Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 90
Frobenius method for regular singular points

So, we have seen how if you have an ordinary point, you know there is a differential equation and if you consider an ordinary point and look for a power series expansion about an ordinary point, there is a you know region of convergence in which this series is guaranteed to converge. And you are going to have two independent solutions right. So, this is the statement of a theorem which we just became familiar with. And we looked at a concrete example, how this plays out.

So, in this lecture, we will look at how you know there are certain kinds of singular points; not all singular points are terrible singularities, you know there are relatively benign singularities which are called regular singular points.

If you have singular points which are regular in nature, we will discuss what that means. Then it is still possible to come up with a valid power series expansion right. And the method associated with this, goes by the name of the Frobenius method which also we will discuss in this lecture ok.

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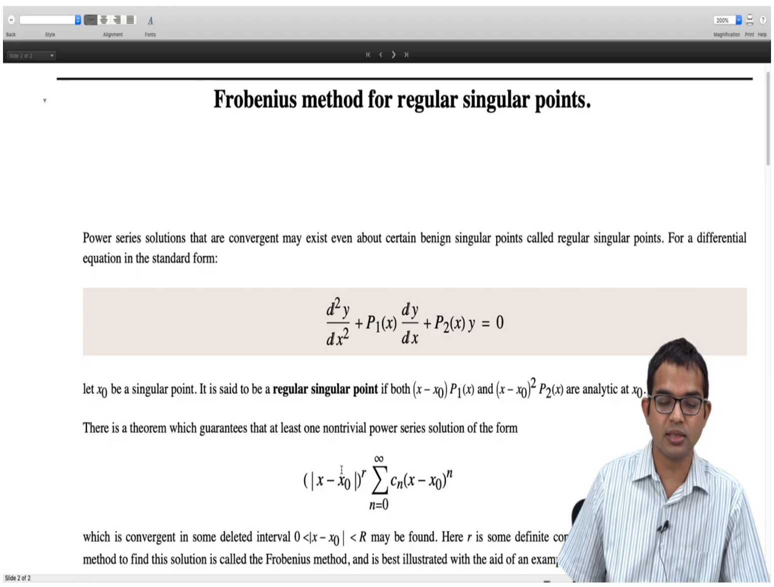

So, the differential equation we are looking at which can be brought into the standard form is d squared y by d x squared plus P 1 of x d y by d x plus P 2 of x times y equal to 0. Suppose, x naught is some singular point.

So, the question is, will we be able to do a power series solution, will we be able to find a valid power series solution now, if you were to do an expansion about such a singular point? Well, it turns out that if such a point is a regular singular point right, then it will be possible.

So, what is a regular singular point? So, as far as this differential equation is concerned, it would be called a regular singular point, if you know x minus x naught times P 1 of x and x minus x naught the whole squared times P 2 of x.

If you were to multiply just by x minus x naught the first function and multiply by x minus x naught squared the second function, you should be able to heal it of the singularity. If you know both these functions become analytic and at x naught you know, then it is called a singular point all right, but it is a regular singular point right.

So, this is a special class of differential equations. And now, there is a theorem which says that if this is true, then you can get at least one non-trivial power solution of this form right. So, in fact, you know it is possible to say more and get another independent solution as well, but that is you know those are some details which would be involved going a bit too far. So, we will look at just this very sort of bare statement. So, there is at least one power series solution of this form.

What you have to do is you know take the same kind of power series solution which we had before, but you will have to tag it along with a product mod of x minus x naught the whole power r right. So, this r is something that we have to evaluate and it will not hold for any r, but you will be able to find an r, such that you multiply by this. So, this is the part which makes it singular in some sense right.

So, this is like a peeling off of the singularity. There is a way to do this in such a way that your function is convergent in some deleted interval right. So, at x naught of course, the series will not converge. It will diverge at x equal to x naught, but arbitrarily close to it, you will be able to find a power series solution which is convergent. That is the statement of this theorem right.

So, we will again not bother too much about the details of this statement, but we will just use it as you know something on the backdrop of which we can do a calculation.

Well, we will look at a concrete example, and see how this plays out. So, the method to find the solution is called the Frobenius method right. So, the theorem just tells you that there is a solution and then you have to find it right. So, the Frobenius method is best illustrated with the help of an example. So, let us look at a concrete example.

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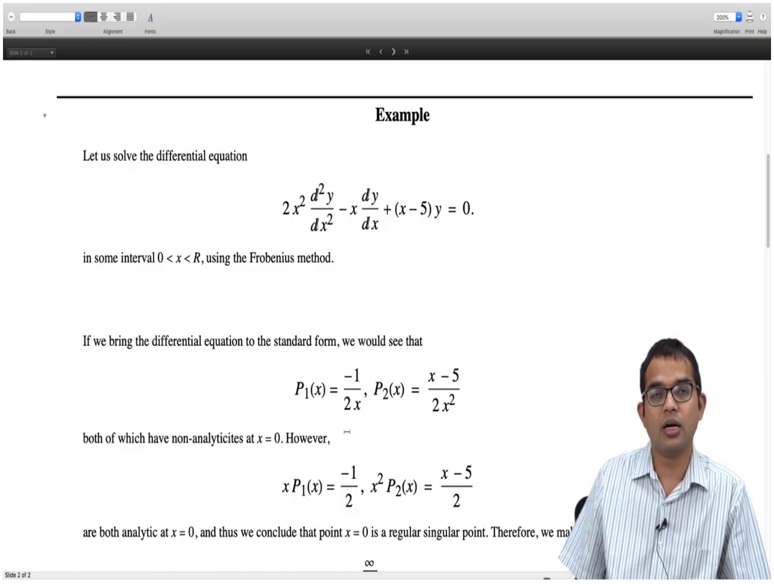

So, we have this differential equation which looks very simple right. So, it is remarkable how you know such very simple looking differential equations. It is a homogeneous differential equation, you know the functions involved also seem rather simple, and yet there is a lot of you know technique involved even with such a simple differential equation.

So, if you wish to find a solution to this problem, if at x equal to 0 of course, there is a problem right. So, this is a singularity. There is a singularity at x equal to 0. But luckily it is a regular singular point. Why is that? So, we can check P 1 of x, if you were to bring it in the standard form, it will be minus 1 over 2 x, and P 2 of x will be x minus 5 divided by 2 x x squared.

So, if you multiply P 1 with x, it becomes minus 1 over 2 which is you know it is a happy function; P 2 of x if you multiply by x squared, then you know you cure it of this singularity of order 2, which is there you know there is a you know pole of order 2 as it is called so.

Anyway, you are able to cure it of the singularity by multiplying by x squared, so, which becomes x minus 5 by 2 which is a polynomial on the right hand side. So, therefore, both are analytic at x equal to 0. And thus, we conclude that x equal to 0 is a regular singular point.

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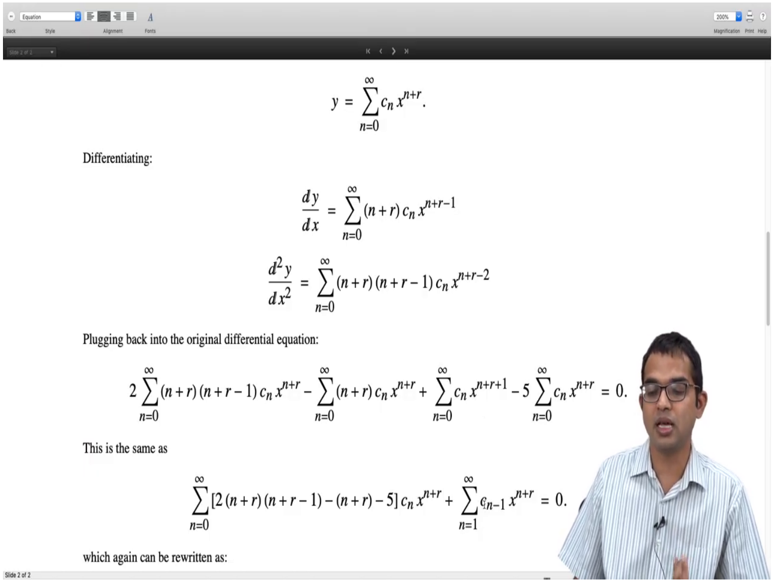

So, if we are trying to do a power series expansion about a regular singular point, the theory tells us that we must look for a solution of this form right. You have to multiply throughout you know take a regular power series like c n times x to the n, but multiply throughout by some x to the r right. What is r is something which needs to be determined. We will see how to find it.

So, in this case, it will just become x to the n plus r right. So, r in general can even be a complex number right, but in our case, we will see that it is some real number for the concrete example that we are looking at. But it is just a remark, which I am making that in general, it can be a complex number and there is a lot of sophisticated theory around it, but we will content ourselves with looking at you know, a simple concrete example of how this plays out.

So, we differentiate it as usual right. So, convergence is guaranteed in some you know deleted interval. And so, differentiation term by term we will work out, so, then summation over n n plus r times c n times x to the n plus r minus 1. Then if you take a derivative once again, you get n plus r times n plus r minus 1 times c n x to the n plus r minus 2, and this also needs to be summed all the way from 0 to infinity.

Now, you should plug back these two expressions and the first one. All these three expressions back into the original differential equation, then you have you know 2 times, so, it is just 2 x squared, so, x 2 times x squared you know will give me back x to the n n plus r, then I have x times d y by d x minus x times d y by d x. So, that again will give me back x to the n plus r. Then I have plus x minus 5. So, I can separate this out, and write you know just the x part first and then the minus y part.

So, when I do x times x times y, I will get n plus r plus 1. And then when I do minus 5 that will just give me x to the n plus r. So, I have all these terms added up to 0 right. So, now, there is a lot of you know rearranging and some simple algebra connecting together and bringing things into a convenient form right. So, what we do is first of all write 2 n n plus r times n n plus r minus 1 minus n plus r minus 5 all of this you know is multiplied by c n times x to the n plus r.

So, it is conveniently brought together. And then I have also this, so, this is not minus this is a plus. So, there is a plus. So, instead of writing it as c n times x to the n plus r plus 1, I can write it as you know I can make this standard transformation n plus 1 is equal to k. So, then it will become you know k plus r c k minus 1, so, which is you know where the k will now go from 1 to infinity. And then in place of k, I put back n.

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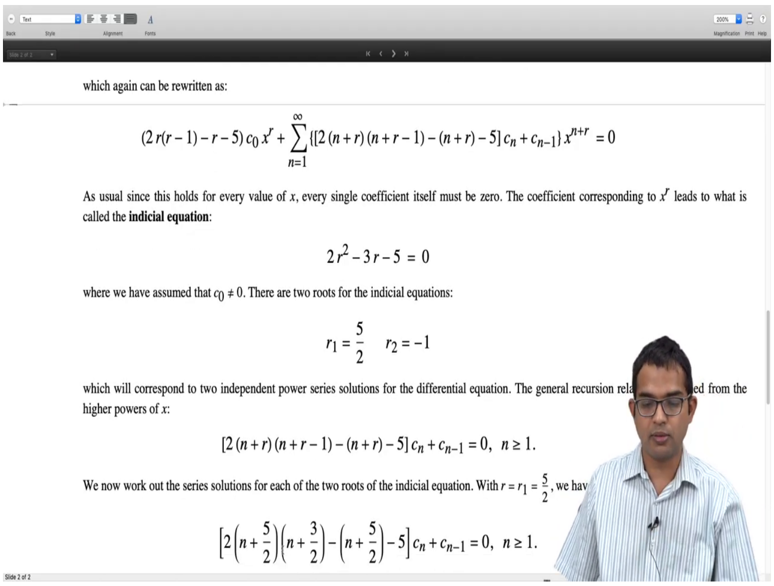

So, I have n going from 1 to infinity c n minus 1 x to the n plus r. So, I have everything now, in terms of this x 2 n plus r, but the first of these is going all the way from 0 to infinity, whereas the second one is going from 1 to infinity.

So, what I will do is I will pull out all the stuff involving just 0 and write it, you know first, and then I have another summation involving you know which where the summation goes all the way from 1 to infinity. So, the 0 term will be just, so when I put n equal to 0, I get 2 into r into r minus 1 minus r minus 5 times 0 0 times x to the r right.

So, this whole stuff is you know, it corresponds to just x to the r, there is a coefficient times x to the r, and then I have a summation over n, you know where I have x to the n plus r here. And this coefficient just is just all of the stuff plus also c n minus 1 right. So, it is fairly straightforward. And as usual, we demand that you know term by term this is going to be 0 for this to hold, since, it holds for any value of x.

And so, the first thing we do here is as follows. This is where the slight deviation comes from the previous type of method which we were looking at, namely power series expansions about an ordinary point. So, if you are doing it about a regular singular point, you must first solve for what is called the indicial equation. After all, we do not know what r is right, so that in fact is solved here.

So, assuming that the c naught is not 0, it is this guy which has to be 0 right, because you know coefficient this entire coefficient must go to 0. So, this means, 2 r squared minus 3 r minus 5 must be equal to 0 right. So, this is called the indicial equation. If we solve for this, you know we will come to know what are the values of r, where for which this kind of a thing power series can be a solution.

It turns out that you know you will have a quadratic equation, because we are dealing with a second order differential equation of this kind. So, you will get a quadratic equation which in general can have roots which are complex. But in our case, we will have I mean the particular example that we are considering is a rather simple one.

So, here, we will see that in fact, r you know there are two roots r 1 is equal to 5 by 2, and r 2 is equal to minus 1 right. So, both are real roots. So, there is quite some theory around you know what happens if you know the roots are the repeated roots, and if the roots happen to be complex, one of them will be you know, the two will be conjugate to each other, and how the solutions will play out and so on? We will not get into these details.

And also, there is some theory about what happens if you know r 1 and r 2 are both real, but if the difference between them is a is an integer or not and so on right. So, there are some subtleties involved. We have this statement about how the theorem tells you that there is at least one non-trivial solution, but you will actually be able to find two independent solutions.

So, sometimes you may have to work with what is called a you know logarithmic singularity has to be put in and so on. So, we will not go into those details at this point. So, we will just work with this very concrete example, where things are quite simple and in fact, we will get two linearly independent solutions here, one corresponding to r 1 and other corresponding to r 2 ok. So, how do we, what is the machinery here?

So, we will first use r 1 as the root and then we will use r 2 as the root, and then work out this recurrence relation right. So, the recurrence relation comes from the next terms for n greater than or equal to 1. We see that 2 times n plus r times n plus r minus 1 minus n plus r minus 5 times c n plus c n minus 1 must be equal to 0 for n greater than or equal to 1. If I look at the first solution, first root r 1 equal to 5 by 2, then we have 2 times n plus 5 by 2 times n plus 3 by 2 minus n plus 5 by 2 minus 5 times c n, whole thing times c n plus c n minus 1 equal to 0.

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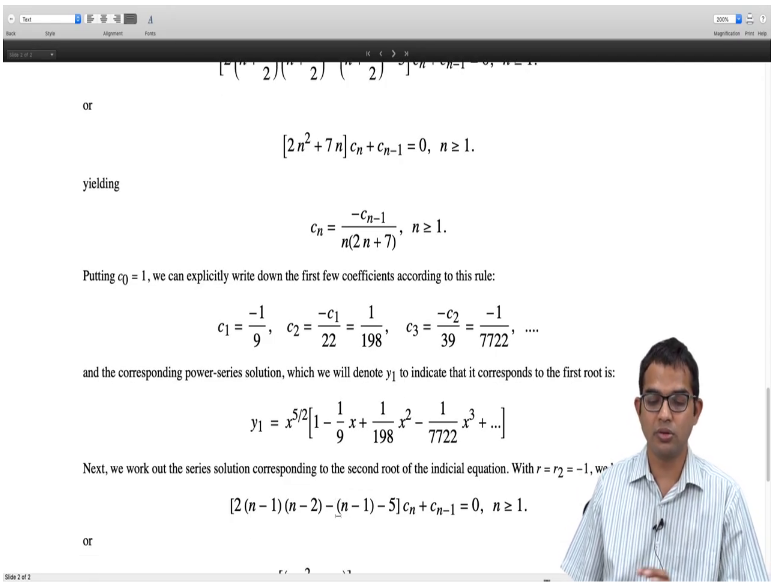

Which immediately, you can simplify this it will lead to just 2 n squared plus 7 n times c n plus c n minus 1 equal to 0, or equivalently you can actually go ahead and write c n as minus c n minus 1 divided by n times 2 n plus 1 for n greater than or equal to 1. So, since, you know c naught, so, if you know c naught, you can work out c 1; from c 1 you can get to c 2 and so on.

So, let us put c naught equal to 1 for simplicity here right. So, we are looking at the general solution. So, you will see later on that you know we will bring in these free constants later on. We will work out two independent solutions, you know putting c naught equal to 1 in each of them, and then we will take some arbitrary linear combination of these two will also be a solution of the will be the general solution.

So, if you put c naught equal to 1, then we get c 1 according to this relation is minus 1 over 9; c 2 is minus c 1 by 22 which in turn can be written you know, because c 1 itself is known it will go to 1 over 198; c 3 again you write it in terms of minus c 2 divided by 39 according to this relation and then but c 2 is known. So, therefore, you just plug it in, and then you get alternating sign as you can see one of them is going to be positive, then x is negative and so on.

So, this is going to result in a power series solution, which we will denote y 1 to say that it is the first root that this corresponds to. So, y 1 is x to the 5 by 2 right. So, you can pull out that factor x to the r which is x to the 5 by 2 in this case times 1 minus 1 over 9 times x plus 1 over 98 times x squared minus you know this stuff times x cube so on right.

So, there is a, you know it is basically like a power series, but you have this extra factor which is sitting here right. So, it is a you know it is a device by which you have somehow peeled off, just the right factor such that you can still have a convergent power series in the vicinity of a regular singular point.

And likewise we can also work out the series solution corresponding to the second row. So, now if you plug in r 2 equal to minus 1, so, then we have the recurrence relation to be 2 times n minus 1 times n minus 2 minus n minus 1 minus 5 times c n plus c n minus 1 equal to 0.

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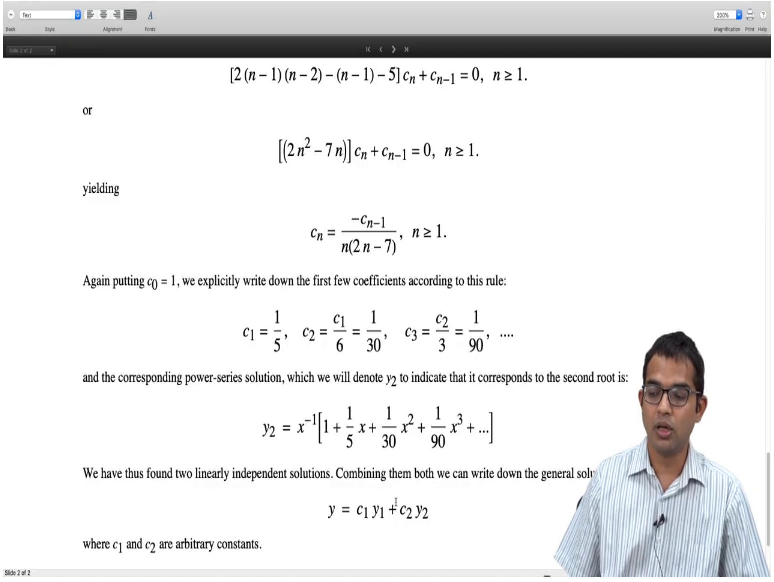

You rearrange all this. You know a lot of simplifications will come about. And you can show that in fact, c n is now, minus c n minus 1 divided by n times 2 n minus 7, earlier we had 2 n plus 7. So, now, you have 2 n minus 7 for n greater than or equal to 1.

Again we can you know put c naught equal to 1, and write down the first two coefficients. c 1 will turn out to be just 1 over 5, you can check this, it is plugging in this formula. So, c 2 can be written in terms of c 1 which is already given to be 1 over 5.

So, you get c 2 to be 1 over 30, c 3 will turn out to be 1 over 90. And then your negative terms will start after that right, you can check here first you know c 1, c 2, c 3 all the way up to c 3 are all positive, because you see that you have a 2 n minus 7, and n is not sufficiently large, but once n becomes large you will start getting negative terms as well.

But let us stop here and look at you know, just the you know just a representative set of terms, and we will not look at higher order terms. So, we have y 2 is equal to 1 over x. So, there is this factor x to the minus 1 which comes out right that is the, you know that is how you it is a benign singularity. You can precisely sort of you know it is a concrete understanding of the nature of the singularity itself comes out in this solution. It is 1 over x which comes out outside. And then you have a series which converges.

So, as long as you are not precisely at x equal to 0 in some interval around it, it is a convergent series right. So, once again you can worry about how far this will spread out, what is the radius of convergence and all, but which are all a matter of detail, which we will not go into here.

So, we have thus found two linearly independent solutions; one of them we called y 1, and the second one is y 2 which is x to the minus 1 times 1 plus 1 over 5 x plus 1 over 30 times x squared plus 1 over 90 times x cube plus so on. So, it will give you some negative terms right.

You can if you wish, look at a few more terms. But the key point is that we have two independent power series which are both convergent in some deleted neighborhood around the regular singular point. And we can tie them together, you know you can bring in an arbitrary coefficient c 1 you know take it along with y 1, and c 2 along y 2 with y 2 and write down the general solution as c 1 y 1 plus c 2 y 2. After all, it is a second order differential equation, so, indeed you will expect two free constants which is what we find.

So, in this lecture, we have seen a demonstration of how if your singular point is benign and specifically, if it is what is called a regular singular point, then it is still possible to you know look for a power series solution around this point, in some deleted neighborhood at that point itself you do not have a solution.

But in the neighborhood of that point in a deleted neighborhood of that point, there is going to be at least one non-trivial you know power series solution you can find, and you know there is more sophisticated theory of about how you know these indicial equations works out what happens, if you have more complicated roots and so on which we have not gone into the details.

But hopefully with this example, we have got a flavor for the type of ideas which you know which go into finding solutions. Some of these get used when we look at things like Bessel functions etc. There are standard special functions, lots of special functions that are connected to these kinds of series solutions. But hopefully we have got a flavor for this technique in this discussion ok, then that is all for this lecture.

Thank you.