Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 89
Initial value problem power series solution

So, we have seen what an ordinary point is as far as a differential equation is concerned, and when you are looking for a power series solution about an ordinary point. And we have also seen that there is this theorem which tells us that if you are expanding about an ordinary point, then you know convergence is guaranteed in some region of convergence about the ordinary point.

So, we will look at an application of this right, specifically for an initial value problem where you know initial conditions are given, and we want to use the power series method to expand about an ordinary point. So, we already did some kind of an expansion about an ordinary point as far as the Schrodinger equation was concerned. But in this lecture, we will look at how to do this for an initial value problem ok.

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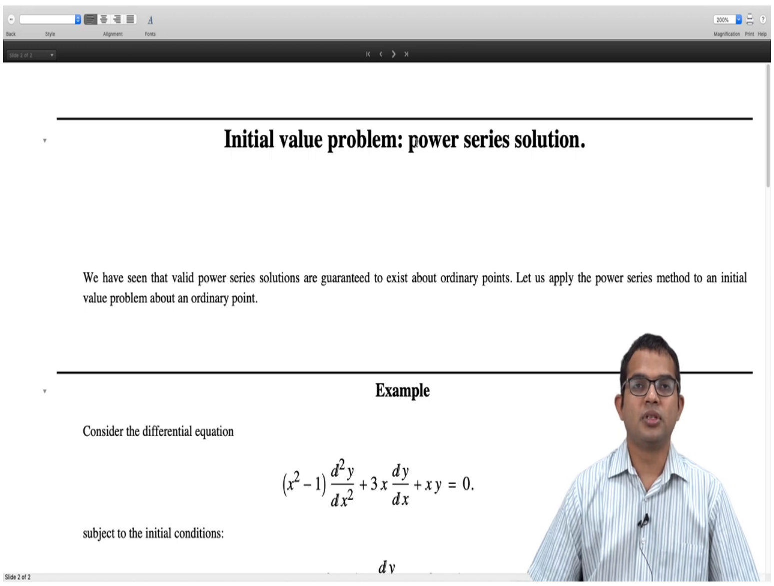

So, the theorem tells us that if you have an ordinary point, then you know there is a convergent series. So, you can go ahead and use the same standard technique. So, you write down the series. Then you compare coefficients term by term, and then write down a recursion relation and solve.

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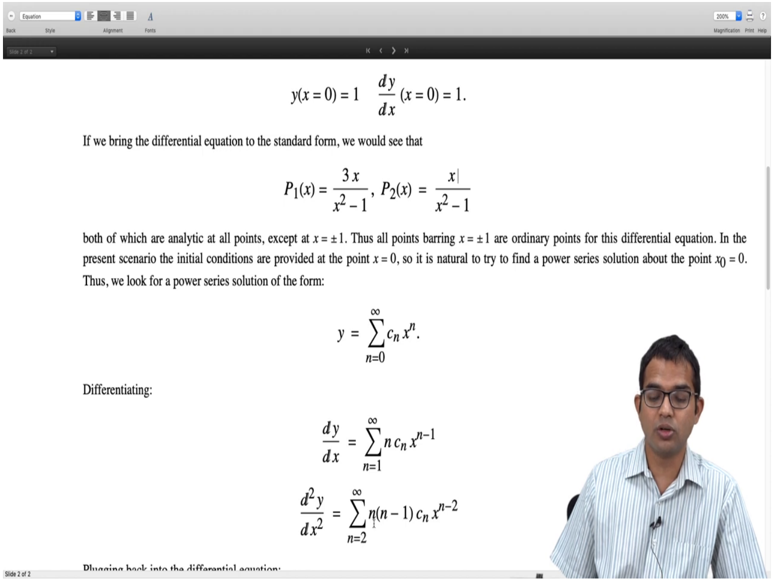

So, now suppose you have an initial value problem which means let us look at a concrete example. So, you have something like x squared minus 1 d squared y by d x squared plus three x d y by d x plus x times y equal to 0. And initial conditions are given to be at x equal to 0, y is 1, and the derivative at x equal to 0 is also given to be 1.

So, if we were to bring this to the standard form, then we would have P 1 of x is equal to three x by x squared minus 1, and P 2 of x is equal to x y divided P 2 of x would be just x by x squared minus 1. So, y is not part of this function. So, x by x squared minus 1.

So, we see that both of these functions P 1 and P 2 have singularities at x is equal to plus or minus 1. At every other point, both P 1 of x and p 2 of x are well behaved functions. So, they have no difficulties other than these two points x is equal to plus 1, and x equal to minus 1. Therefore, every point other than these two, x equal to 1 and x equal to minus 1 is an ordinary point as far as this differential equation is concerned. .

And since for us in our case you know the initial conditions are specified at x equal to 0 it is natural for us to come up with a power series expansion about x equal to 0 right. So, if we had had conditions about some other point, then we could have you know in some sense we could have shifted the origin to that point right, so that is the technique. And then the differential equation will undergo a small change and then you know the method that we are adopting here will follow through.

So, what we would like to do is you know the theorem guarantees that since we are trying to expand about an ordinary point, you know we will be able to find two linearly independent solutions, it is a second order differential equation. And so indeed so we look for a solution of this form y is equal to summation n going from 0 to infinity c n x to the n.

So, it is very similar to the problem of the Schrodinger equation, a solution of the Schrodinger equation for the harmonic oscillator problem which we did you know some time ago. So, differentiating we get d y by d x is summation you know n times c n x to the n minus 1; although summation still goes from 0 to infinity, you can remove n equal to 0 term because you know n equal to 0 will kill it.

So, you can say that it starts from 1. And likewise when you take the second derivative you can say that the summation starts from n equal to 2 because the first two terms you know vanish one of them because of the factor n, the other will vanish because of the factor n minus 1.

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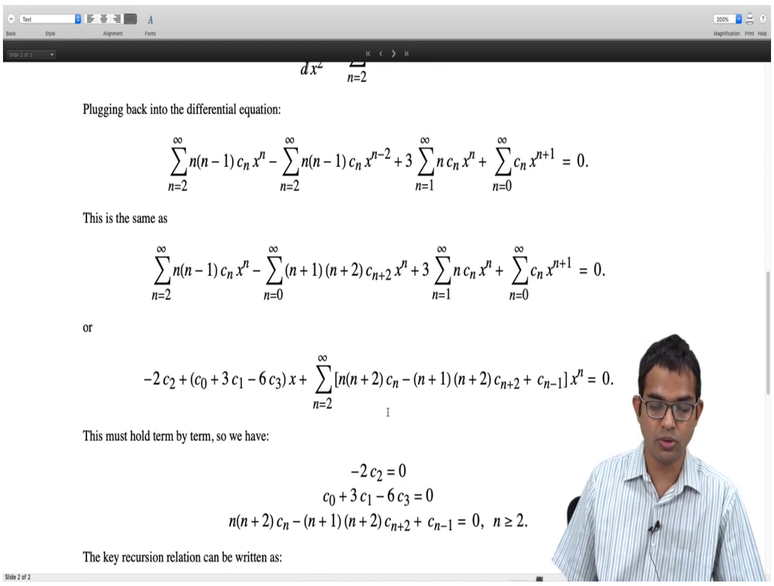

So, plugging all these back into the differential equation, we have you know d squared y by d x squared is as is minus so there is an x squared minus 1 alright. So, the x squared when it multiplies with this will give me x to the n minus 1. So, I have expanded this. So, minus so this is just minus 1 times this as is I have written this one as it is. Then I have a plus 3 times x.

So, 3 times x times this will give me this becomes x to the n, and n c n x to the n. Then I have plus y which is just x times y. So, I have x to the n plus 1 here summation over c n x to the n plus 1 equal to 0. So, this is the same as saying I have you know the first term I write it as it is.

The second term, so instead of writing the summation going from 2 to infinity, what I have done here is to make a change of variable. If you wish, I can write n minus 2 is equal to k, so I have x to the k. Then n, n is k plus 2, so c n will become c k plus 2. And then wherever I have n, I must replace it with by k plus 2.

So, I have k plus 2 minus 1 that will be k plus 1 times k plus 2. And then the sum will now go from 0 to infinity because n minus 2 is k. So, k will go from 0 to infinity. But it is just a dummy variable which is getting summed over. So, in place of k, I can put n back in and I have the second term. Then I have plus 3 times, I write this term as it is n going from 1 to infinity n c and x to n, then I have this term n 0 to infinity I write it as it is c n x to the n plus 1 equal to 0.

So, now I will collect all these terms n equal to 0 and n equal to 1 only some of these have it. So, I will just pull out all those you know terms involving just n equal to 0 and n equal to 1, and then I want to write everything else together in one you know sum, from n equal to 2 all the way up to infinity I will just combine all of that and write it as x to the n.

So, also I noticed that in fact this last term I could write it as c n minus 1 x to the n right, so that is what I have here and that will start from n equal to 2. So, in other words, I have put you know and also I pull out one of these, so you see that here it starts from x to the. So, if I make n plus 1 as k I have x to the k and n is equal to k minus 1, and then k will start from 1 right. And then the first term I bring it over here, so that will be c 0 x which I have here, and everybody else is absorbed into this second term.

So, I can check that I have n into n plus 2. Well, I mean I can also go ahead and do some simplifications here. So, I have n into n minus 1 plus yeah c n minus 1 is as is, and n into n plus 2. So, I have so this is the first term. And then I have a minus n plus 1 times n plus 2 c n plus 2 x n the second term indeed, yeah. So, I have you. So, I have combined this term, this term with this plus 3.

So, if I do this, then I have n into n minus 1 plus 3 which becomes n into n plus 2. So, I will just combine the first and the third term. So, then I have n into n plus 2 c n minus n plus 1 times n plus 2 c n plus 2 plus c n minus 1. And then there is all this stuff involving these first two terms, there is just minus 2 c 2 for it is the constant term. And then I have also you know just pulled out all these stuff corresponding to just x right. So, you can verify that this is true, and this whole thing must be equal to 0.

So, now comes the key point of you know these arguments which is that term by term every coefficient must be 0 because this holds true for any x. Therefore, I have minus 2 c 2 is 0, then I have c naught plus 3 c 1 minus c 3 0, and I also have this you know recurrence relation which I can get n into n plus 2 times c n minus n plus 1 times n plus 2 times c n plus 2 plus c n minus 1 must be equal to 0 for all n greater than or equal to 2 right.

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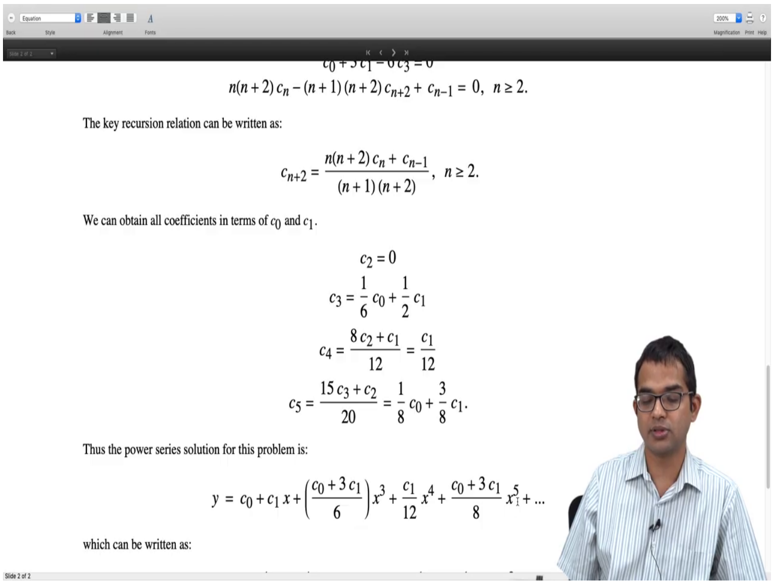

So, an equivalent way of writing down this you know this recursion relation is to write c n plus 2 in terms of c n and c n minus 1. So, I have n into n plus 2 times c n plus c n minus 1 divided by n plus 1 times n plus 2 for n greater than or equal to 2 that is c n plus 2 4.

So, we can obtain all the coefficients in terms of c naught and c 1. If we have c naught and c 1, then we put so we have already got c 2 to be 0. So, c 3 is c naught by 6 plus c 1 by 2 right.

So, this is just the second equation. And then after that, c 4, I can get using this. So, if I put n equal to 2, I will get c 4, you can check that this will just turn out to be c 1 over 12. And then c 5 will turn out to be so you have 8 c 2 plus c 1 by 12, but c 2 is 0 right.

So, therefore, you will have c 4 is dependent only on c 1, so c 1 over 12. And then I have c 5, c 5 will turn out to be a function of c naught and c 1. So, 1 over 8 c naught plus 3 by 8 c 1 you can check this right. So, all I am doing is explicitly applying this recursion relation, and then you have found the higher in terms of the lower and then which in turn is connected to its lower and so on.

So, in general you will be able to write down you know all the terms are specified in because we have this recursion relation. So, let us write down this explicitly for the first few terms all the way up to the 5th power.

So, then I have c naught plus c 1 x plus c naught plus 3 c 1 divided by 6 times x cube plus c 1 over 12 x to the 4 plus c naught plus 3 c 1 by 8 times x to the 5 plus so on, which can be written as you know two separate terms, one involving c naught and the other one involving c 1.

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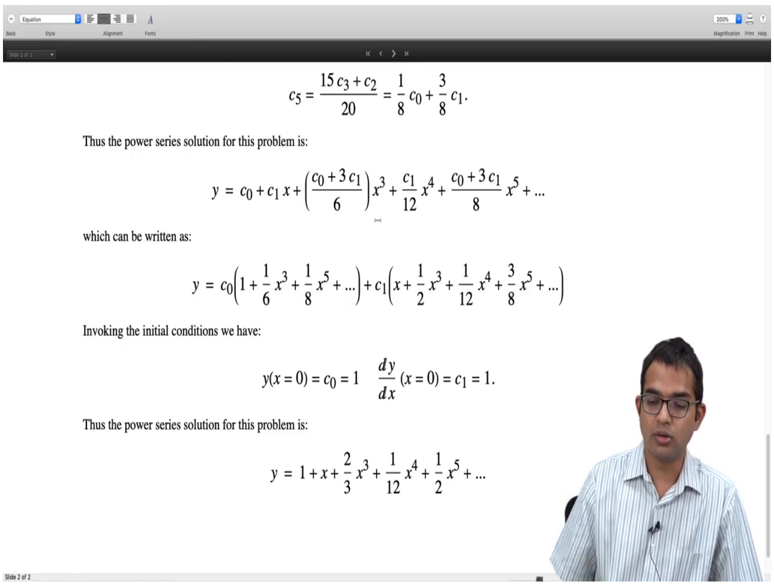

So, I can think of these two series. One of them is c naught times 1 plus x cube by 6 plus x to the power 5 by 8 plus so on that is one series. And the other one is c 1 times x plus half x cube plus x to the 4 over 12 so on that is another series where c naught and c 1 are free parameters. So, in fact, each of these is separately a solution because you can put c 1 to be 0 for example, and, so this is also a you know an independent solution. And likewise the other series is also an independent solution.

But, in this particular case, we are given initial conditions. So, in fact, we can work out c naught and c 1. So, y of x equal to 0. If I put x equal to 0 right, so we assume that there is convergence in some you know radius of convergence about the origin right. So, we would not even worry about finding this radius of convergence, but it exists.

But in fact, in this case we can even tell what that radius is expected to be. So, the rule is that if you know if you are expanding about an ordinary point, so and if one of them so you just simply look at these functions P 1 of x and P 2 of x right.

So, since this is going to have a singularity at x equal to 1 and x equal to minus 1, so the radius of convergence if you were to do the Taylor expansion of this function itself about the origin and this function itself about the origin. Both of these Taylor expansions will have you know radii of convergence. So, the smaller of these two radii's radii at least that is going to be the radius of convergence for the power series expansion that you are looking at.

But anyway we are not going to go into all the details of this. So, for as far as we are concerned, we are interested in you know using this machinery. So, there exists some radius which we do not need to bother about in the for as far as this course is concerned, about how to compute this radius of convergence, but let us just keep in mind that it exists.

And within this radius of convergence, so indeed if x equal to 0, you know all these terms will just go away and you are just left with c naught. So, y of x equal to 0 is just c naught which is given to us to be 1, so c naught itself is 1. And likewise if you take a derivative, you can do term by term differentiation you know because of this convergent series. So, then you have d y by d x is of x equal to 0 is also c 1 and that is you know also given to be 1.

So, c 1 is 1, c naught is 1, thus the power series for this problem is just 1 plus x plus 2 by 3 x cube plus 1 over, well, I mean I am just plugging in here I have 1 plus x plus you know 1 plus 3 by 6 that is 4 by 6 2 by 3 x cube plus 1 over 12 x to the 4 and so on. So, the key point is that we have explicitly seen that you know this for someone concrete example that if you were to expand you know about some ordinary point, then you know you get two independent solutions.

Both of these are power series which go all the way up to infinity in this case right. And they are convergent and each of them has their own radius of convergence which can be evaluated by standard techniques for evaluating the radius of convergence. You know there are tests available which you might learn about in some other course.

But, as far as we are concerned, we just want to apply this technique you know with the theorem in the backdrop. We have been told that there is a theorem which guarantees that it is a convergent series, and it is also going to give you two independent solutions; because it is an ordinary point as far as the differential equation is concerned. That is all for this lecture.

Thank you.