Mathematical Methods 1

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Linear Algebra
Lecture - 09
Triangle inequality

So, we have seen how one can come up with something called the Cauchy-Schwartz inequality. We first looked at it using two-dimensional vectors and three-dimensional vectors and so on and then, we saw that this can be generalized to two arbitrary vectors in the abstract vectors from linear vector space.

And so, there is another inequality which is called the triangle inequality which is also possible whenever you have an inner product right. So, we are going to derive this for abstract vectors using the Cauchy-Schwartz inequality in this lecture ok.

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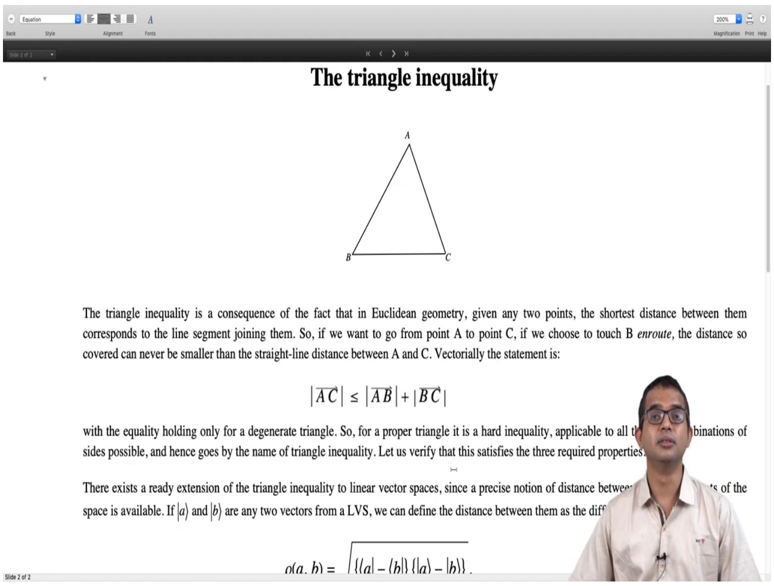

What is the triangle inequality? It is simply the statement that you know if there are in Euclidean geometry right, if you are going from a point A to a point C, then the shortest distance between A and C is if you take the straight-line path right. If you go to C via some other point B right so, you go from A to B and then go from B to C, then necessarily AB plus BC can never be smaller than the direct line AC right.

So, it seems like a completely obvious statement, but it is you know it has its own power and it can be made abstract and it can be extended to abstract vectors which is the subject of this lecture right. So, we can state the same fact with vectors like here. So, now let us see how we can extend this to abstract vectors from a linear vector space. So, let us start with some two vectors A and B drawn from a linear vector space.

And first of all, there is this notion of a distance that we can define between any two vectors in a linear vector space which has an inner product. The moment you have an inner product so, that is one reason why the inner product is important is, it is you know a linear vector space that is endowed with an inner product automatically becomes what is called a metric space, which means that you know it simply means that there is a well-defined notion of a distance between any two points in the space right.

So, in general, you know there are multiple ways of defining distance between two points, it is not a; it is not a unique, it is a just like the notion of an inner product itself is not unique, it must satisfy certain properties likewise, the notion of a distance is also not in general unique right, but let us not go into you know like a formal definition of distance. But let me just tell you that one essential requirement for a distance to be a distance is that it must satisfy triangle inequality right.

So, if you take any three points and then you compute the distances according to the prescription given to you, then triangle inequality must hold right. So, this is like a requirement.

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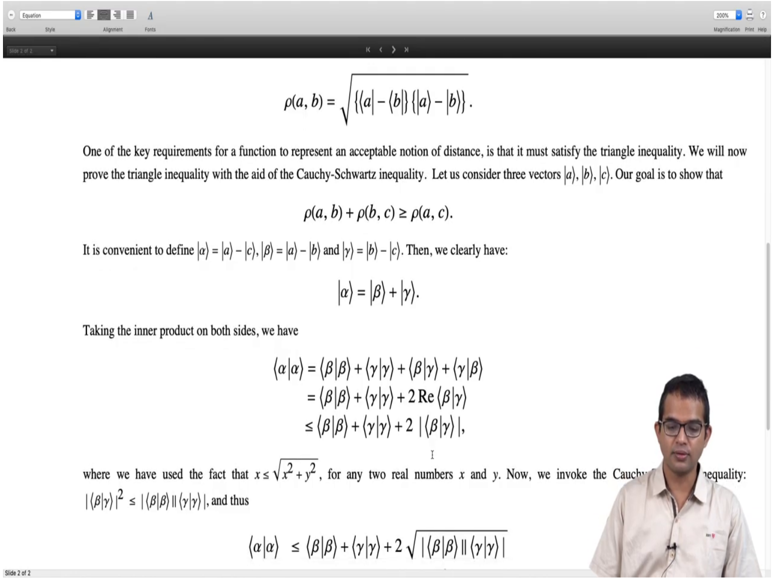

So, let us see how the triangle inequality works out for what is the most natural way of defining distance on a linear vector space where a norm is defined, where an inner product is defined. So, think of these two vectors a and b. So, just simply subtract these two vectors and create another vector. The difference of two vectors is also a vector right because it is a linear vector space. Linear combination of vectors also belongs to the same vector space and then, find the norm of the difference between these two vectors.

So, it does not matter whether you do a minus b or b minus a, you are going to take the norm of this quantity which is just simply defined as the inner product of the difference of these vectors with itself and then, you have to take a square root of this right. And, the square root is always something that you can do because it is a real number and it is a positive number right. So, this is all encoded into the notion of an inner product already. So, it is completely well-defined.

And now, we will show how we can use the Cauchy-Schwartz inequality to prove that this distance will satisfy the triangle inequality and therefore, this becomes what is called a metric space right ok. So, let us see how this works out. So, the goal is to show that if there are any three vectors a, b and c the distance between a and b plus the distance between b and c must necessarily be greater than or equal to the distance between a and c that is the goal right.

So, let us start with some notation. So, we will define the vector alpha as the difference of these vectors a and c a minus c, the vector beta is the difference of the vectors a and b so, a minus b and then gamma is vector b minus vector c.

So, then we clearly have alpha is equal to beta plus gamma right so, it comes from all these cancellations this minus c will cancel with plus c and then you know minus b will cancel with plus b. And, you can quickly verify that you will just have alpha is equal to beta plus gamma is something that you can check. It is evident from the definition.

Now, we will take an inner product of you know of each quantity of this equation both on the left-hand side and on the right-hand side; so, alpha with alpha and then beta plus gamma with beta plus gamma on the right-hand side. So, if you take the inner product, then you have the inner product of alpha with alpha is equal to inner product of beta with beta plus inner product of gamma with gamma plus inner product of beta with gamma plus inner product of gamma with beta on the right-hand side right.

So, we can write it as you know beta, beta plus gamma, gamma plus 2 times real part of beta with gamma. So, how does this come about? This comes about just from the fact that the inner product of a vector beta with gamma is equal to the inner product of gamma with beta conjugate right, gamma beta whole star.

So, and then, you can verify that if you take any complex number and add to it, its complex conjugate, it is the same as 2 times the real part of the complex number that is all that is going on here.

Now, we use the property of complex numbers if you wish, you know the real part of a complex number is necessarily going to be less than or equal to the modulus of this complex number. So, in place of 2 times real of beta, gamma, we will write it as less than or equal to 2 times complex the modulus of this complex number inner product of beta with gamma right.

So, where all I have used is the fact that if you know any two real numbers, the real part is the real number and the imaginary part is a real number. If I think of it as x and y, then surely x must be less than or equal to square root of x squared plus y squared for any two real numbers x and y right, because you know the right-hand side if at all you are doing something or adding to it or if y is 0, then you are going to get back x itself.

So, you can never by doing the procedure on the right-hand side you can never you know remove something from x, you can you are only adding if at alright. So, this is an obvious inequality. So, we will, we have used that already. Now, we are ready to invoke the Cauchy-Schwartz inequality right. So, we already have this inner product of beta, gamma, modulus of inner product of beta, gamma.

So, we know that the modulus of beta with gamma, the modulo of the squared is going to be less than or equal to the; less than or equal to the modulus of you know beta with the inner product of beta with itself times the modulus of gamma with itself, the inner product of gamma with itself. So, that is the Cauchy-Schwartz inequality.

So, if you just plug this in, we will put it in with the square roots. So, then we have inner product of alpha with alpha is less than or equal to inner product of beta with beta plus inner product of gamma with gamma plus 2 times square root of you know the modulus of beta with beta times the modulus of gamma with gamma right.

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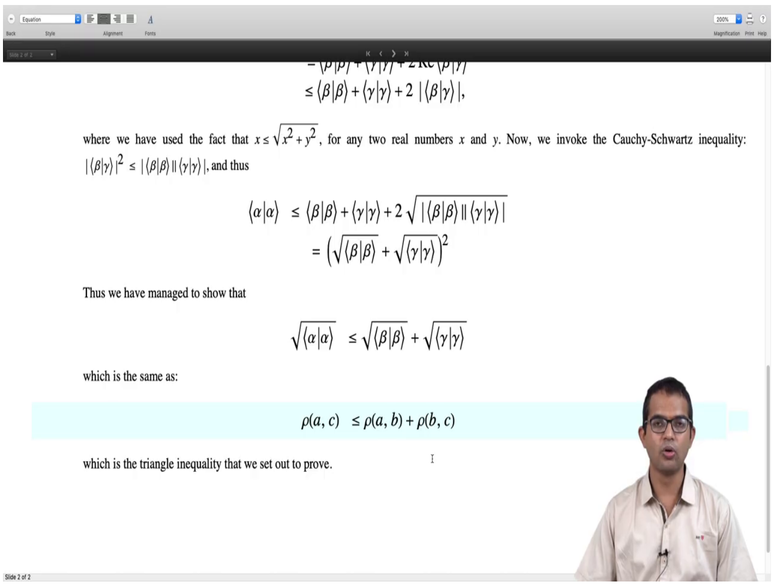

So, now it is in a form in which you can complete the squares and simply write the right-hand side as the square of square root of beta with beta plus square root of gamma with gamma right. So, therefore, and then if I take square roots on both sides, I am basically done now, I have managed to show that square root of alpha, alpha is less than or equal to square root of beta, beta plus square root of gamma, gamma right which is the same as saying rho of a times c rho of a comma c.

So, the distance between a and c necessarily is less than or equal to the sum of the distances a to b and b to c right. So, this is the triangle inequality. So, which is the inequality we set out to prove. And therefore, this notion of a distance defined like here is a legitimate notion. There are two other properties which are obvious like the distance between a point and itself is 0 according to this definition and so on.

So, given that triangle inequality holds, the space becomes a metric space. So, every linear vector space endowed with an inner product definition, makes it a metric space. So, that is what this lecture was all about.

Thank you.