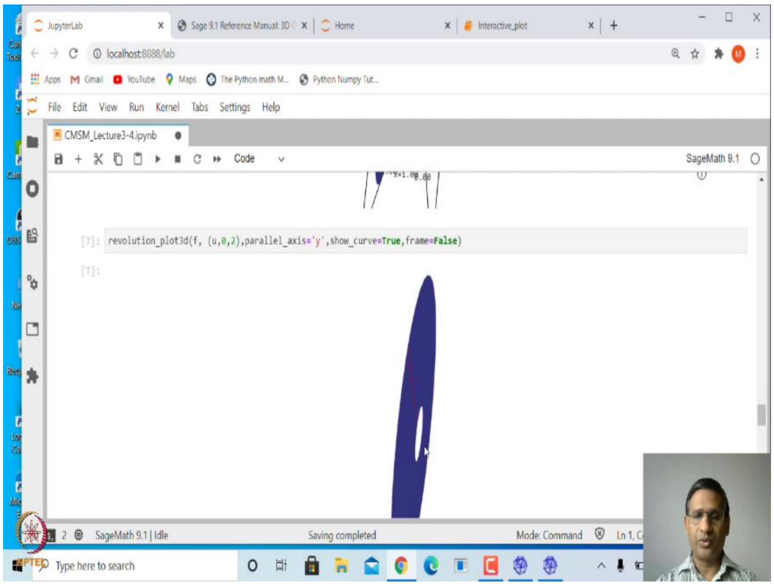

So, if you instead of parallel axis x, if you say parallel axis y and rotate, then you will see here this is only the disk that is what you can see here. Right .. yeah.

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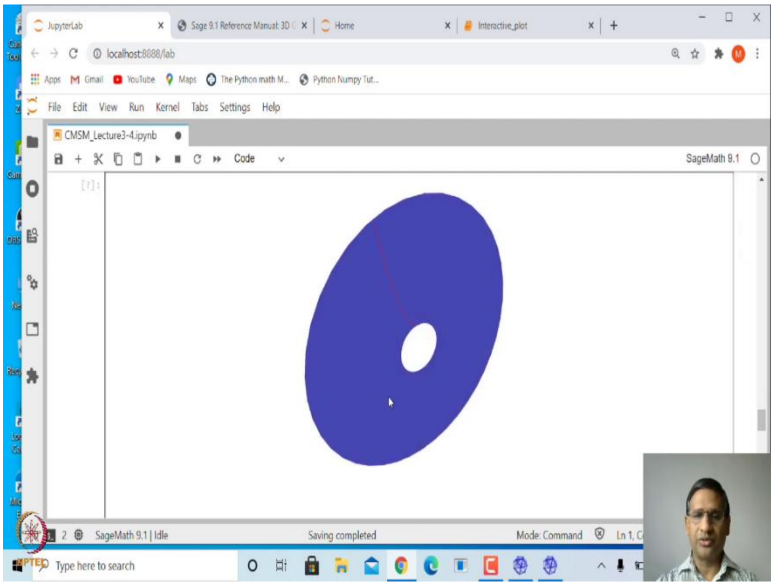

So, and here, in earlier one we have plotted the surface along with the frame. Here we have said frame equal to false that is why you do not see any frame. Advantage of keeping the frame is that you also know the axis etcetera; in this case you do not know; what are the axes ok.

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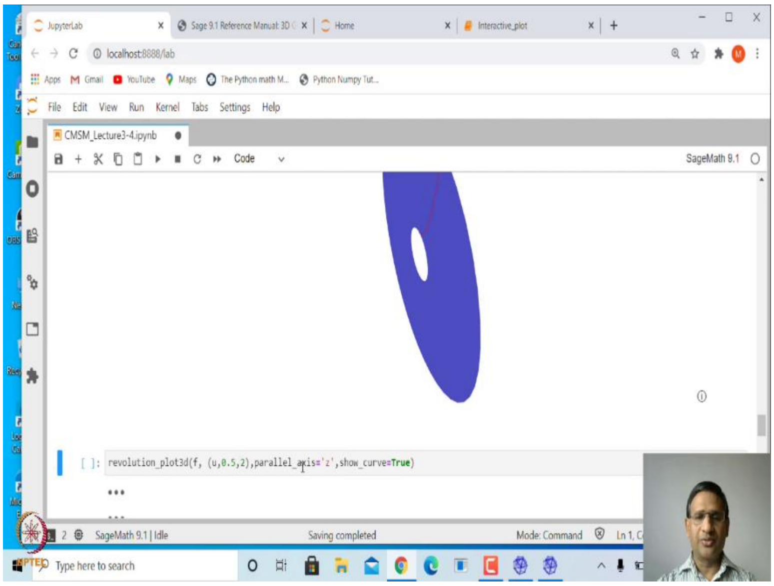

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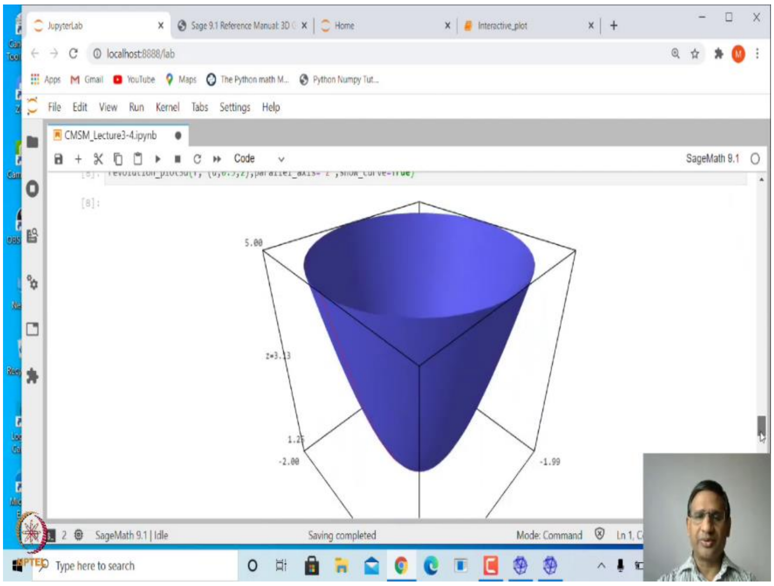

Similarly, you can rotate about z-axis. So, in this case, the surface which we will get is something like this yeah.

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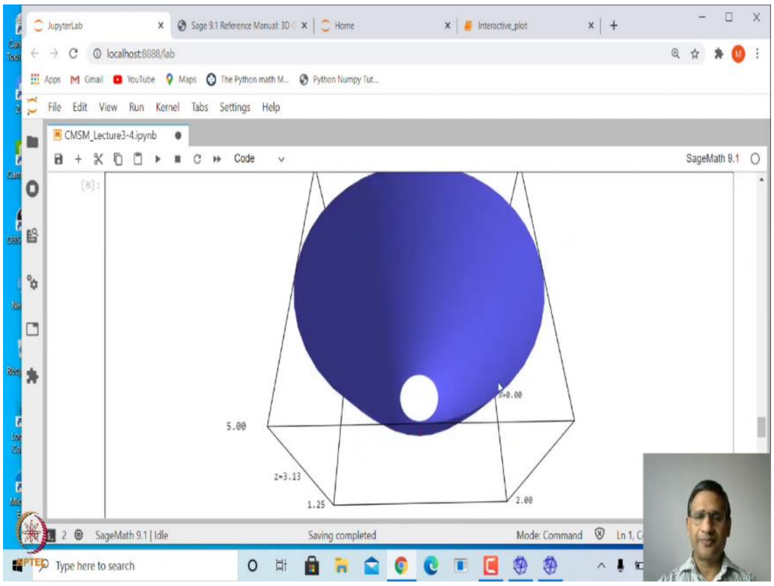

So, this is here actually let me just see here we have plotted from 0.5 to 2.

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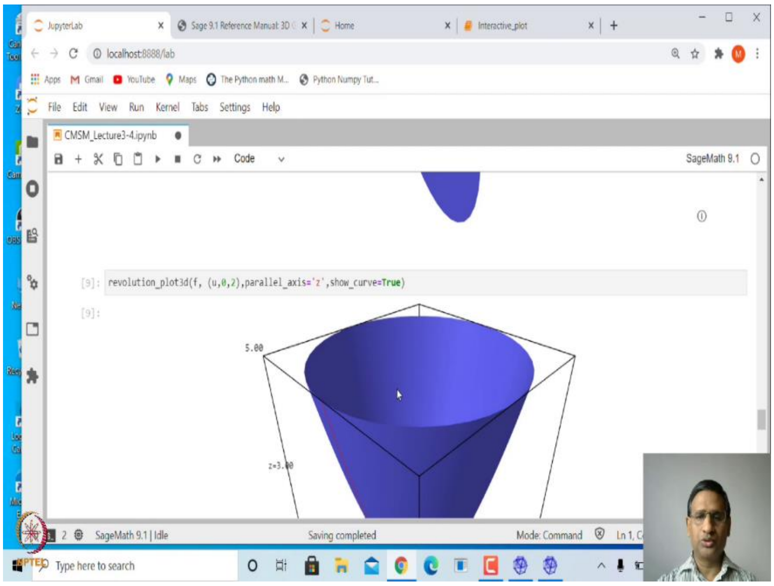

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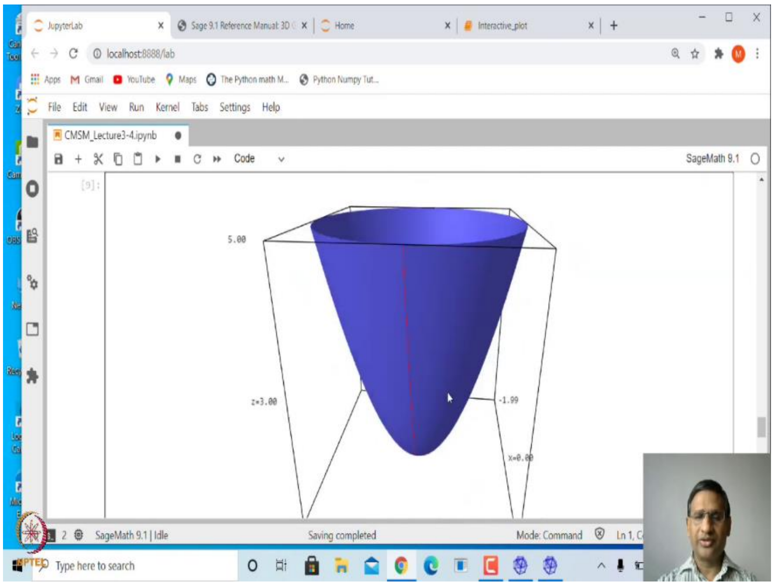

Let me plot from 0 to 2. And this is a kind of what is called paraboloid right. So, you are rotating this parabola about z-axis.

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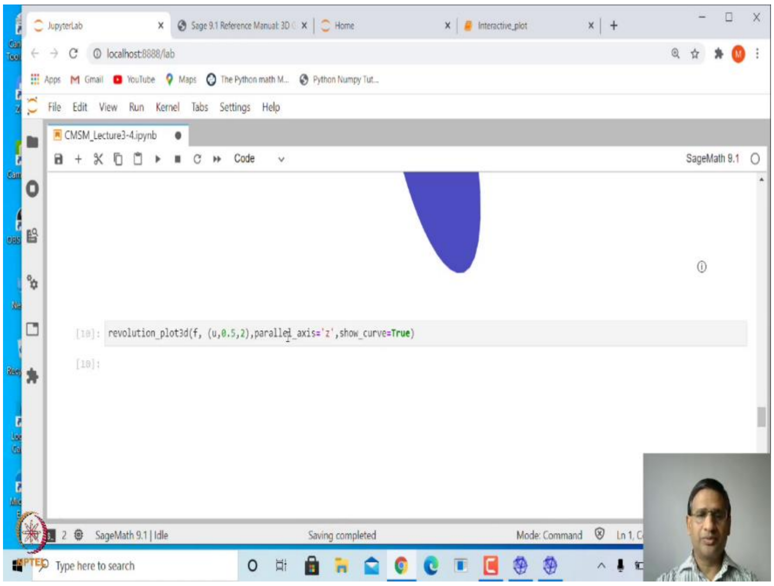

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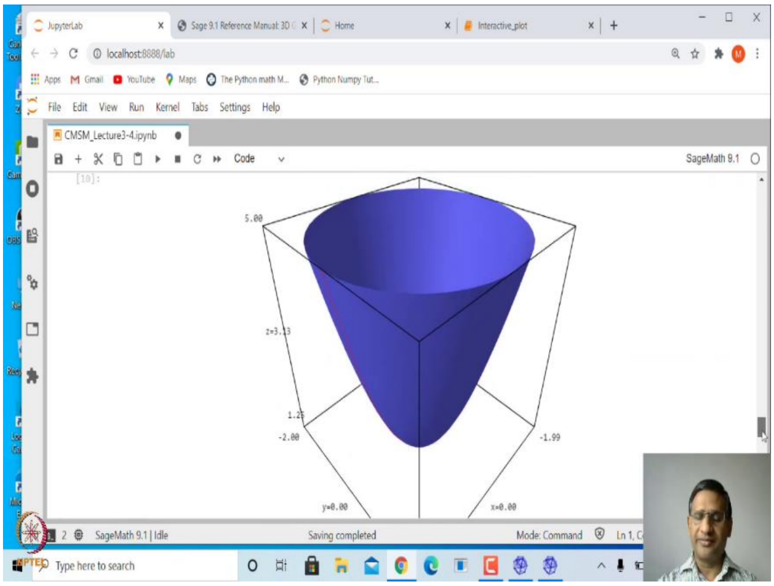

And in case, we plot from 0.5 to 2, if you vary from 0.5 to 2, then this portion of this curve from 0 to 0.5 will be left.

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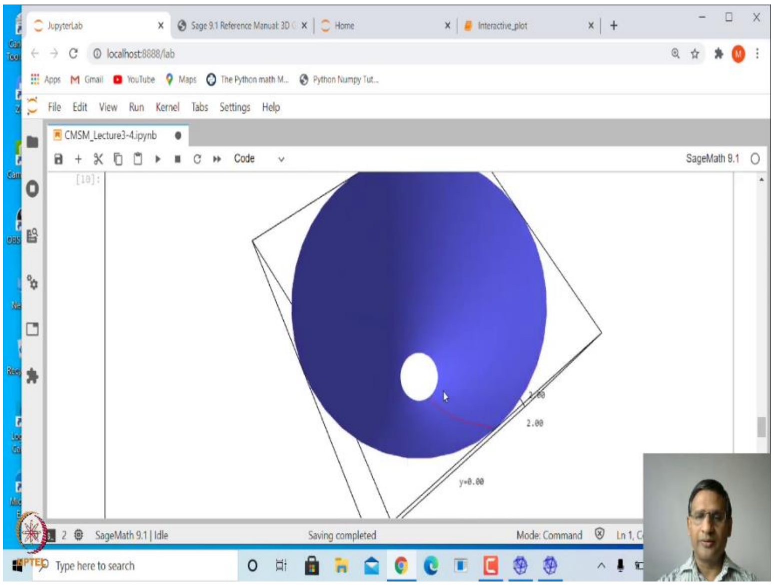

And as a result you will see a hole here that is the portion which is omitted from 0 to 0.5, right, so that is how you can plot surface of revolution about various axes.

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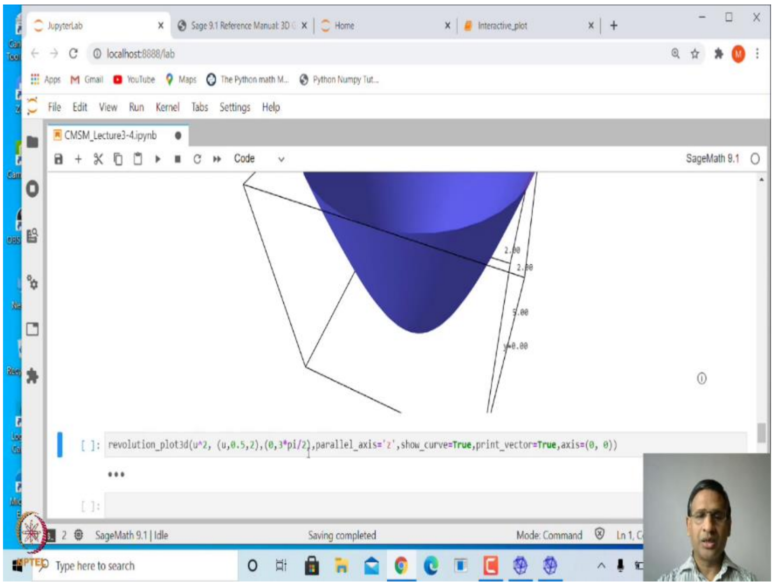

You can even mention the angle by which you are rotating. So, by default it rotates by an angle, this curve it rotates by an angle 2pi, i.e. 0 to 2pi. But suppose I want to rotate only from 0 to 3pi/2 that is by or let us say by angle pi.

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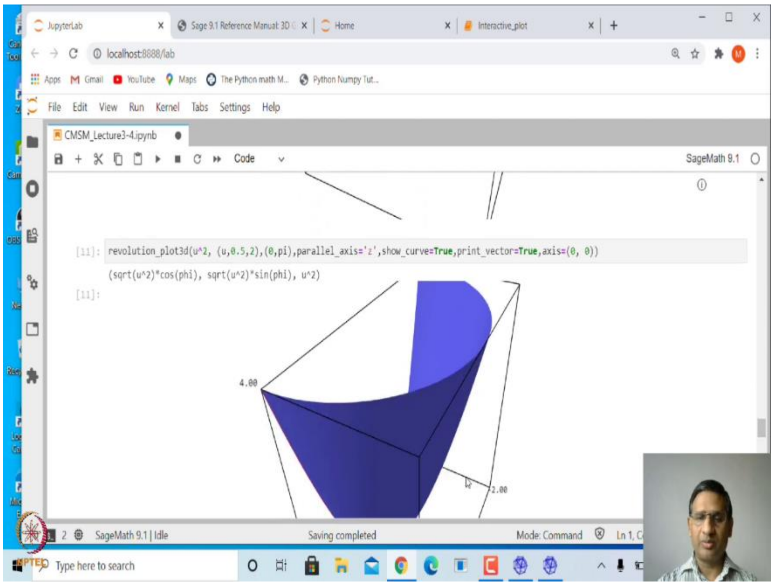

Then, you have to just mention this 0 to pi and everything else is same axis is parallel equal to z and curve show. And this print vector underscore vector equal to true will tell you what is the parametric coordinates of the surface of revolution.

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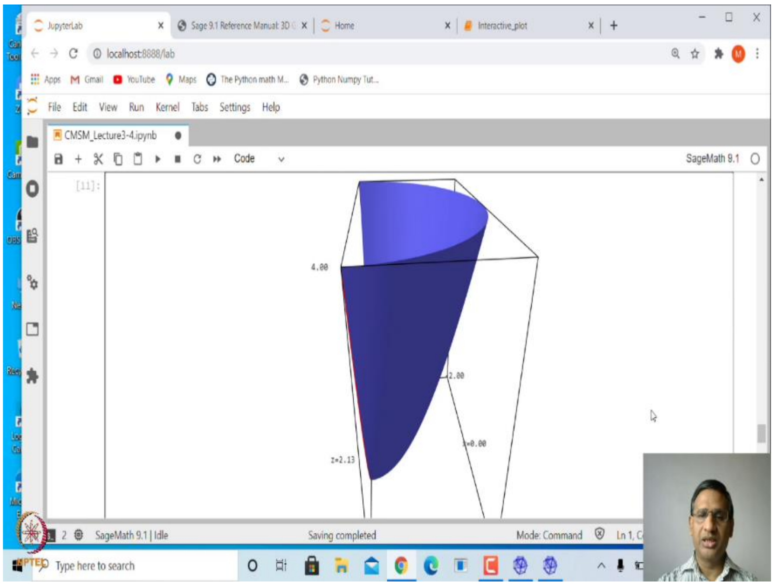

So, that is what you can see here. So, it has been rotated by an angle pi that is what you see here half of the paraboloid. And, this is the parametric equation of the surface in terms of u and phi. The phi is the angle of rotation right. So, that is very nice, whatever by angle by which you want to rotate you can do.

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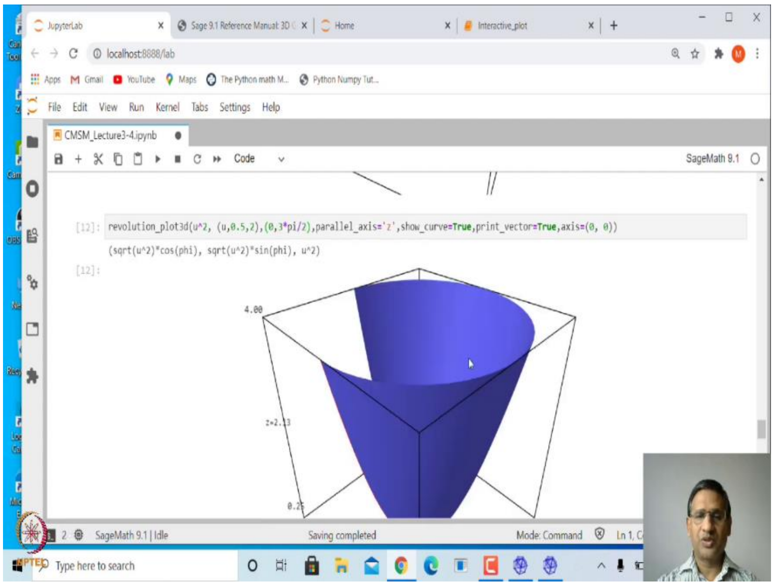

So, if I say 3 pi by 2, then it will be three-fourth of this paraboloid right.

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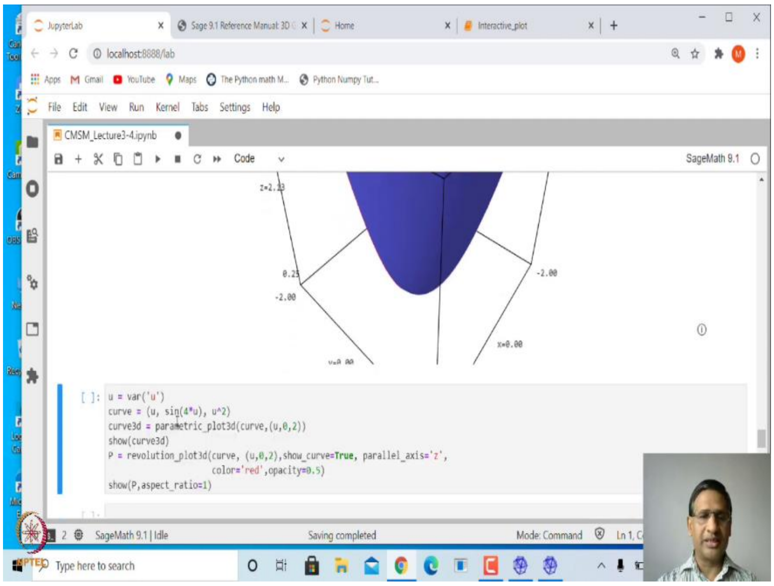

Let me give you one more example. Here the curve is taken as in 3d which is first coordinate is u, second coordinate is $sin(4u)$, and third coordinate is u^2. Actually, this example is taken from the help document itself. And this then you plot the curve 3d.

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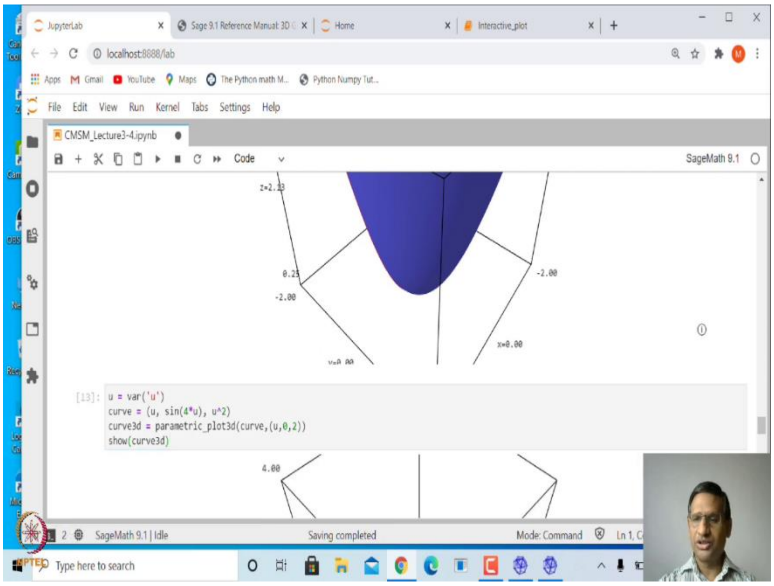

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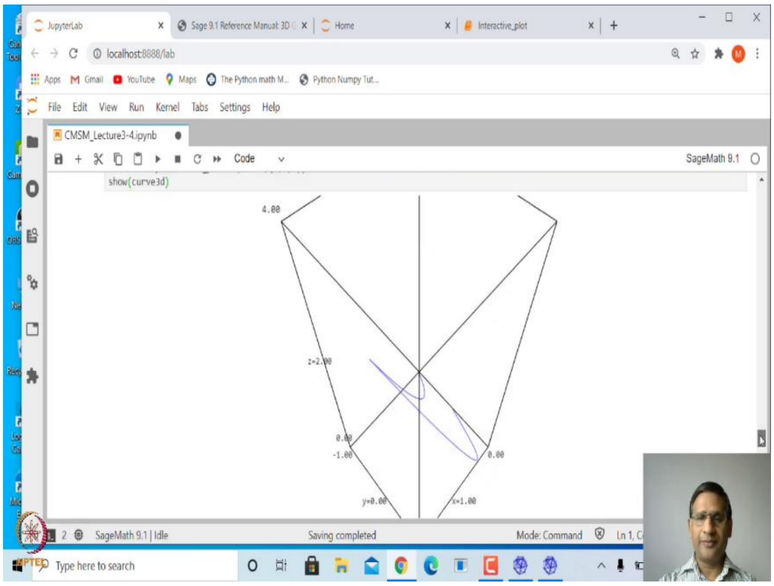

And let me ask it to show the curve first. Let us first show the curve. So, this is the curve, this is the curve in 3d. Again, let me rotate and show you that is the curve.

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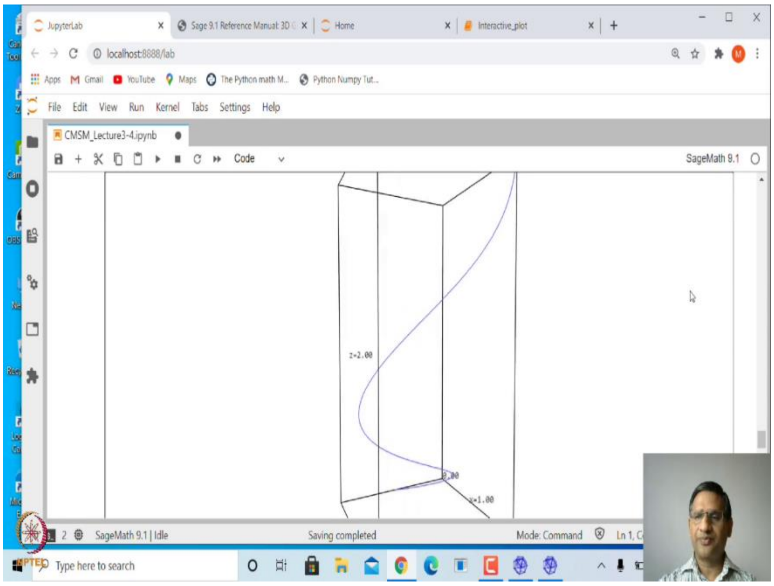

And this we want to rotate about let us say about z-axis. And then what do we get?

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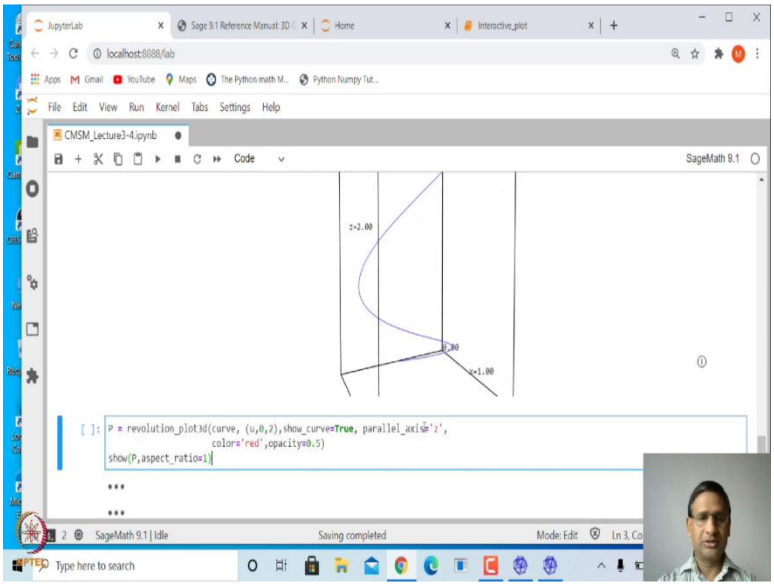

So, let me say this is a revolution underscore plot of the curve. Let us say u takes values from 0 to 2, and show the curve, the color is red. And opacity = 0.5. So, by default opacity is 1. When you reduce the opacity, this surface will become transparent ok. So, let me run this. And then you ask it to show this p, and aspect ratio is 1 which is ratio of x y with z will be 1.

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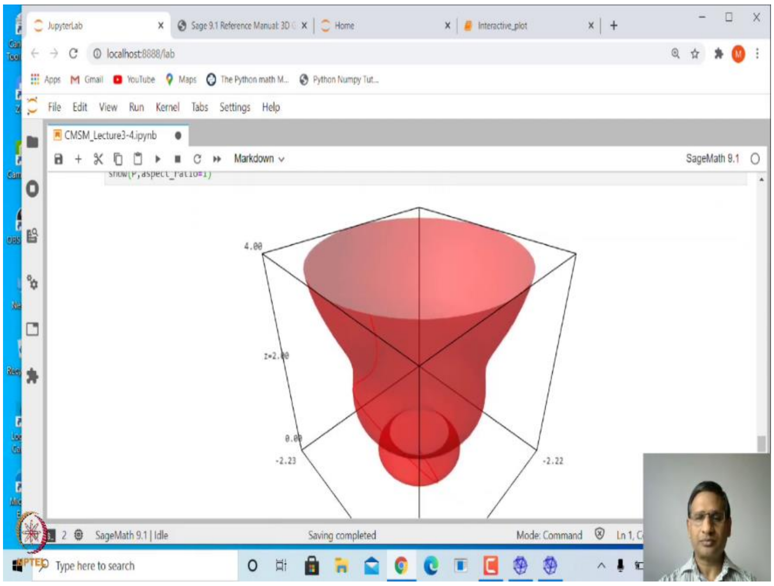

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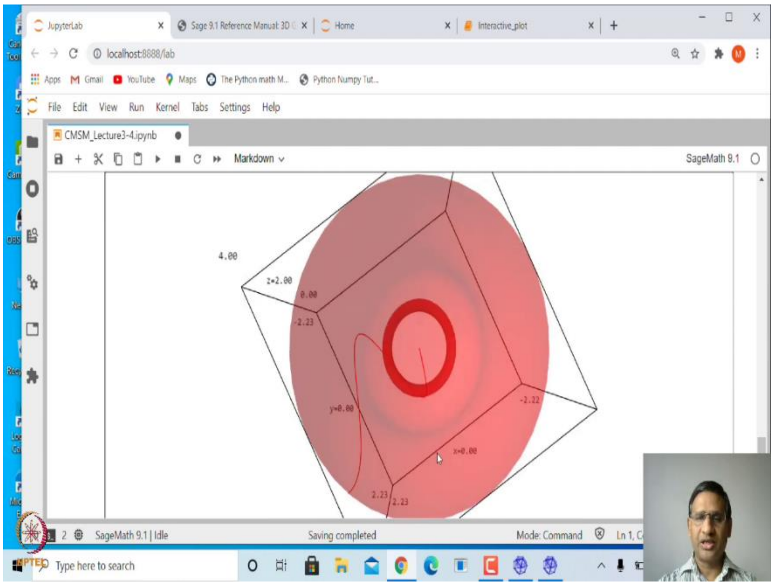

So, that is the surface this looks very nice surface, and it looks like a very nice cup right. So, you can change the color. You can see here the color of the curve along with the surface.

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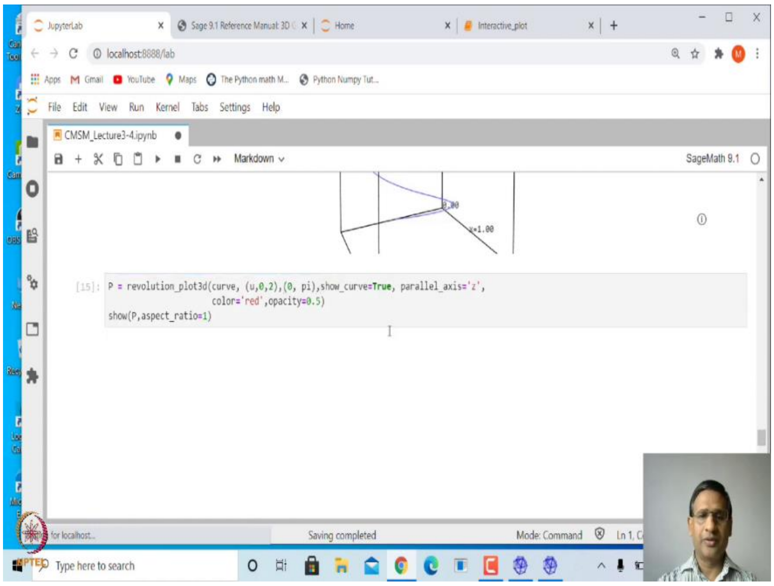

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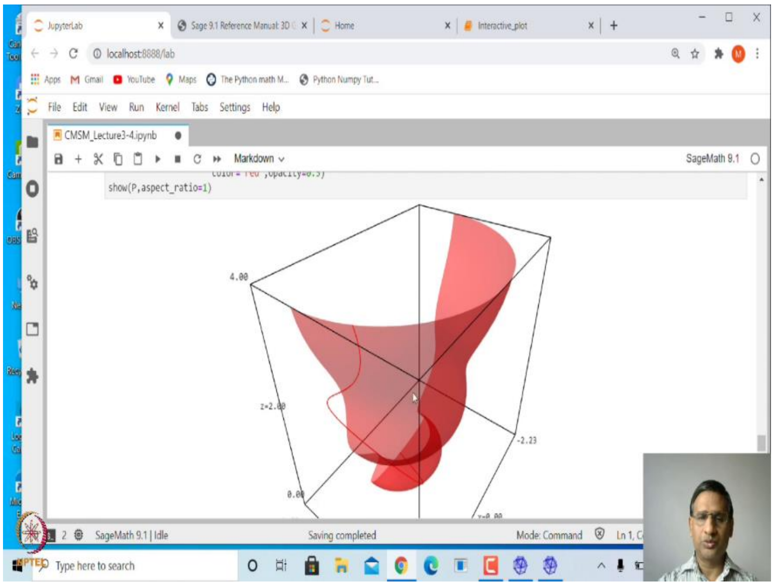

And if you want to rotate only between 0 to let us say 0 to pi, then only half of the cup will be shown that is what you can see here right. So, this is a very nice curve.

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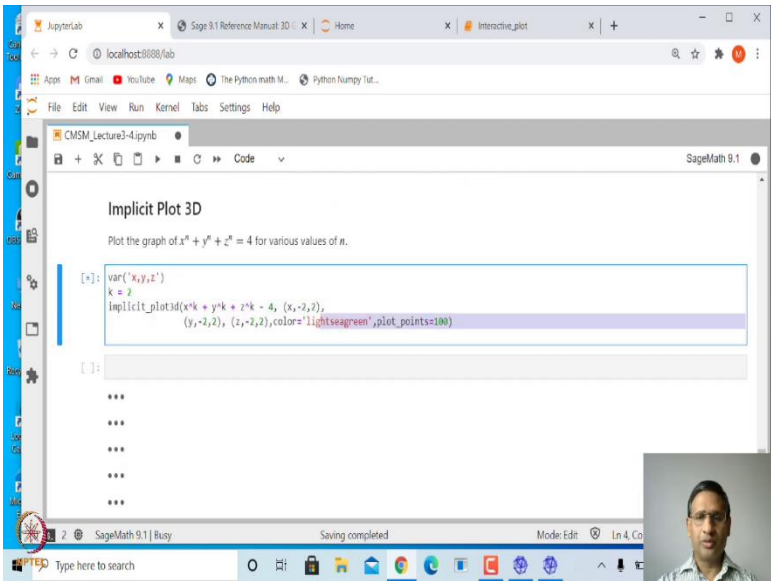

Similarly, you can plot a surface, which are defined implicitly or implicit 3d, this is a function $f(x, y, z) = 0$. So, suppose we already in the previous lecture we saw how to plot graph of $x^n + y^n = 1$ and for various values of n. Now, we can extend this to $x^n + y^n + z^n$ is equal to instead of 1, you make it 4, and then for various values of n.

So, let us try to plot this. So, in this case, we need to declare three variables x, y and z. And let us say we instead of n, I have called here k, let us take k equal to 2. So, if I take k equal to 1, this is $x + y + z = 4$ which will be a plane.

But k equal to 2 will give me $x^2 + y^2 + z^2 - 4$ equal to 0 and x and y varying between -2 and 2, z also varying between -2 and 2. So, this will be actually a sphere. So, and this color is light green – light sea green. And we are plotting 100 points. So, let us run this you will see a sphere. It is taking little bit time right. And then we will change the value of k, and then we will see what kind of surface we get.

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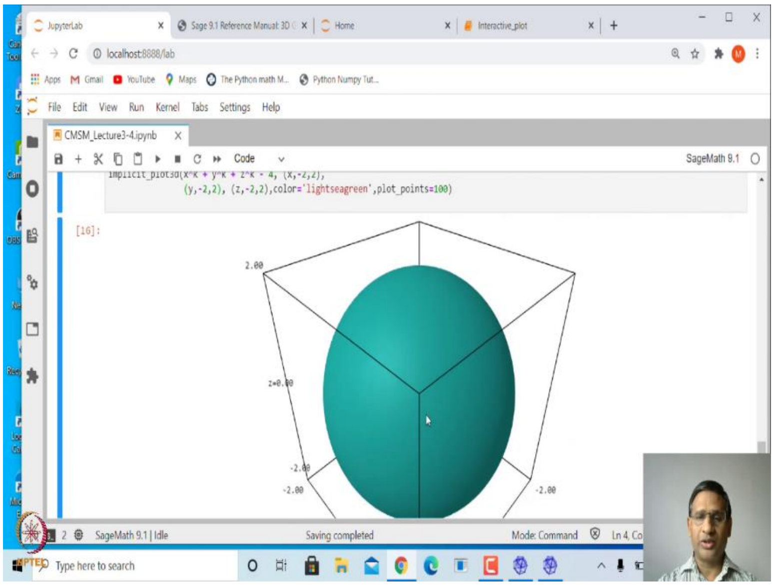

So, if you roughly remember what we got in case of $x^n + y^n = 1$, you should be able to visualize what kind of surface we are going to get. So, this is a sphere which is $x^2 + y^2 + z^2 = 4$

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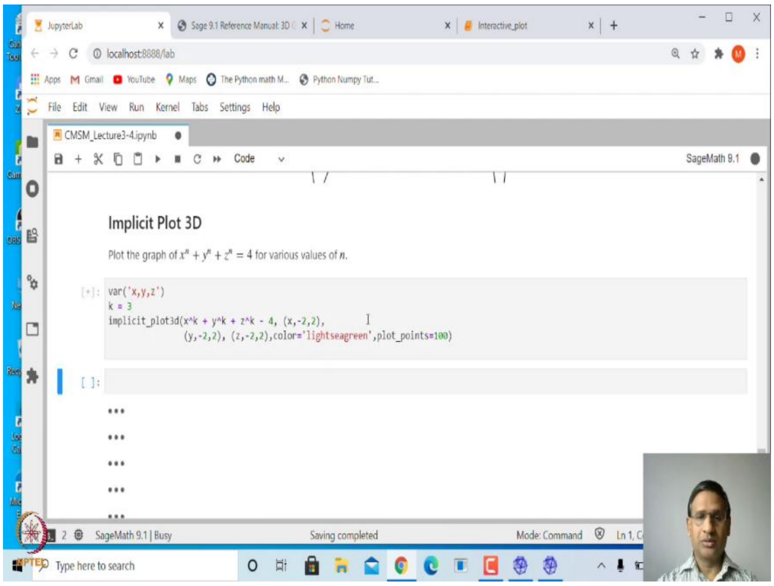

Now, let us change k from to 2, let us change it to 3. And in this case, you should be getting a kind of not a closed surface, the even one will give you close surface whereas the odd ones will give you not a closed surface. So, let us look at what is the plot we are getting. Again, this will take time because this implicit plot is slightly more involved.

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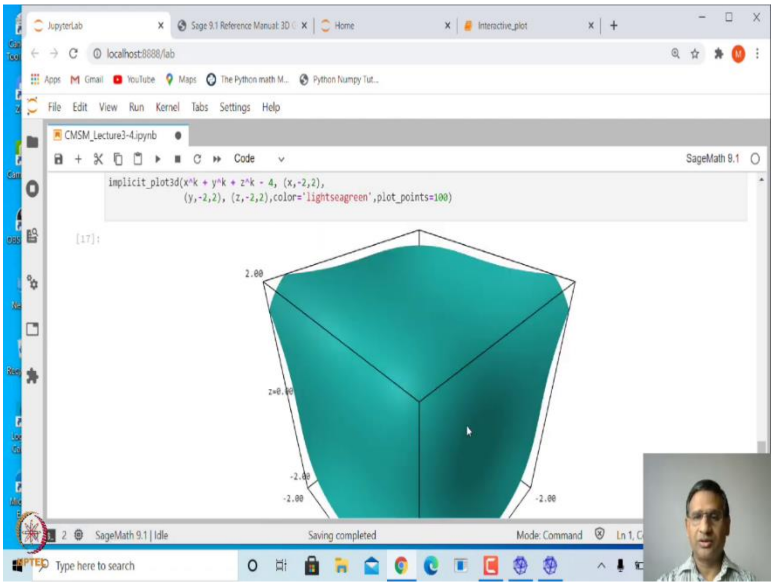

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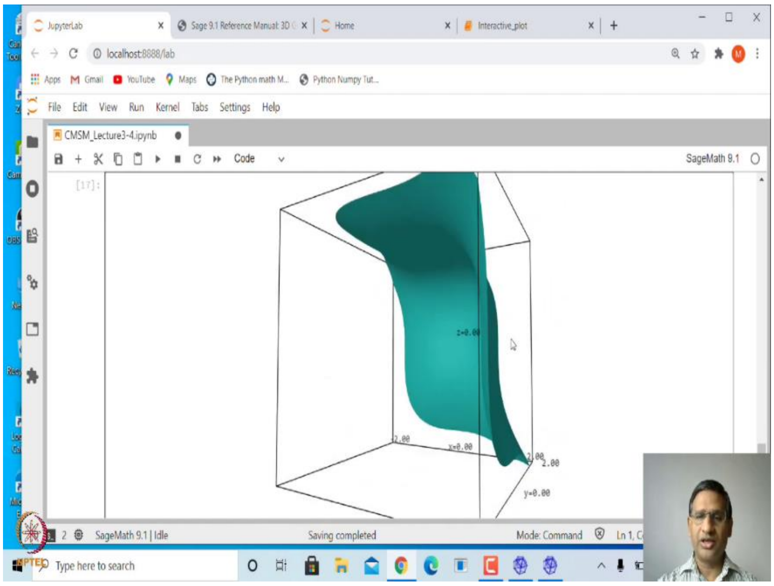

So, this is the surface. Again, you can see here this is a kind of cap right. So, a kind of you can think of this as a hat right.

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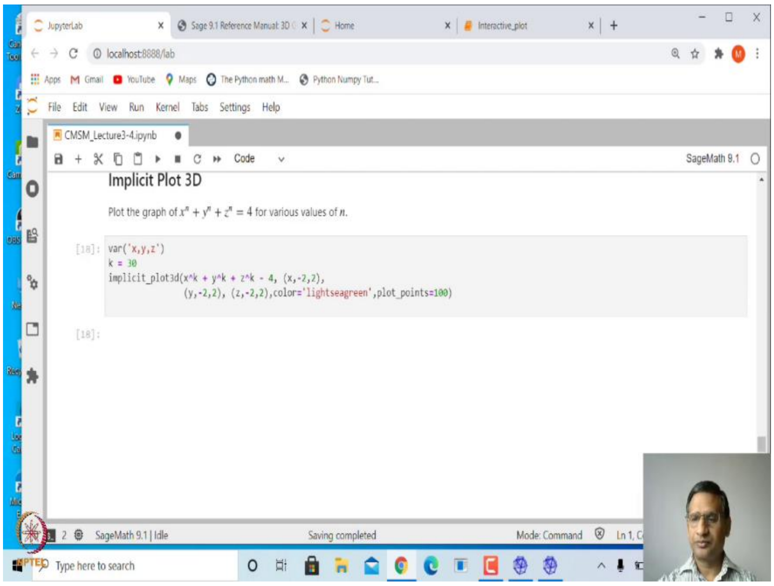

Now, similarly, if I look at for example, instead of 3, if I say let us say 30 and then try to plot it may take even more time, but you should get a closed surface. It may have some corner, yeah, it is like a cube right.

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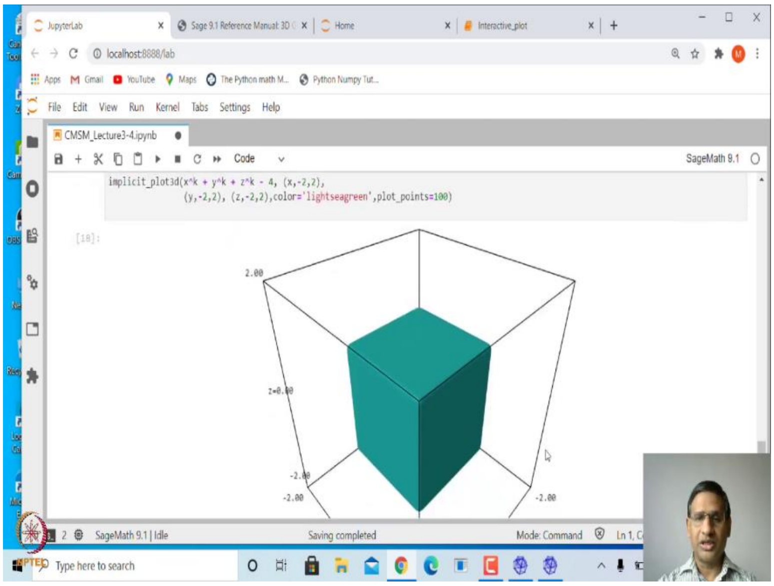

As you increase this; the value of n and if it is even, this sharpness of this is will increase.

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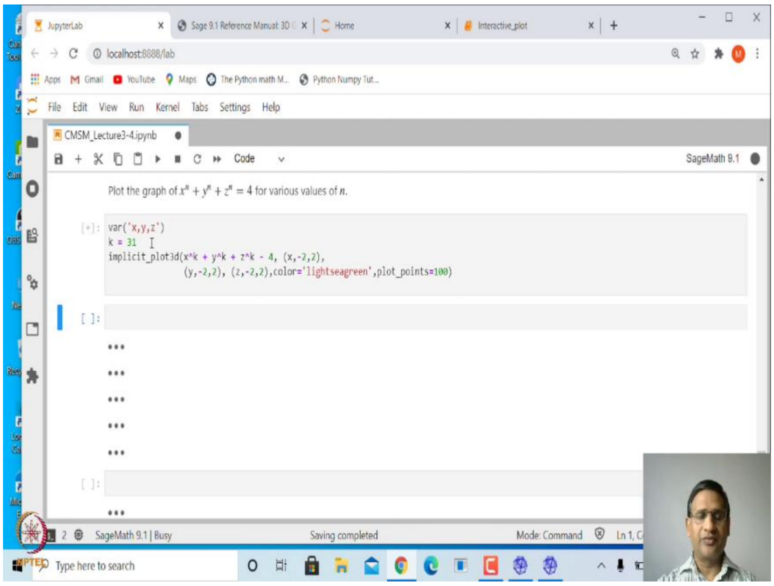

And in case of odd, let us say for example, 31 will give you again a surface which is not closed; it is a kind of cap. But it will have some kind of edge right.

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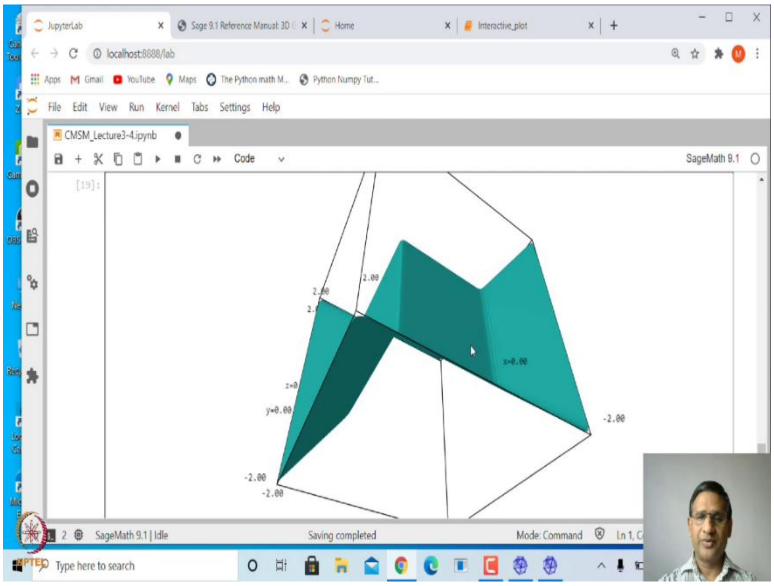

Again, you can see here check this, this is a again some kind of a squarish cap you can think; of course, this does not look that good. It may not look good on our head, but this is also a kind of cap right.

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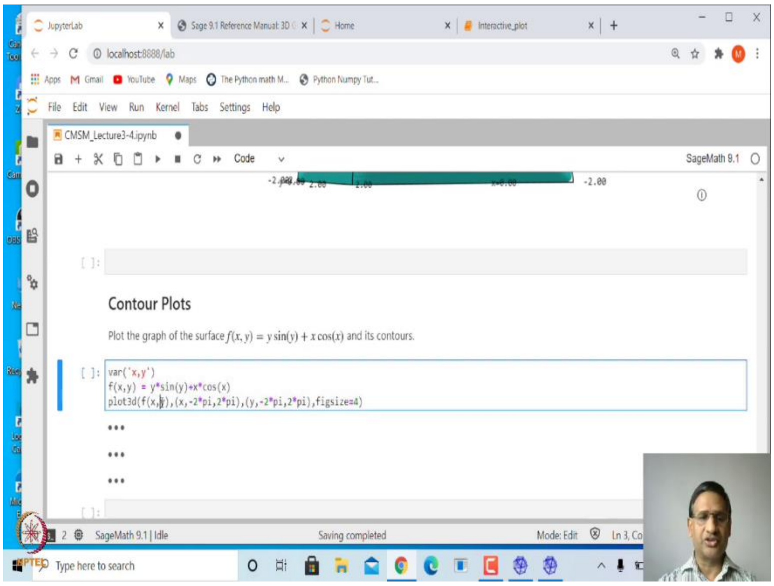

So, you can try to plot various other implicitly defined functions in 3 dimension.

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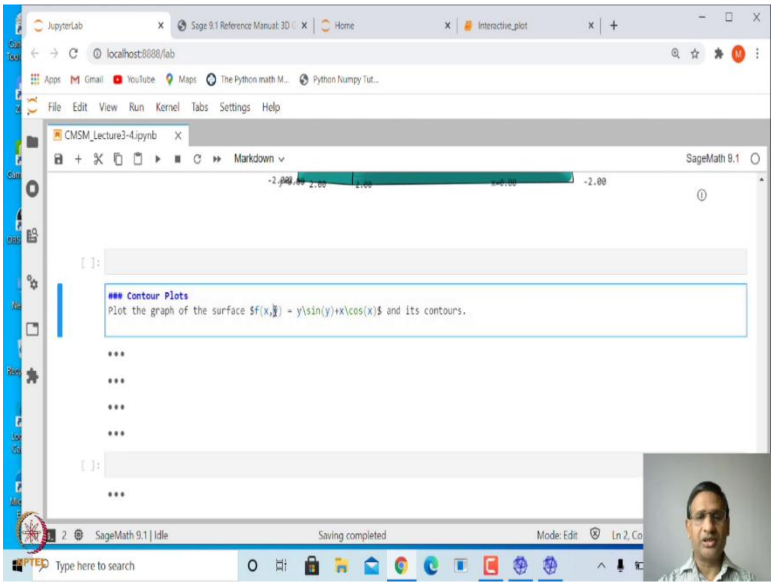

Now, let us say we if we plot graph of this function, let us say $f(x,y) = y.sin(y) + x.cos(x)$ suppose let us plot graph of this function.

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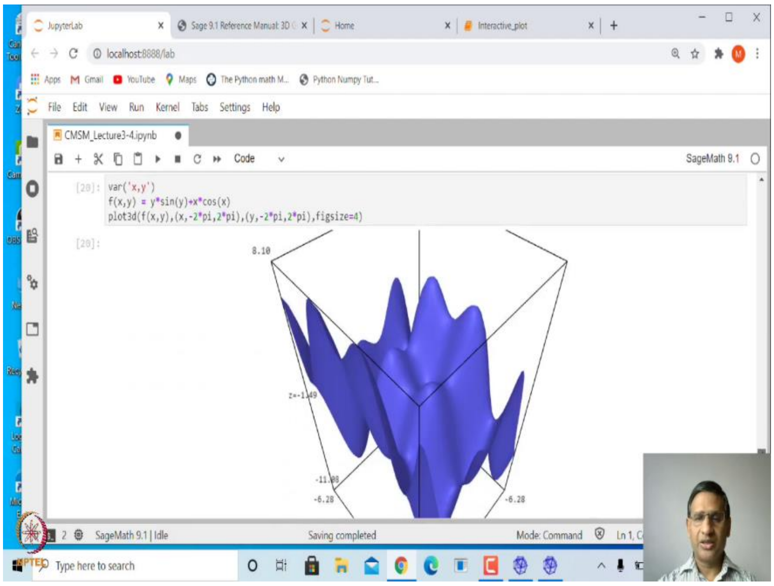

Now, when you plot the graph of this function, this is the surface. Now, how does one build this surface? So, if you actually take the slice of this surface by z planes various z planes or planes parallel to x-y, then you will get curves. This curves you project it on x-y plane they are called contours.

So, Sage has inbuilt function to plot various contours. And once you know what are the contours of a function $z = f(x, y)$, and you can put all these contours one over the another, then you will be able to build this surface, so that is how surface is actually plotted.

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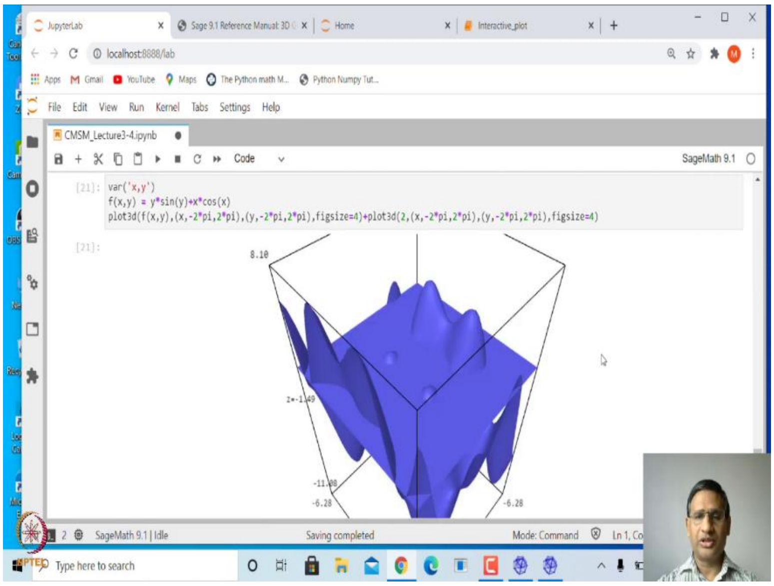

So, let us see how we can plot the surface. So, for example, in this plot let me add say plot3d of z is equal to let us say I will call z is equal to let us say 2 plane z = 2, and let us plot this yeah. So, let me put color here in different color.

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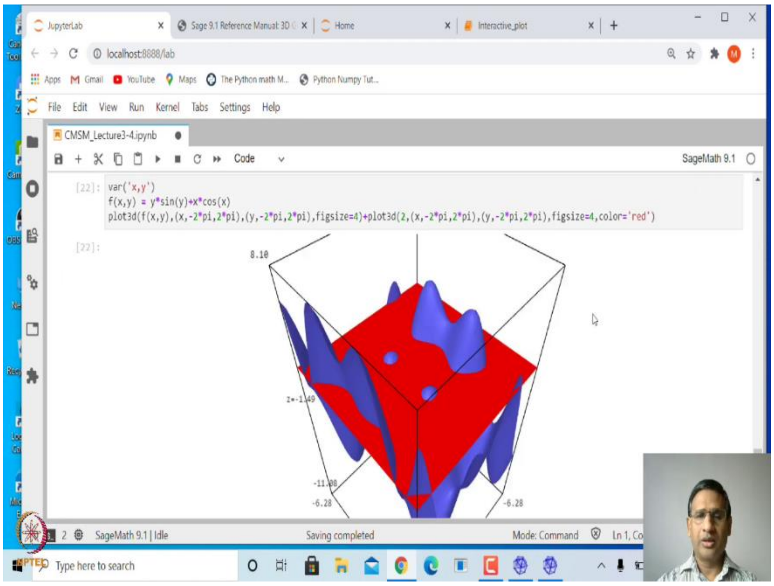

So, color equal to let us say red inside single quote right. Now, you can see here, this is the intersection of this plane by z = 2 plane and the intersections are going to be curves. So, this when we project these intersections which are curves which we project on x-y plane, this curve is called contours right.

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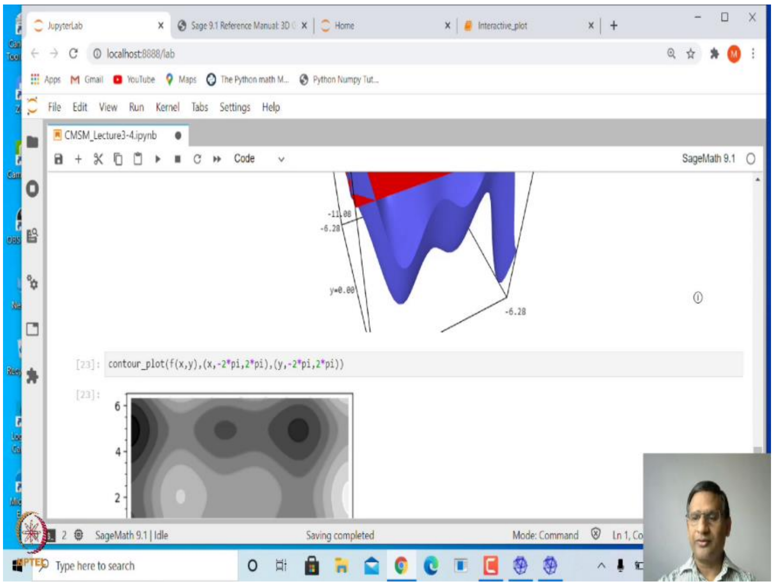

So, let us plot these contours. So, Sage has inbuilt function called $contour\_plot()$. And again you supply the same option like control plot of $f(x, y)$ and x going from -2pi to 2pi, y also varying from -2pi to 2pi.

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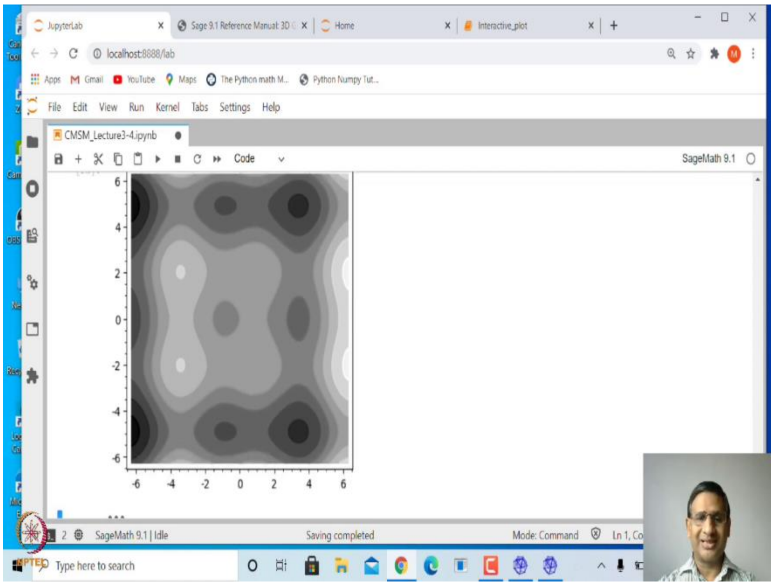

So, when we do that, it will plot default some contours. But this has actually shading inside. The shading depends upon the height of the contours, so that is why we can say graded shading.

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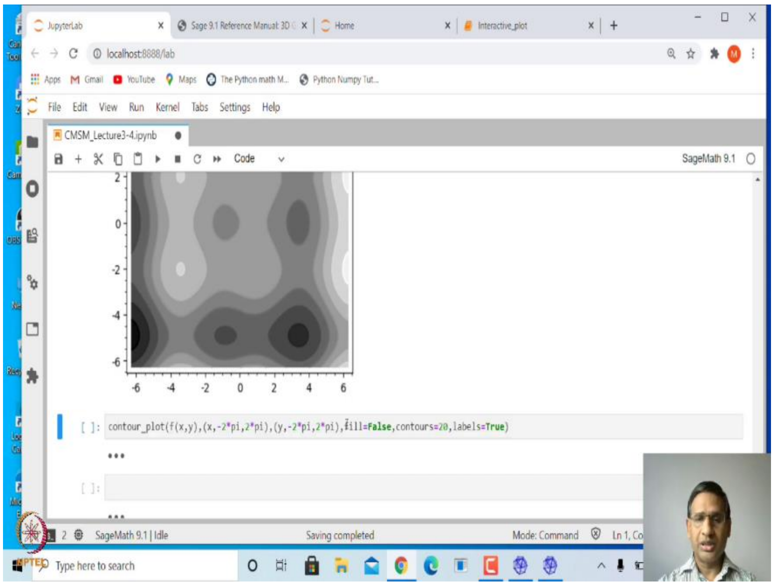

Now, you can even say fill equal to false, in that case these fillings will disappear and you can also increase the number of contours. By default, it has plotted few contours. And but if you want to plot more contours, so for example, if I say 20 contours what will happen?

In this particular domain, this will take value of z minimum value of z and maximum value of z. And that minimum to maximum this interval will be divided about equally into 20 parts, and those 20 contours you will see right.

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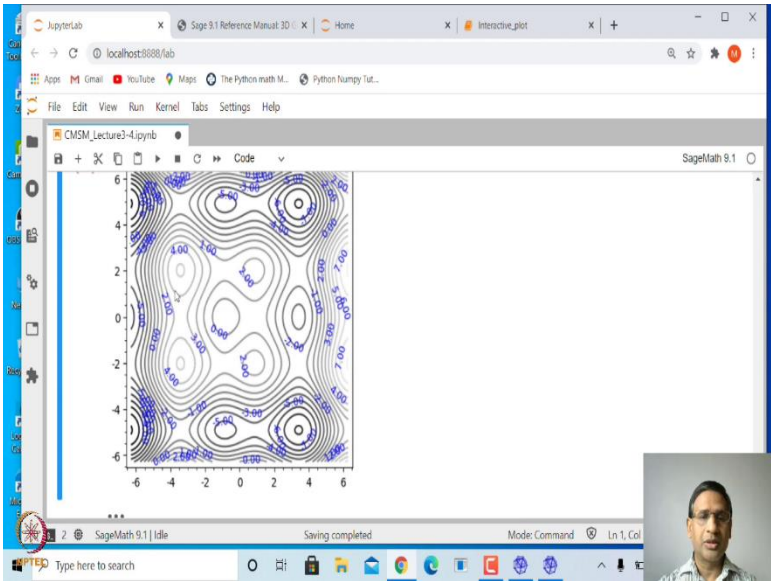

So, let me yeah. So, these are the contours and in fact, I said label equal to true, so at each one, it also prints a label. So, for example, if you look at this particular curve, this is z = 2 plane, the z = 2. So, $f(x, y) = 2$ and this one, and then there will be other one.

For example, there should be another counter part of this, so somewhere here, yeah. So, this is the one z = 2, there is another side, z = 2 right. So, you can plot these contours at various level.

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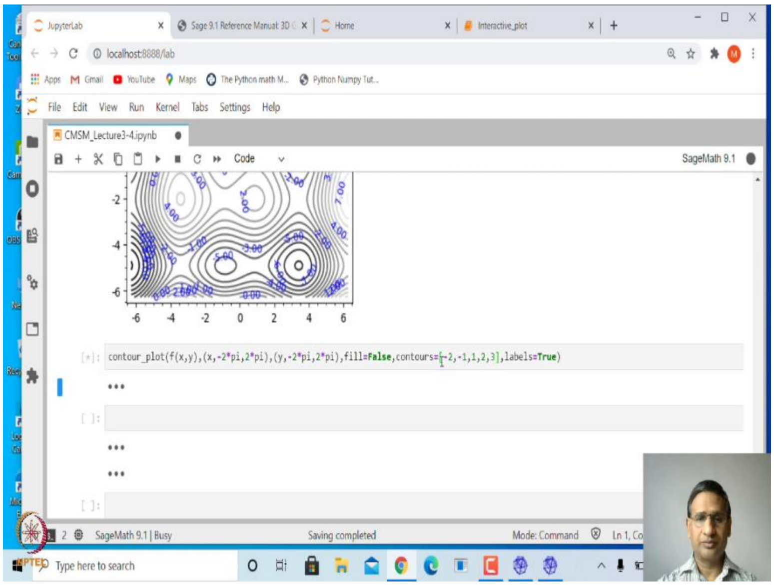

In fact, if you can give this label also, you can mention for example, let me just copy this and insert and let us say we want contours at let us say -2, then at -1, at 1, at 2, and then let us say at 3.

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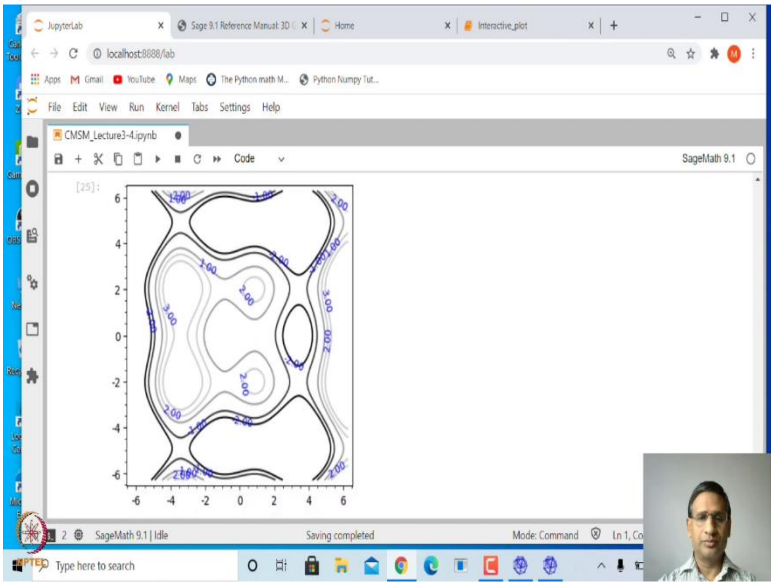

So, only these contours will be shown ok. So, one is at -2, then at -1, then 2, 1 and so on right.

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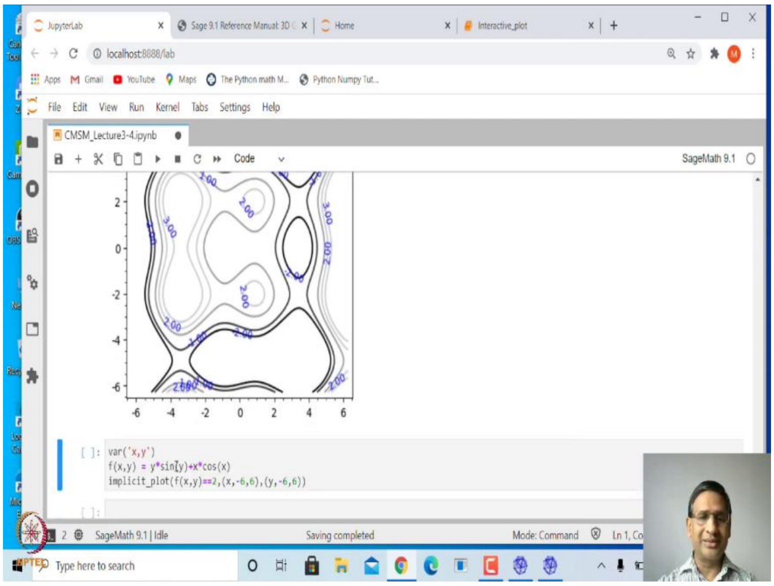

So, and that you can obtain just exactly by you can use implicit plot and $f(x,y)$ equal to 2.

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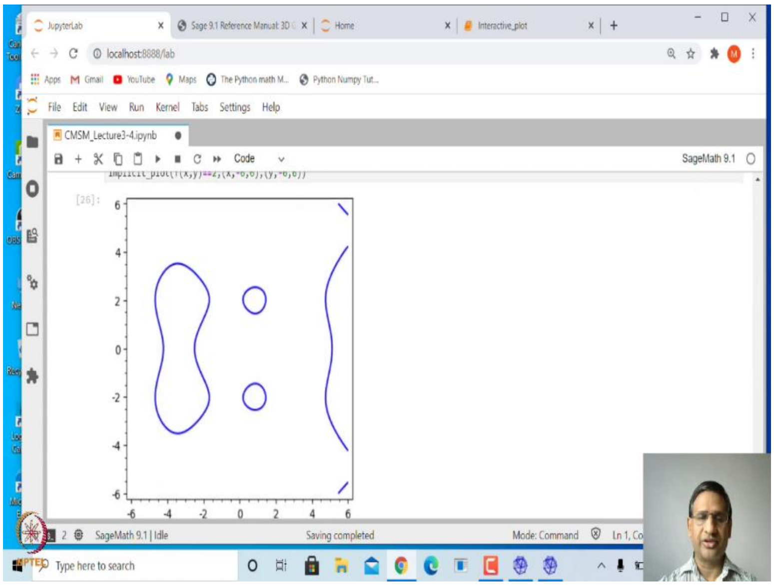

And then when you plot this exactly what you will see. So, for example, $f(x,y) = 2$ plot, and if you compare this the curve with this z equal to 2 contours here, this exactly what you got. Of course, here it was plotted between -2pi to 2pi; here I plotted between -6 and 6, so right ok.

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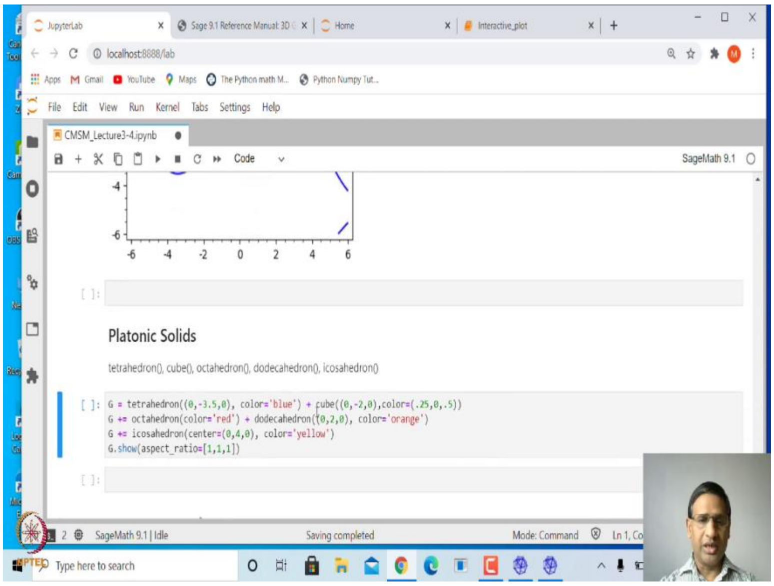

So, next is many of you must have already seen there are actually exactly five platonic solids. And these platonic solids they are called tetrahedron, cube, octahedron, dodecahedron and icosahedron. So, these, these, these things you can plot in SageMath. There are inbuilt functions. This tetrahedron round bracket – empty round bracket, cube or empty round bracket, this will plot these platonic solids.

So, let me just plot all these platonic solids together. This is again this particular code is being taken from the help manual. So, first it is plotted tetrahedron in blue color. And in this you add cube. And then in that G, you add octahedron and dodecahedron. And then you finally, you add icosahedron and then plot this together.

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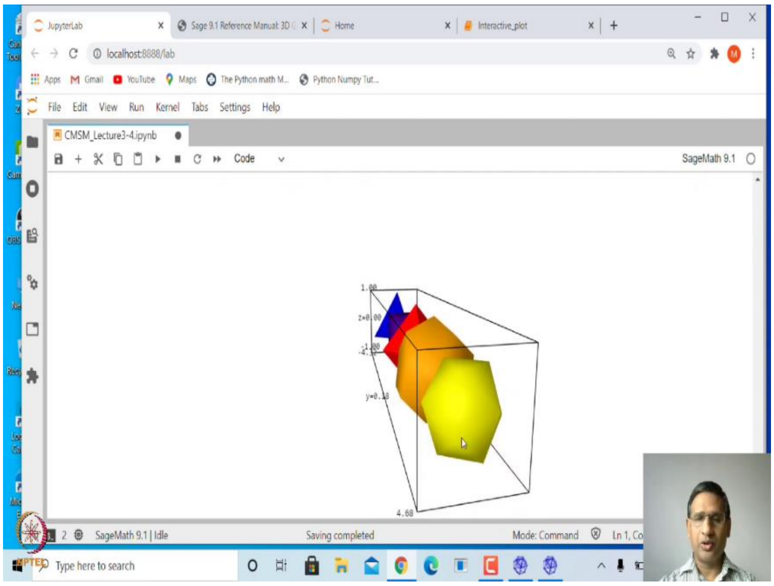

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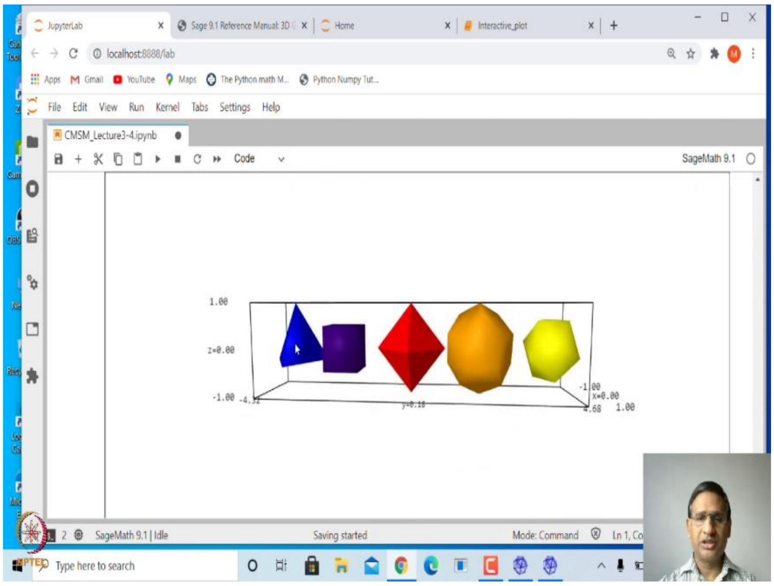

So, this is the plot of all these five platonic solids, this is tetrahedron, this is cube, octahedron, this is a dodecahedron, and this is icosahedron right. You can plot separately also.

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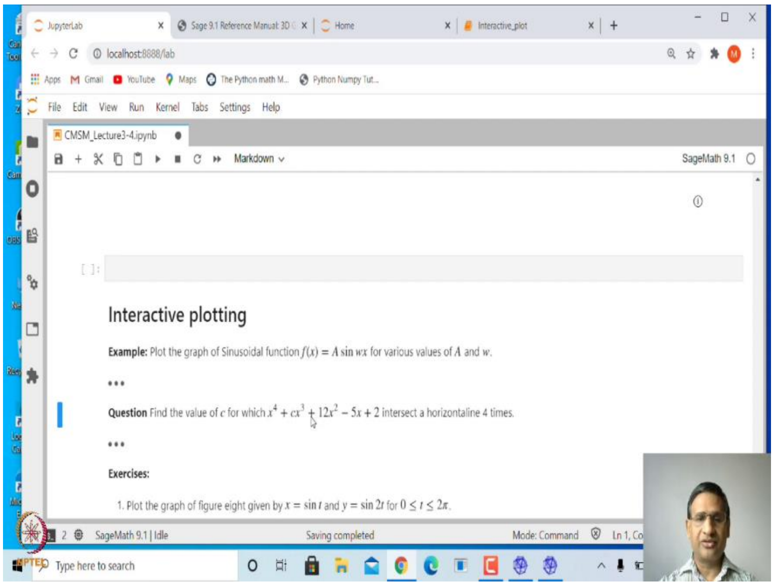

Now, another important thing that you can do in SageMath plotting there is something called interactive plotting. So, you can create animations. You can plot curve with in which there are various parameters. For example, if I ask you to plot graph of $ax^2 + bx + c$ as a, b, c varies.

So, you can have a slider for a, slider for b, slider for c. And you can observe how the graph changes at which as you change a, b and c. So, let me give you just couple of examples, but this one is let us plot graph of sinusoidal curve which is f(x) equal to A into sin omega x for various values of A and omega.

And another example that we will look at let us say for example you look at this curve $x^4 + cx^3 + 12x^2 - 5x + 2$. So, this will be a curve in the plane. And this is c is a parameter, c is varying. So, for what values of c you can find let us say a horizontal line which intersects this curve 4 times.

So, that is the this, but I will show you this in Jupyter Notebook. Here I am using for my convenience JupyterLab Notebook; however, this interact has some kind of bug in with interactive plot in JupyterLab.

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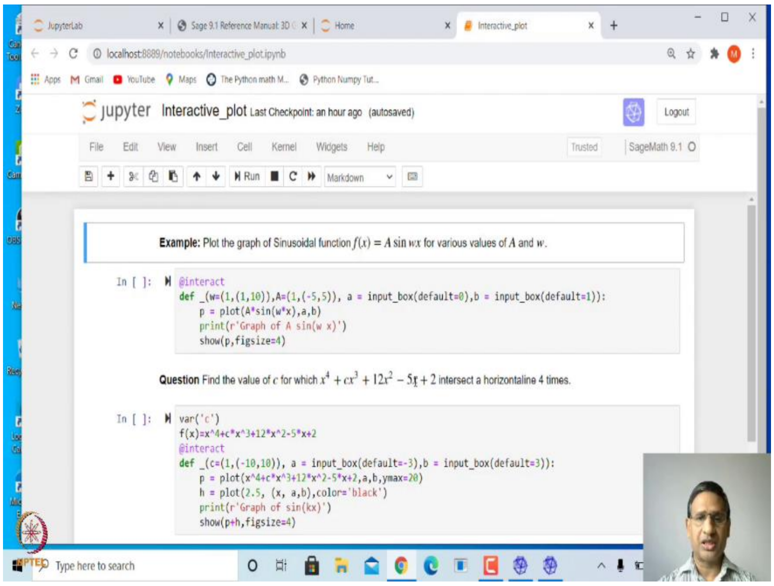

So, let me show you the same examples in Jupyter Notebook. So, how does it work? So, here you need at the rate interacts that is the function which creates interactive plot. And after that I have created a function def and without any name, so I am just putting here underscore. And w, w takes value between 1 and 10, and default value is 1.

Similarly, A, it will create a slider for A with a default value 1 and it varies from -5 to 5. And this graph is plotted between a and b, where a takes values, default value of a is 0, default value of b is 1. So, between 0 and 1, it will plot initially. And what to plot? So, p is equal to plot A into sin omega x, a going from a to b, and then just print some label and then ask it to show this.

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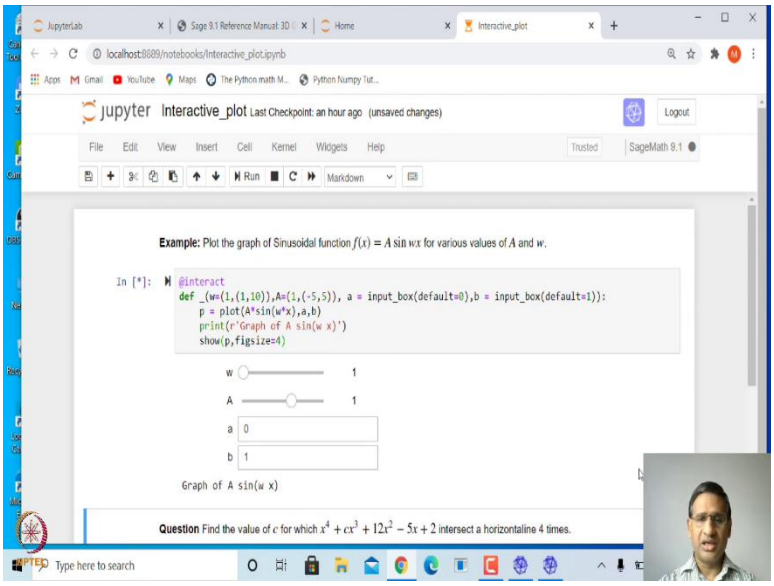

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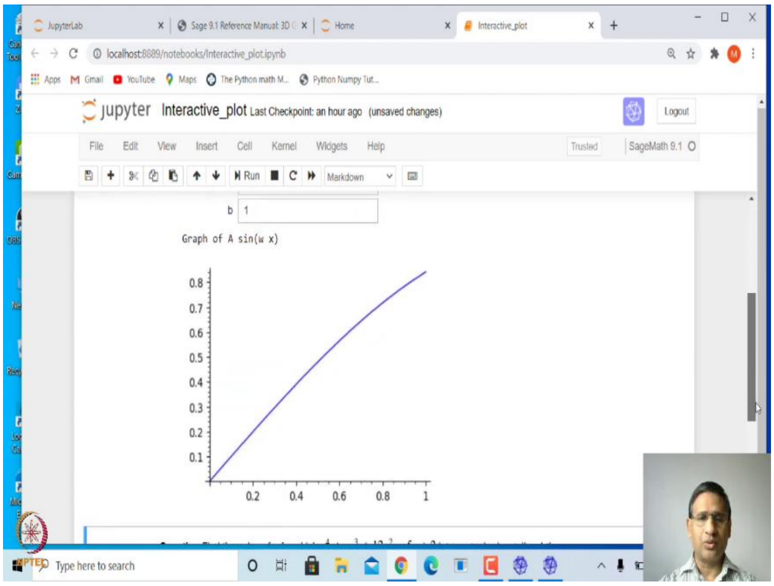

So, let me run this. When I run this, you can see here, it has plotted the graph of the function f(x,y) = A sin omega x between 0 and 1. Now, I can change this value of omega.

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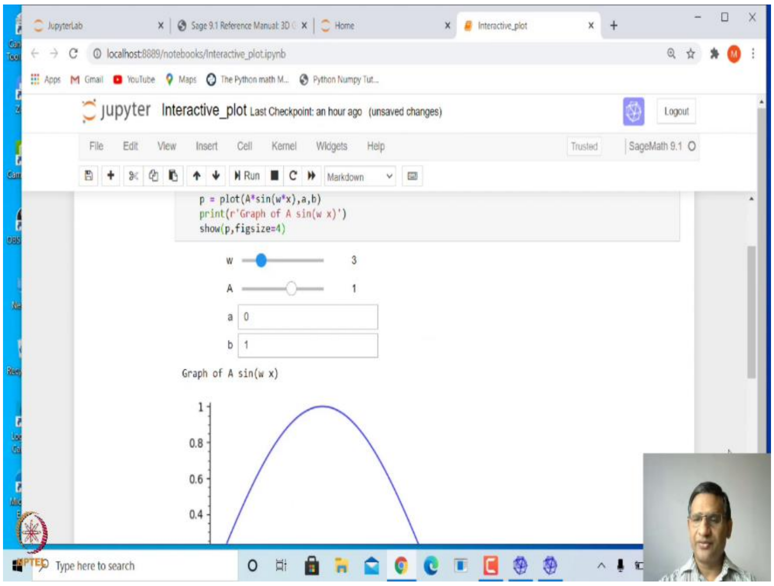

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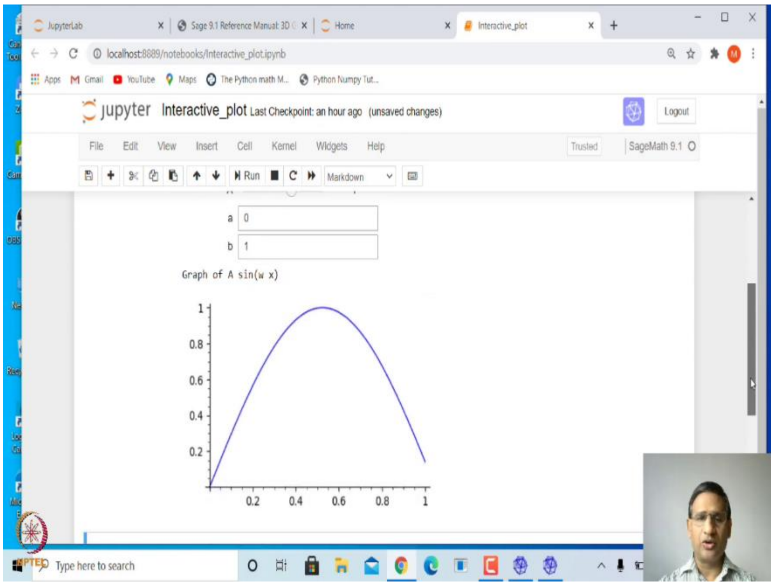

Let me say if I change the value of omega, this the curve will automatically change.

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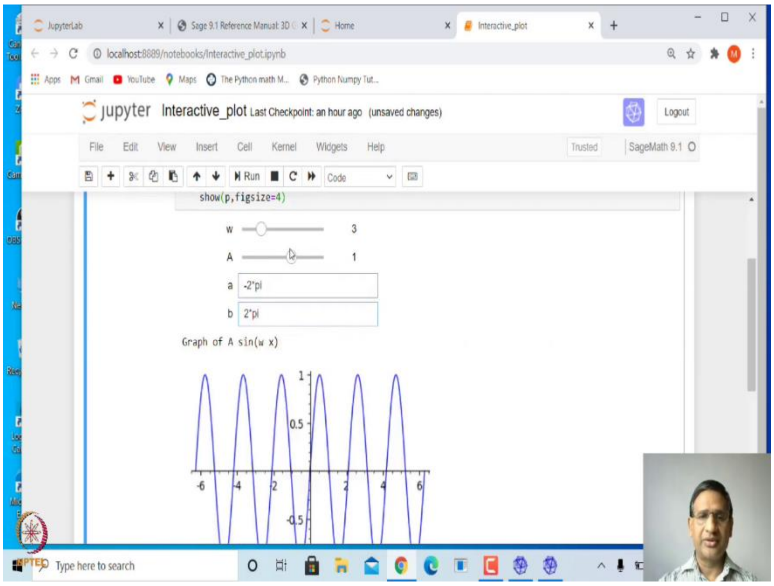

If I change the value, let me first change a, let us say create a from -2*pi, and b is also b is also 2*pi let us say 2*pi, so that is the graph.

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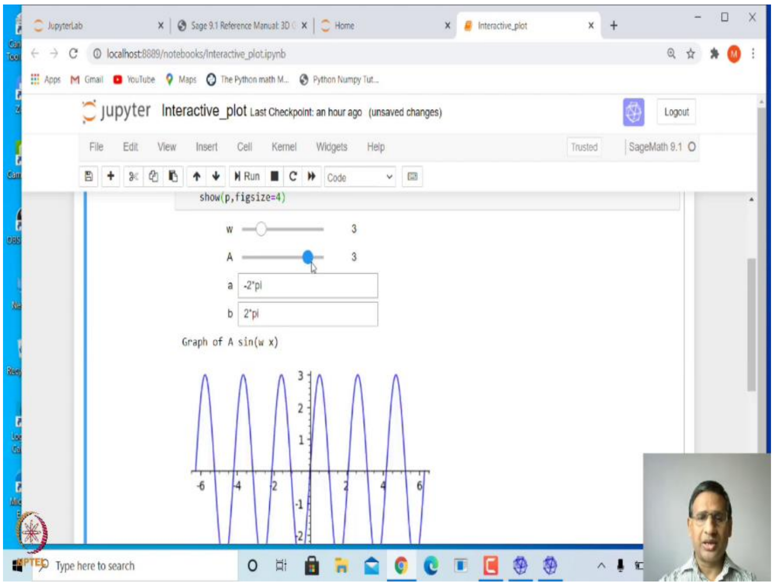

And if I change A, A is in the product. So, if I have A more than 1 then the frequency will increase, I can have a negative also.

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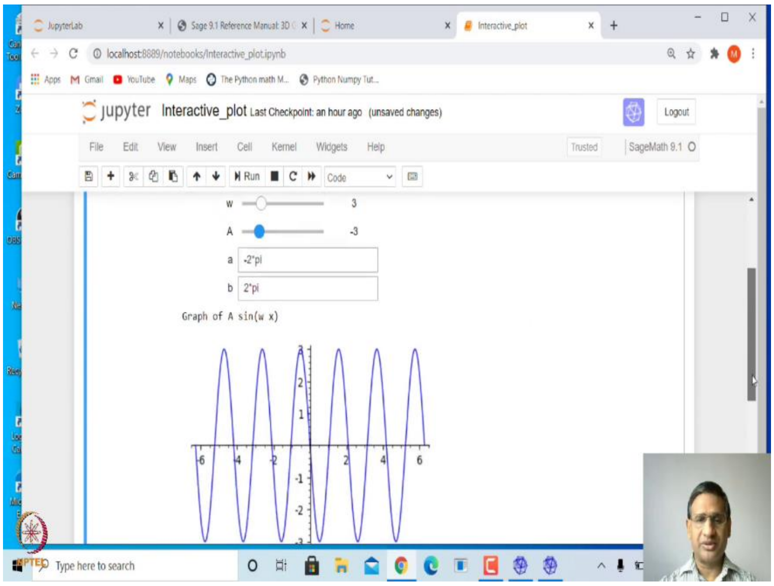

And like that you can change the parameters and this will create an interactive plot. Similarly, for the previous for the other problem, what is that problem? You want to find plot the curve this.

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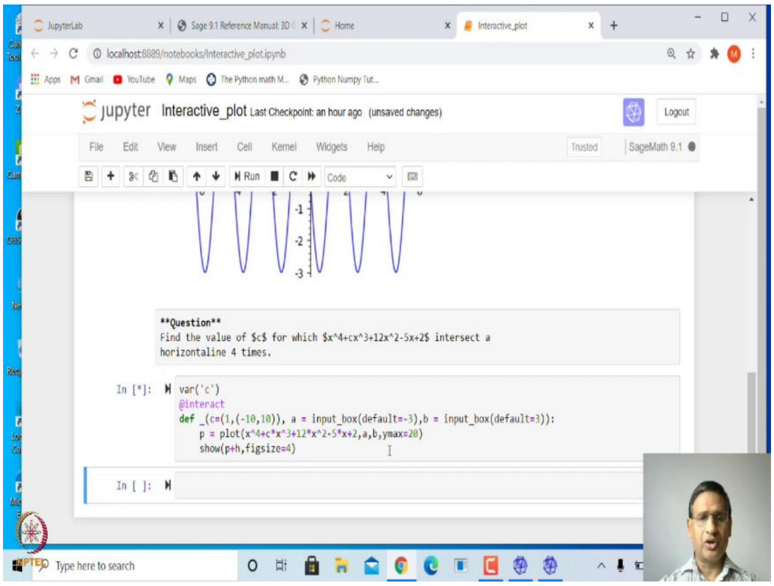

And find or check whether some horizontal line can intersect this curve 4 times. So, first again the example is same. f is defined as, so this actually is not needed. So, again where c is a variable here and c takes values default value as 1, but it varies from -10 to 10, a value default value is -3, b default value is 3.

And then plot p which is $x^4 + cx^3 + 12x^2 - 5x + 2$ between a and b, and maximum value of y is equal to 20. Because as you change a and b, this value this may vary quite a bit. So, we have restricted y value to be 20. You can also restrict minimum value equal to something right and then add a horizontal line at some point. So, for the time being, let me get rid of this. So, this also I let me not have this right.

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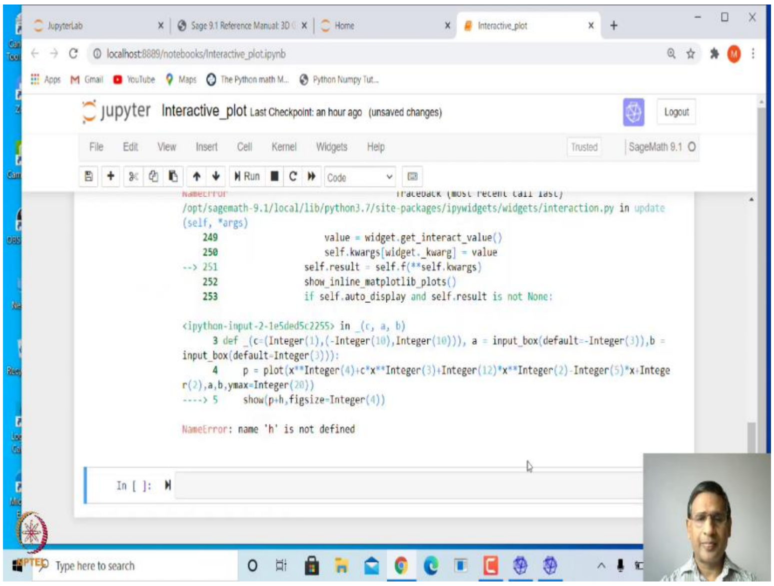

So, let us just ask it to plot this, h is not defined. So, this h is I deleted. So, it should not be there right.

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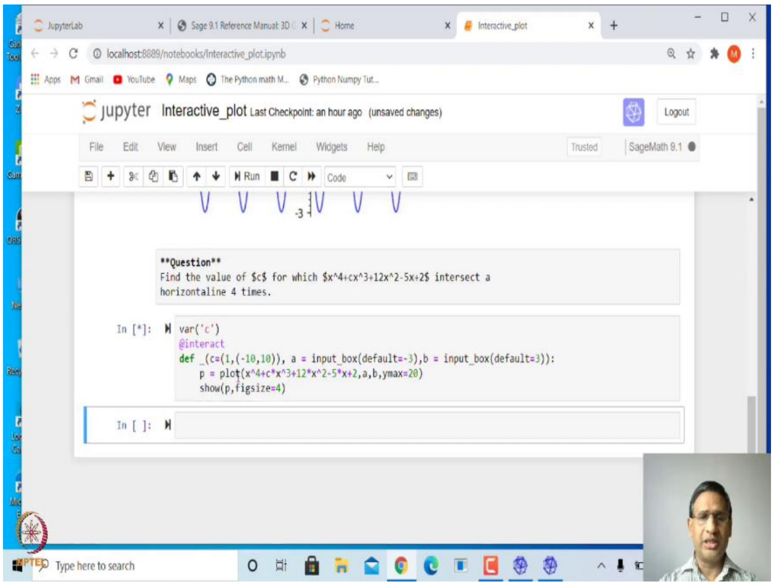

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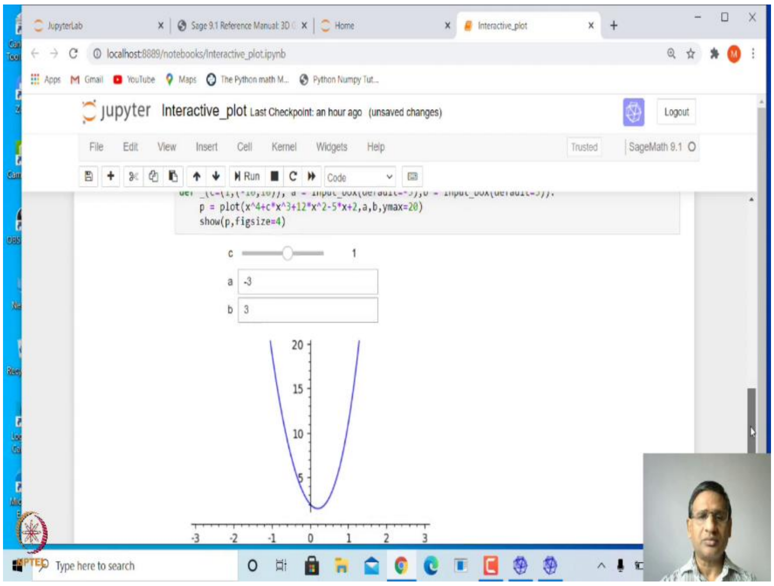

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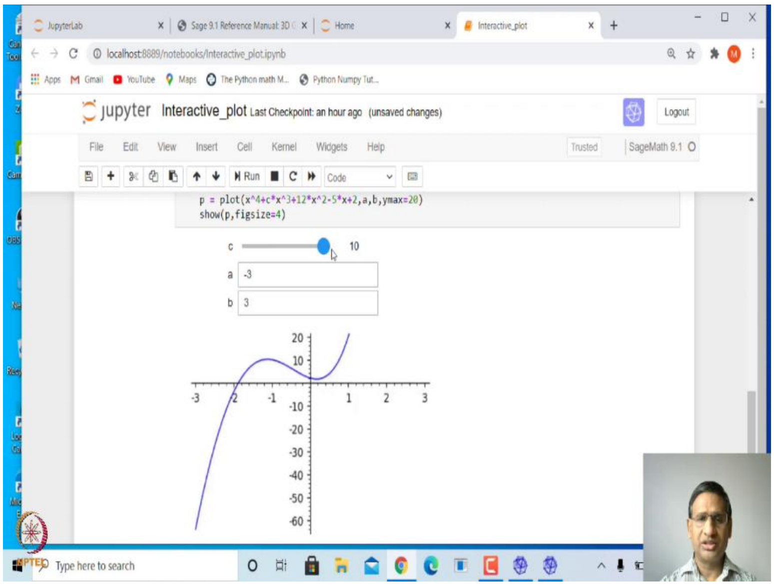

So, this is the graph and let us say if I change c, if I change c for example, if I change c is equal to something this is how it is changing.

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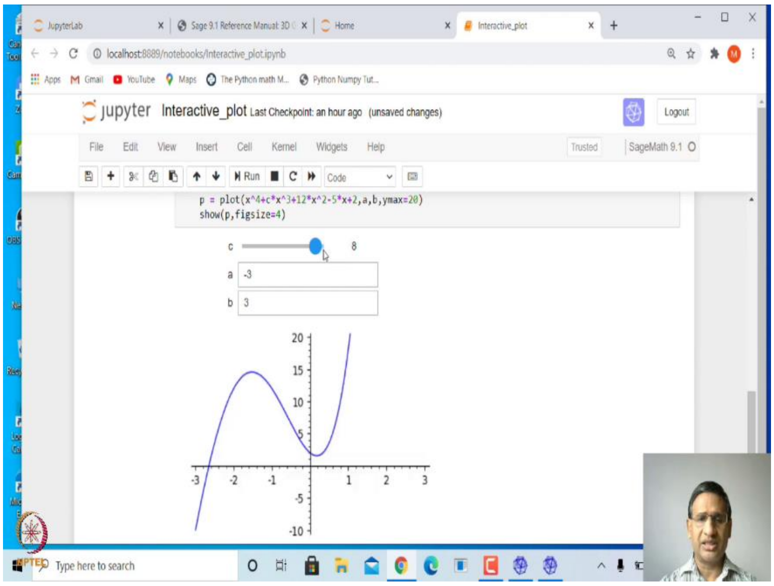

So, up to 10, I do not think any horizontal line will intersect 4 times. Now you can see that there is for c equal to 8, for example, I can have a horizontal line between this and this. So, you can find a range. So, for example, I can take a horizontal line at z at y equal to 5, and this will intersect 4 times.

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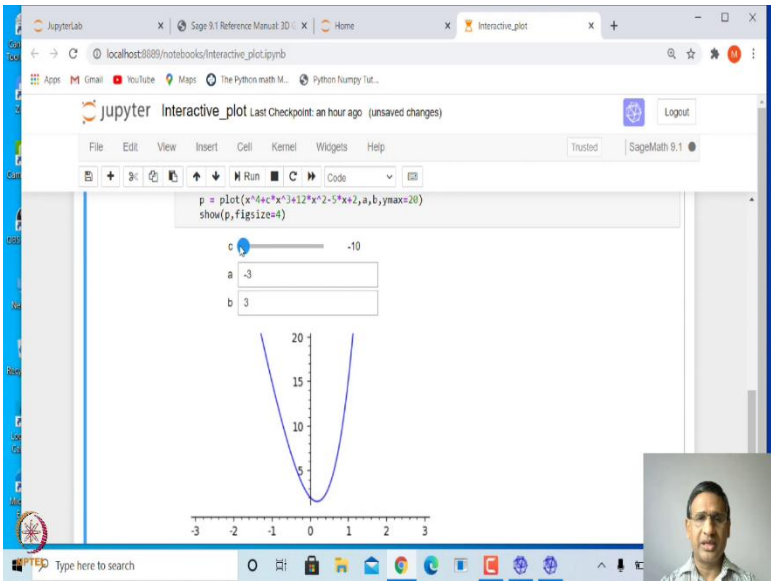

Similarly, if I look at negative side; if I look at negative side, then also something similar may happen ok. So, now, it is still changing, it has become slightly slow.

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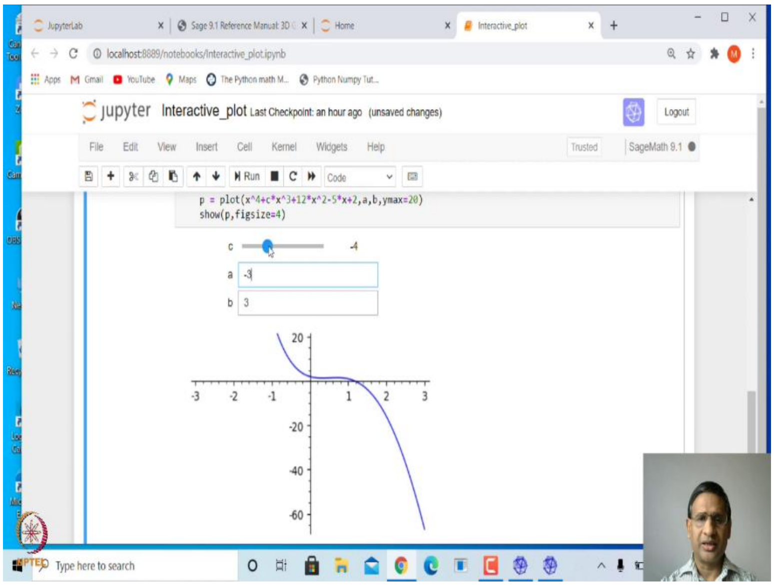

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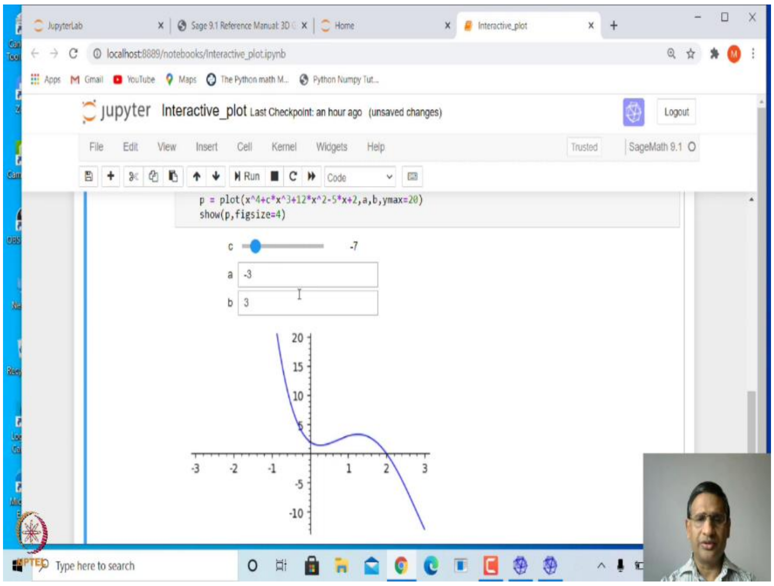

Yes, something, yeah. So, if I change, yes. So, if I, c = -7, then also you will see for example, I can say about parallel line passing through let us say about 3 should intersect this twice or 2.5.

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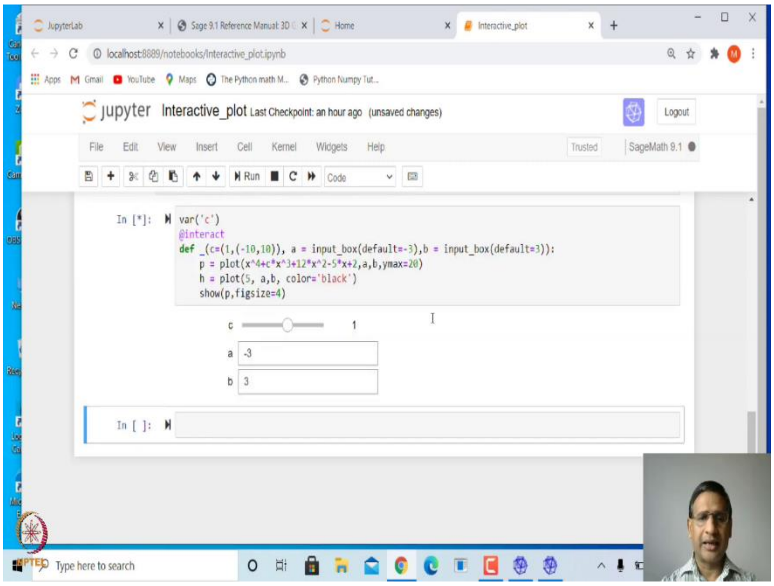

So, let us add that line. So, for example, if I add one line here that is what we had. h is equal to plot let us say something like 5, and again between a and b, and then let us say color = black.

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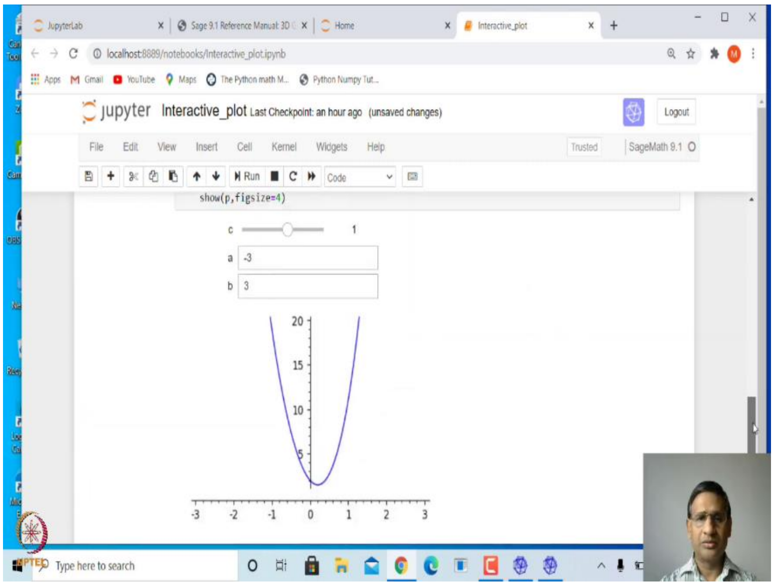

And yeah, no, I have not added in this. So, I have to say p + h right.

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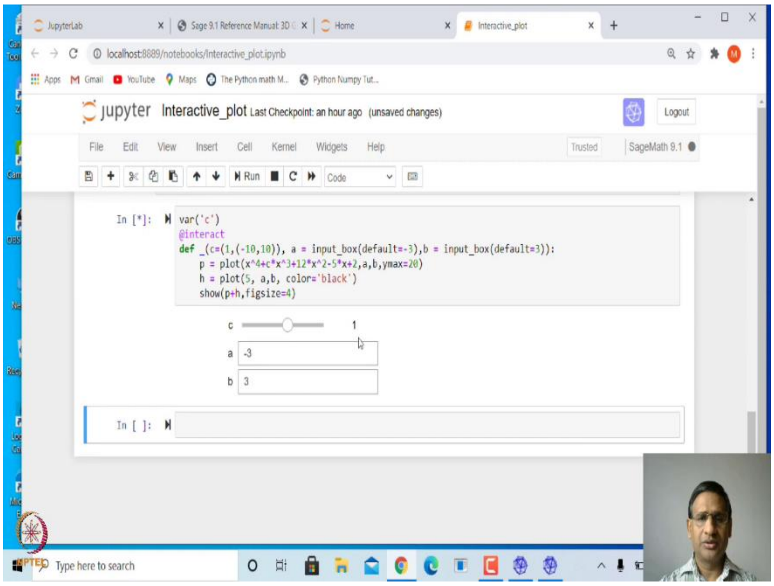

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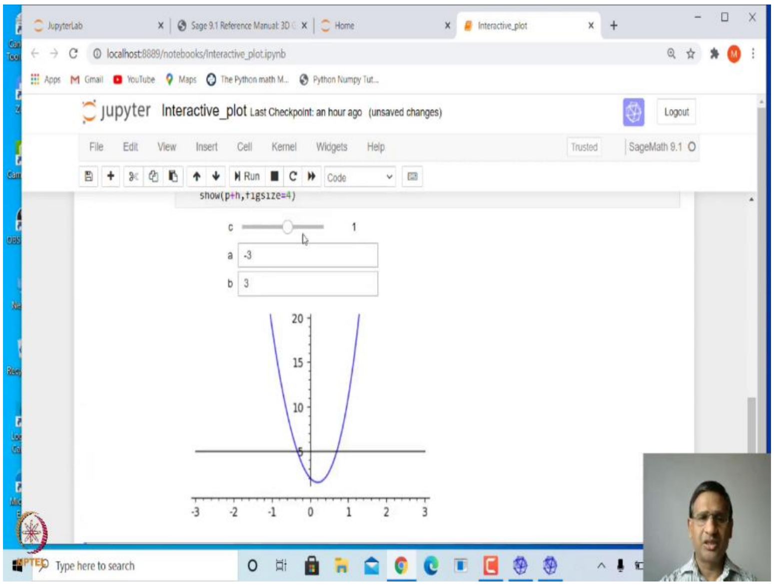

So, now, you can see here.

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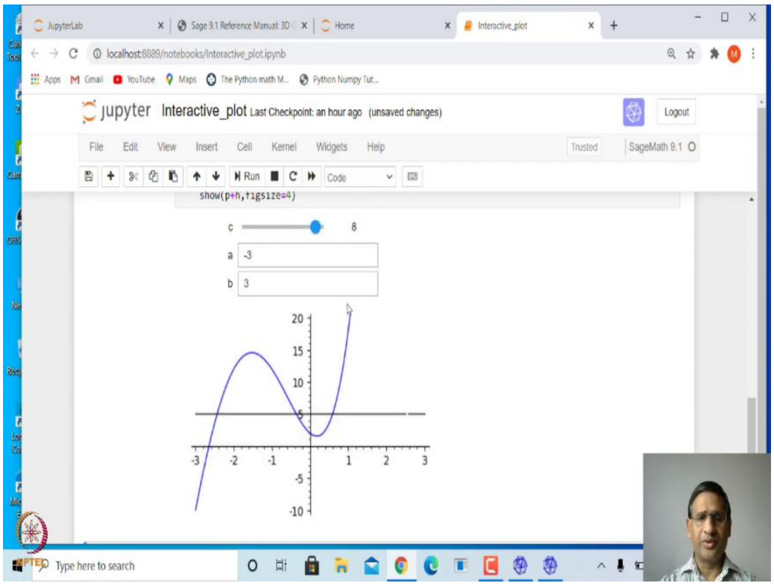

And when we change this value of c, we saw around 8 yeah. So, when we have c = 8, this intersects this curve only 3 times not 4 times ok. I am sorry, this intersects only 3 times actually.

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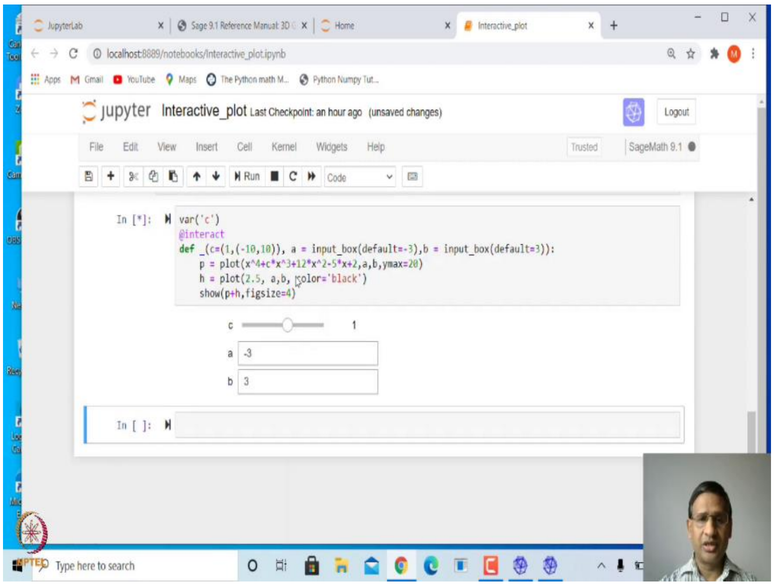

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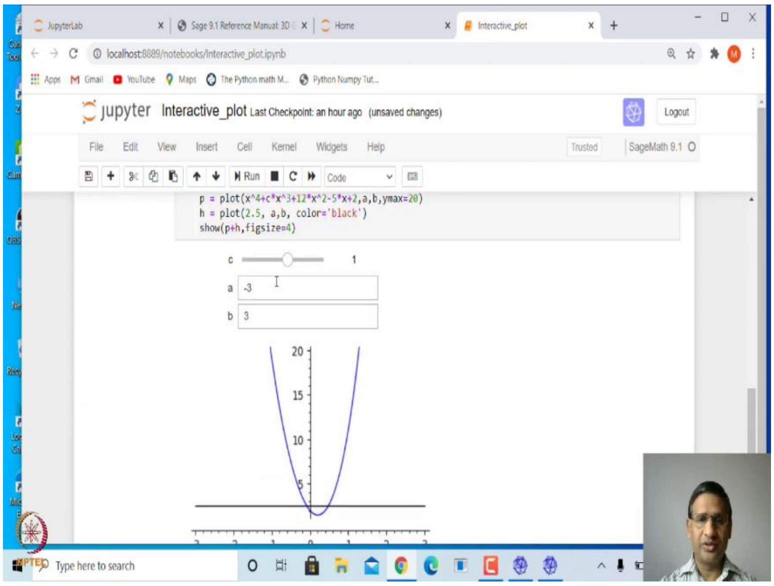

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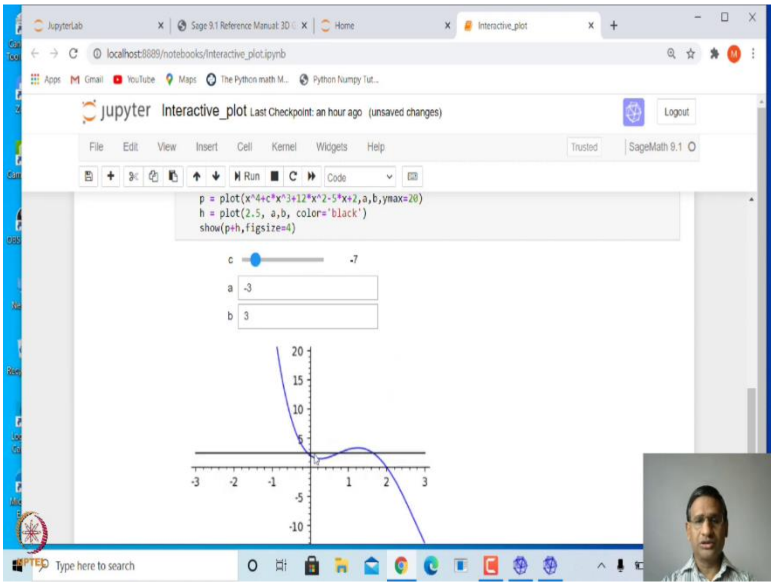

So, if I take negative value, and then let us say take this as 2.5 and then plot this, and let us move this c to negative somewhere -7 is what we saw yeah. So, this is again intersecting; this again intersects only 3 times not the 4 times right. So, I should change this problem to intersecting only 3 times actually.

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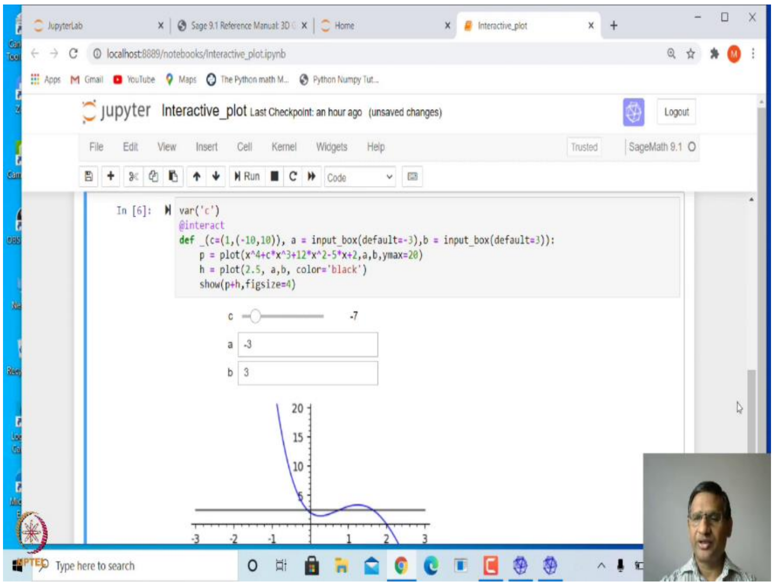

It is intersecting 3 times right. You may explore when will it intersect 4 times whether it will intersect or not that you can find out actually. So, this kind of problem this is actually explorative problem. So, can be very easily explored using this kind of interactive plots in SageMath ok.

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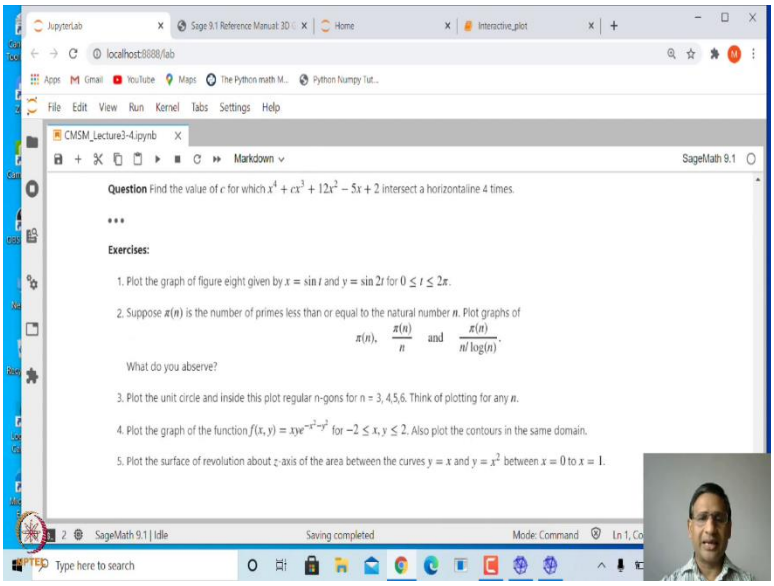

So, that is the plotting 3 dimensional graphs in SageMath and I will leave you with some simple exercises. First try to plot graph of a function which is a figure 8. This we did in Python actually. This is $x = sin(t)$, $y = sin(2t)$, t going from 0 to 2pi. And another exercise is let us say pi n represents the number of primes less than equal to natural number n.

And then this pi n, we plot this graph of pi n for let us say n going from 0 to or 2 to say 1000, and also plot graph of pi n divided by n and then also pi n divided by n divided by log n, and also try to observe what happens to these ratios when you plot.

The next one is plot unit circle inside this plot try to plot regular n-gons with n equal to 3, 4, 5, 6 and so on, and this you will see that this is actually an approximation of pi right. So, then next one is plot graph of 3d surface $f(x, y) = xye^{-x^2-y^2}$, for x y varying between -2 and 2.

And the last problem is plotting the surface of revolution about z-axis of the area between the curve y equal to x and y equal to x square between x equal to 0 and x equal to 1. So, this is going to be some kind of area. And when you plot these two curves it is going to create two surfaces, one inside the another. So, it will be a solid actually right. So, that is the set of exercises.

So, I am sure all the other plots, etc., as and when we come across when we do actual this calculus etc., we will again plot them. But you can also refer to the user manuals which I told you for 2d and 3d both. You can explore there are lots of examples in this which can be used ok.

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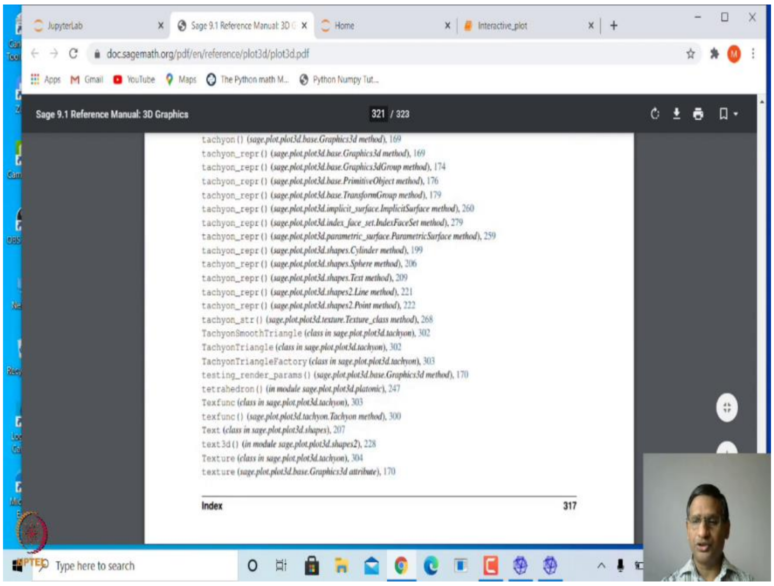

So, thank you very much. I will see you in the next class.