Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 87
Power series method

So we have looked at a number of different methods for solving differential equations. So, we have seen how you can somehow get the general solution for a homogeneous differential equation. Even if you put some you know some function on the right hand side and make it an inhomogeneous equation there is a way to solve this problem.

There are two methods we considered to work out this kind of a problem. One of them was the variation of parameters method and the other one was the Green function method right. So, all of this is possible only if you have some a priori way of getting a general solution for the homogeneous differential equation. So, it turns out that even very simple looking homogeneous differential equations cannot be solved you know in closed form right.

So therefore, we discuss a method which is called the Power series method in this lecture which will enable us to find, when possible, a power series solution for homogeneous differential equations. So, in this lecture we will you know with the help of a very familiar example, we will illustrate how this method plays out ok yeah.

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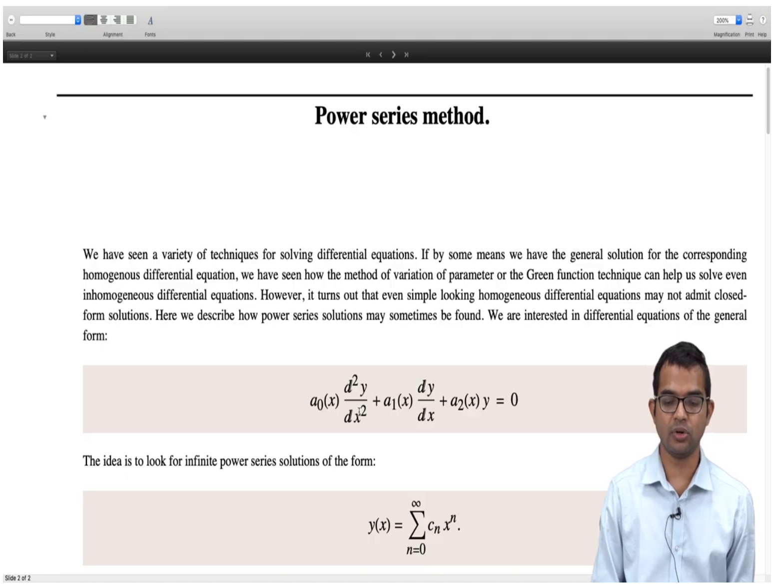

So, like we said you know in general homogeneous differential equations, if you can find a general solution by some method right you have access to you know in homogeneous differential equations as well as we have seen. But even the homogeneous differential equation itself can be a hard problem many a time and so the kind of differential equation.

We are interested in working out the following ODE: a naught of x some function of x you know sits there d squared y by dx squared plus a 1 of x dy by dx plus a 2 of x y is equal to 0. So, it is homogeneous because the right hand side is 0, but the difficulty comes because a naught a 1 and a 2 you know are functions of x, if they had been constants then of course we know how to solve this problem.

Sometimes even if very simple functions are involved, the differential equation may be a hard problem. So, in general there won't be closed form solutions for this kind of differential equation. But you know the technique that we will describe here today involves making Ansatz of this infinite series form.

So, we asked the question if suppose there is an infinite series right. So, we know that you know functions have you know Taylor series expansions right, which are valid in some interval and so on right. So, at this point we will not go into you know discussions of convergence and so on right.

So, you know, just suppose there is a Taylor series like you know expansion of this kind of power series expansion, which is a solution to this kind of a differential equation. How can we; how can we work such a series if it exists right. So, that is the type of question we will address in this lecture with the help of a familiar example.

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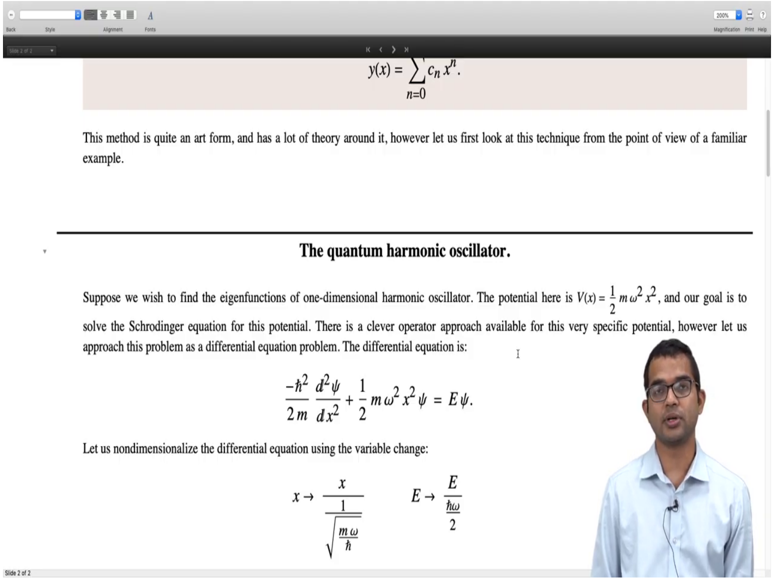

So, what we will do is we will look at the quantum harmonic oscillator right. So, we wish to find the Eigenfunctions of the 1 dimensional harmonic oscillator problem. So, the potential we have is V of x is equal to half m omega squared x squared and our goal is solve the Schrodinger equation corresponding to you know this potential right.

It is a very familiar potential and this problem has been you know surely most of you have encountered this problem and probably you have seen a clever you know algebraic approach operator approach to solve for this. And but there is also a differential equation way of solving for this problem which is our you know the, which is the approach we will describe in detail right.

So, this is described for example, in Griffith's textbook right. So, here it is. It is like a textbook example of you know a differential equation and how we can apply the power series method to work the solution alright. So, you know the potential that we are considering here the harmonic oscillator potential is of vital importance as it appears in all kinds of contexts.

So, sometimes it may not be exactly of this form x squared, but the idea is that you know lots of potentials are you know restoring in nature. And so the simplest approximation for a you know a stable equilibrium setup is to make it is to take the quadratic potential. And that is why you know this is of such great importance to understand this problem.

So, for sure the quantum harmonic oscillator is important - the harmonic oscillator itself we have seen the classical harmonic oscillator itself we spent quite some time a lot from the point of view of differential equations. But now we see that the Schrodinger equation applied to the quantum harmonic oscillator also leads us to a very important problem.

So, this time around it will tell us about how to use the power series method right. So, there is no doubt that this problem is of great importance and it appears in all kinds of contexts. So, it is good to understand you know the solution well ok. So, the first step is to non dimensionalize this right.

So, we start with the Schrodinger equation, which is you know h psi equal to e psi. So, what we are doing here is solving for the time independent Schrodinger equation right, from which you can also solve the time dependent Schrodinger equation.

So, the time independent Schrodinger equation is h psi equal to e psi and h in this case is given by minus h cross squared by 2 m d squared by dx squared plus the potential which is half m omega squared x squared. So, this is the differential equation which we need to solve.

So, the first step is to non dimensionalize, so that you remove you know in some sense you are removing all the physics away and treating it as a pure mathematical problem. And then you put the physics back in and. So, we use you know this non dimensionalization. So, the natural units for length here you can check is 1 over square root of m omega by h cross.

So, what you do is whenever you have x you replace it by you know x divided by this quantity and where you have E you replace it by E divided by h cross omega by 2 right. So, I mean you could use some other variable and so on right. But it is convenient for us to just work with x itself right.

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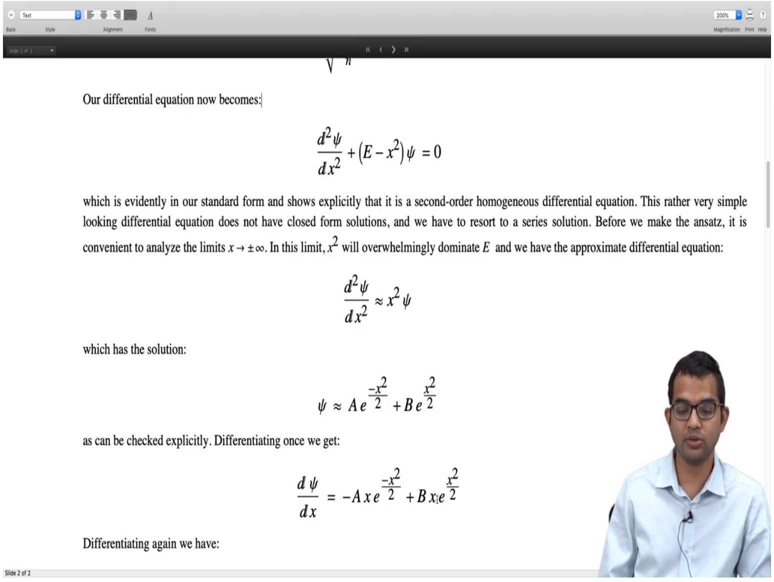

So, now with the understanding that the x that we are dealing with here is a non dimensional x and the E that we are dealing with here is a non dimensional E. So, you see this differential equation with all these constants and everything you know just becomes a simple differential equation, because you have you know pulled away all these constants.

So, you have d squared psi by dx squared in these new units plus E minus x squared is times psi equal to 0 right. So, you know you have this minus sign so this object goes to the right hand side and then you have this becomes a minus sign that is why E is positive here. So now, you see that this equation is evidently in the form that we first described.

So, there is a second order you know this second order derivative first order derivative is missing. So, there is a constant associated with the first one and the only function that appears is just E minus x squared right. So we would of course know how to solve this problem, if there were no x squared. If it is a it were just E of course that is the simple harmonic oscillator problem the classical harmonic oscillator problem right.

So, it is a I mean that it is then you which is not you should not think of it as a wave function, then right it is a familiar problem from classical harmonic oscillator if there were no x squared. But x squared is present at which is the potential energy term and this is you know it is surprising that you know such a simple change to one of these coefficients makes it such a difficult problem.

It is no longer possible to write the solution down in a closed form way and so we must find records to a power series solution right. So, but before we do that it is useful to you know bring in some intuition some physical intuition here. So, and look at what happens you know for very large x right. So, this is a wave function that we are looking for and wave function must satisfy certain properties right.

So, we can massage this differential equation into a slightly different form you know while keeping it completely general in this new form, but it is a more convenient form for the purpose of this power series solution right. So, the idea is to look at what happens to this differential equation when x becomes very large, either positive or negative you see that this x squared will make sure that you know this term E minus x squared will become basically minus x squared.

You know x squared will dominate our E and so you are left with just the differential equation approximately becomes d squared psi by dx squared is approximately equal to just x squared psi right. So, if you can find a solution for this you know we can look at what these wave functions will be like at large x positive and negative. And so it is a little bit of guesswork. We can now figure out that the solution should be of this form E to the minus x squared by 2 right.

So, the way to see this is that if you take a single derivative you know you would want you know an x to be a factor of x to come in and a second derivative should bring in another factor of x. So, then you start guessing that maybe something like E to the x squared will work out then you say ok, let us say maybe e to the alpha times x squared and then you try to fix the alpha which will turn out to be either plus half or minus half in this case right.

So, that is it is a kind of a guesswork game and then you argue that you know suppose we make this as are you know an ansatz for this differential equation it is an approximate differential equation and so this turns out to be an approximate solution which becomes a good solution for large x right.

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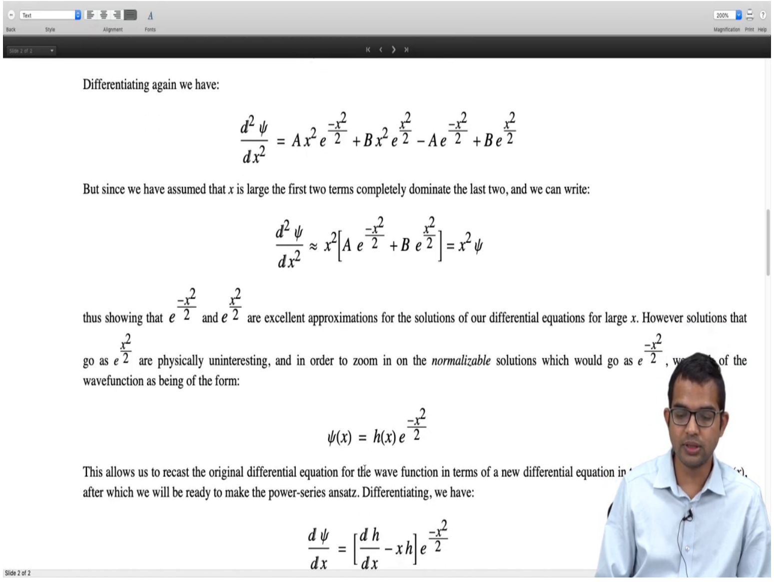

So, a way to check this is to just differentiate it. So, if you do d psi by dx then you get minus Axe to the minus x squared by 2 plus Bxe to the x squared by 2, if you differentiate it again then you get Ax squared e to the minus x squared by 2 plus Bx squared e to the x squared by 2.

You know the first you know two are basically like what you are interested in, but you also have these other two terms minus Ae to the minus x squared by 2 plus Be to the x squared by 2 these come from differentiating with respect to x in each of these terms. And so the argument is that you know for large x, since you have x squared sitting here in the first 2 terms. These terms are going to completely dominate over the third and the fourth.

So, you can ignore these two terms right the last two terms you can ignore, well because they are small in comparison to the first two and then you also observe that the first two terms are really you know the same as x squared times psi. So, indeed for large x this is an approximate solution for your problem. So, in other words in fact you have e to the minus x squared by 2 and e to the x squared by 2 are both you know approximations for this you know solutions of a differentiable for large x.

Now, comes the physical argument. So, we argue that you know we are not interested in solutions whose asymptotic behaviour is e to the x squared by 2, because then these are solutions which will become very large for large x and so there will not be normalizable solutions right. So, we since were interested in wave functions Eigenfunctions for these for the harmonic oscillator problem.

So, for it to have physical meaning we know from quantum mechanics that these Eigen functions must be normalizable. So, this requirement for normalization means that we are not interested in these solutions which you know fall off as e to the x squared by 2. But rather we will be interested in solutions which have this e to the minus x squared by 2 right. So therefore, what we do is we look for solutions of this form.

So, instead of just directly searching for the solution for the differential equation for psi of x, we say that psi of x itself will be some function times you know e to the minus x squared by 2 you know which is a kind of a feeling of operation which one does. You say that ok there it is we want you know e to the minus x squared by 2 these are desirable types of solution.

So, why do not we explicitly put this in by hand right? You could have also done it without doing this, but it turns out that the differential equation that you get in h is a convenient differential equation which we will be able to relate to a standard differential equation. So, that is why this is a useful thing to do right. So, we will see you know as we go along in this lecture how you know this differential equation in h instead of differential equation in psi is a convenient you know differential equation to work with, ok.

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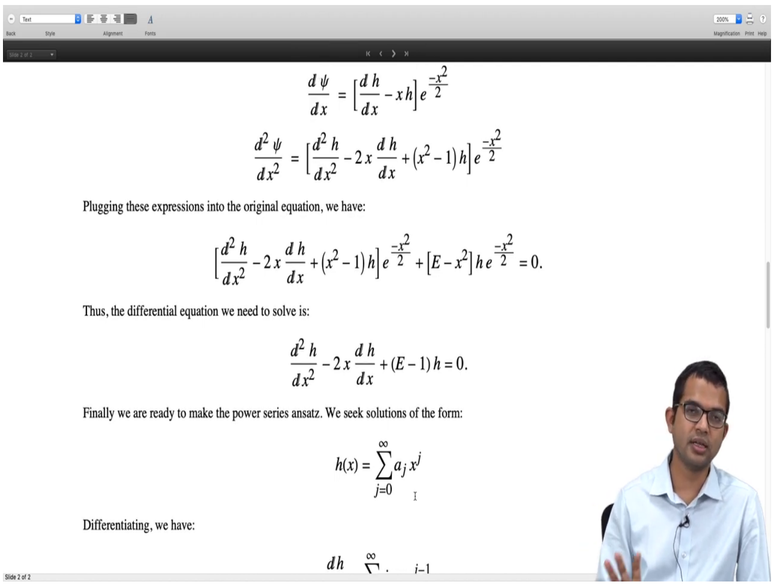

So, what we do is yeah so now we have to work out first of all what the differential equation for h is and then solve it using the power series (Refer Time: 14:11) So, you know at this point it is completely general right. So, it is like all we have done is we have used some physical intuition to you know to think of solutions which have a plausible form and then we are looking for solutions of that form and in fact we will work we will look at a different differential equation.

But actually so far we have not done any approximation, there is no loss in you know there is no you know there is no nothing fundamentally that has been lost. So, far it just we have it has a it like a transformation in your instead of solving one problem your solving another problem, which you know which can get back to the solution the first problem itself right. So, what is that new problem? The transformation is obtained by first doing these 2 derivatives.

So, d psi by dx is dh by dx minus x times h the whole times e to the minus x squared by 2 you know. So, that is the thing about these factors of e to the minus x squared by 2 if you take a derivative you still you know leave such factors around after every differentiation.

And so you can pull out this entire factor when you take another derivative once again you get either d squared h by dx squared or you know if you are differentiating the second term you if you differentiate with respect to you know h. So, then you get x times dh by dx e to the minus x squared by 2, but you could have also done it with the first term when you are differentiating with the second one which will give you minus x times dh by dx e to the minus x squared.

So, if you collect both of these you get minus 2 x dh by dx and then you also have this x squared which comes about when you take a derivative of this guy e to the minus x squared by 2 for the second term. And then you also have a minus so minus h, so if you take a derivative with respect to x.

So, that will give you just h times e to the minus x squared by 2 if you take a derivative with respect to e to the minus x squared by 2 that will give you know x that will come x squared minus 1 right. So, you can convince yourself that indeed this is the first derivative and second derivative then you go ahead and plug back these quantities into the original differential equation right which is this one.

So, d squared psi by dx squared plus e minus x squared the whole times psi. So, then we see that you still have these factors which are sticking around which you can cancel right, because this needs to hold for all values of x. So, indeed d squared h by you get a simple simplified differential equation in x which is also second order in sorry differential equation in h, which is also a second order differential equation in h.

So, we have d squared h by dx squared minus 2 x dh by dx plus e minus 1 times h equal to 0, this x squared will cancel with this minus x squared and you are just left with e minus 1 times h is equal to 0.

So finally, we are ready to make our power series Ansatz. So, we seek solutions of this form h of x is equal to summation over j aj x to the j right. So, we assume you know that this is a you know convergent series and it has all these nice properties. So, you can take derivatives of this function.

And then you know term by term you can differentiate term by term, so the convergence remains unharmed right. So, we assume this at this point right we are not going into detailed discussion of these convergence issues at this point because we want to just illustrate how the technique works.

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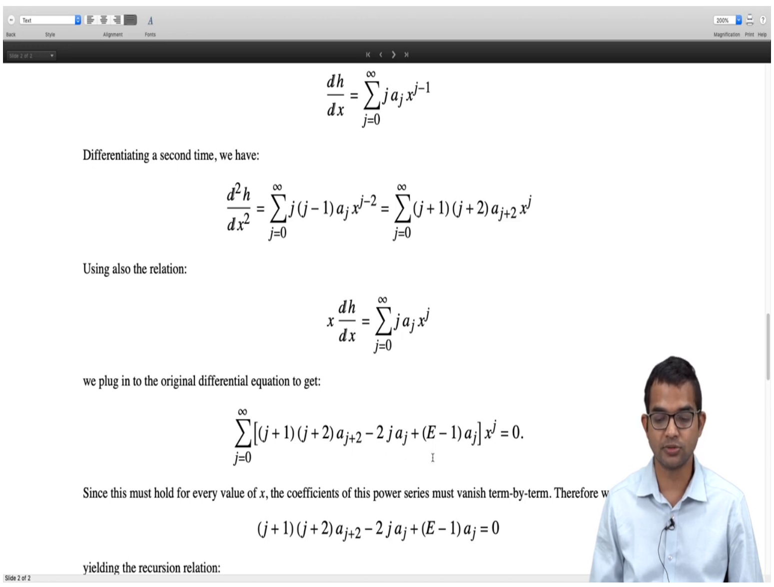

So, assuming that this holds so you have dh by dx is equal to j times a j x to the j minus 1. So, what we want to do is you know make this ansatz and plug this into your differential equation. So, we will actually have to work with x times dh by dx. So, we will work that out - we leave this series expansion for dh by dx as it is. But if we differentiate a second time we get d squared h by dx squared is equal to summation over j j times j minus 1 times a j times x to the j minus 2.

Now, we observe that you know the first term and the second term are real is 0, j equal to 0 will cause the first term to be 0 j equal to 1 will cause this factor will cause it to be the second term to be 0. So, for all practical purposes this summation actually runs from j equal to 2 to infinity and then we can make a change of variable j minus 2 is equal to some k.

So, then in place of j minus 2 you will write k and in place of j you will write k plus 2 in place of j minus 1 you will write if a j equal to j equal to k plus 2. So, it will become k plus 1 and this one will become k plus 2. And then you argue that it is just a dummy variable which is getting some instead of k running from 2 to infinity you can I mean you had j running from 2 to infinity and so k will run from 0 to infinity.

But, you say, in place of k you might as well just call it j again so then you have j going from 0 to infinity j plus 1 times j plus 2 times a j plus 2 times x to the j right. So, we want to bring this you know each of these terms is a summation, but they all should be x to the j right that is the idea.

And then of course since dh by dx has this factor x in this different direction, we can just write it as x times dh by dx is equal to summation over j j times aj times x to the j right. So, now we have expressions series expressions for h of x the second term and the third term, we just plug this in and now we have you know summation over j j plus 1 times j plus 2 times a j j plus 2 minus 2 j a j plus E minus 1 times aj all of this multiplied with x to the j equal to 0.

Now, we argue that you know if this has to be 0 for any value of x. So, this has to be in fact 0 term by term. So, every coefficient must in fact be 0, so we get j plus 1 times j plus 2 times a j plus 2 minus 2 j a j plus E minus 1 a j equal to 0. So, this is very important to them.

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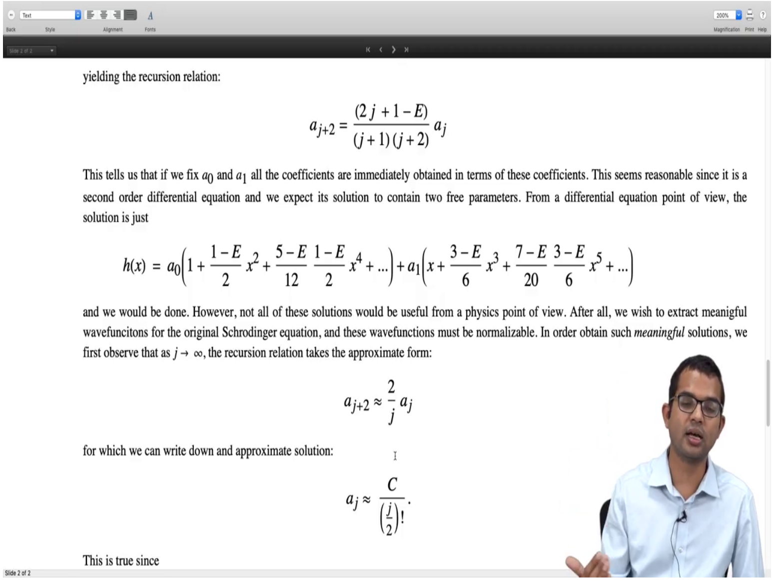

So, because what it does is it gives us a recursion relation it tells us that you know we are looking for a power series in which the coefficients are related by this relation right. So, if you know a j then it tells you how to go to a j plus 2, right. And so what it means is so you see that this factor is in terms of j and E, E is your you know the energy of your system and what it tells you is these coefficients relate a j to a j plus 2.

So in fact, if by some means if you fixed a naught and a 1, so a naught will fix for you a 2. But, if you know a2 you know a4, if you know a4 you know a6 because the same rule can be applied repeatedly. And likewise if you know a1 it will give you a3 it will then give you a5 and in turn a7 and so on right. So, all the odd coefficients are fixed if you know a1 and all the even coefficients are fixed if you know a naught.

So in fact, it is not a surprise that this happens because after all we are working with a second order differential equation, which you know contains 2 free parameters. So, it turns out that you know all the information that is there in the second order differential equation is compressed into this recursion relation involving you know 2 free constants. If you have 2 free constants you know you have the full solution right.

So, in other words we have managed to show that you know h of x is equal to a naught I am just writing out the first few terms times 1 plus 1 minus E by 2, if you have the energy and if you have if you fix a naught and a 1 you have the solution E a 1 minus E divided by 2 times x squared plus 5 minus E by 12 times 1 minus E divided by 2 times x to the 4. So, you have the; you have you know you know 2 of these factors appearing here.

Because one comes from here is just E 1 minus E by 2 you know, because a 2 after all is related a 4 is related in terms of a 2. So, you have to put in a 2 and the stuff and likewise if you work out the next one that will have 3 such factors this whole stuff times another factor. And likewise you see a 1 if you fix a 1 that will give you x you know there is a coefficient corresponding to x.

Then you have plus 3 minus E divided by 6 times x cube plus7 minus E by 20 times 3 minus E by 6 x to the 5 and so on right. So, I have just explicitly written down the first few terms using this recursion relation to show you how you know this operates in practice. So, you know as far as a differential equation is concerned. So, it seems like we have got the solution right for this h if we know h we also know psi right.

But, there is some more physics we have to put into this to make it a meaningful solution. We are not only interested in a solution for this problem. But we are interested in finding a meaningful solution for this differential equation for the Schrodinger equation right. So, in order to see what this means we have this infinite power series does it make sense at all.

So, let us see what happens to this recursion relation when j becomes very large. So, when j becomes very large you see that 2 j plus 1 minus E you can replace this stuff with just 2 j, because j is the only thing which is very large and others can be ignored. So, in place of 2 j plus some minus stuff you can just rewrite this to j and likewise the denominators you can replace j plus 1 times j plus 2 by just j squared.

So, in the end you have 2 j divided by j squared which is just 2 by j. So, basically this recursion relation for large j will just become a j plus 2 the approximately equal to 2 by j times a j. So, some more thought reveals that in fact we can write a closed form expression for a j in term because of this you know this property that a j plus 2 goes as you know a j divided by j and there is a factor of 2 involved as well.

So in fact there is a factorial connection which comes to mind because you know the relationship between a certain number and you know 2 more than that are related by a factor involving that number itself. So, you have to play with this idea that there might be a factor involved and you have to work out exactly what the solution is and then you make a guess. In fact, if you choose to try a j is approximately equal to some constant divided by j by 2 factorial right.

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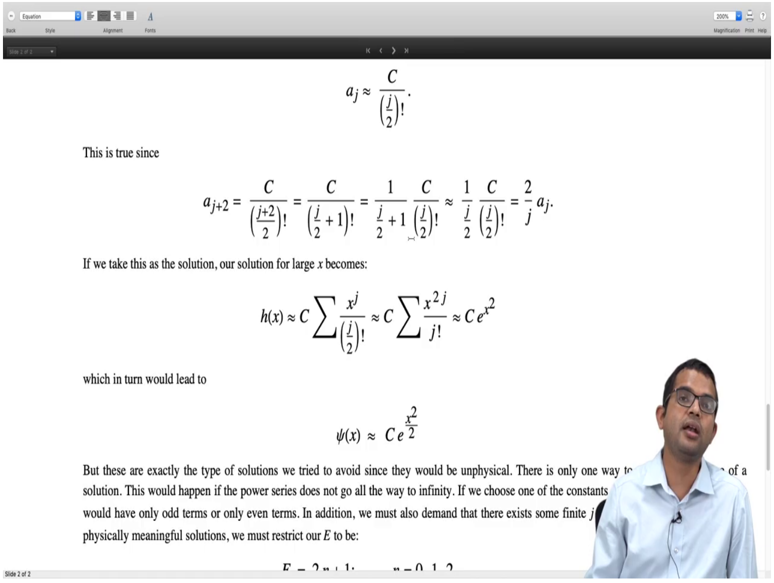

So, this comes because of these two involved here and then it is going to be a solution to this right. So, the way to see this is to just calculate what happens to a j plus 2. So, a j plus 2 will become C divided by j plus 2 divided by 2 factorial, but j plus 2 divided by 2 is the same as j by 2 plus 1 you know.

So, there is a factorial so that you can plug out 1 of these factors 1 over j plus j by 2 plus 1 times C by j by 2 the whole factorial. But C by j by 2 the whole factorial is actually nothing but the solution we have for a j and j by 2 plus 1 since we are going to assume that j is large you can replace j by 2 plus 1 with just j by 2. So, you see that there is approximately equal just 2 by j times a j. So, a j plus 2 is indeed for large j approximately like 2 by j times a j, if you choose a j to be of this form.

It is like some constant divided by j by 2 factorial. So, if you take this to be our solution right; so that means, for large x our solution has this form h of x is you know summation over x summation over x to the j divided by j by 2 factorial, which is the same as saying it is a summation over x to the 2 j divided by j factorial which will work out approximately to be some constant times e to the x squared right.

So, this is just a you know sort of a rough calculation we are doing for what happens to this function h of x for large x, if you know by looking at a j for large j. Now this will lead us to the solution e to the x squared which is actually the undesirable solution, why? Because if h of x goes as E to the x squared then we do this sort of feeling of operation we had done because ultimately we are interested in psi of x.

So, psi of x will be h of x times E to the minus x squared by 2. So, if you put these 2 together you will get psi of x will go as approximately C times E to the x squared by 2 right. So, which is actually the solution we found undesirable. We did not want this type of solution, because it is going to be not normalizable, right.

So, it seems like no matter how you know you choose your a naught and a 1 if the series keeps on going all the way to infinity if you make j to be arbitrarily large it is going to give us a solution which we do not want. So, the only way to avoid this it turns out is you know is to forcefully truncate the series right. So, you have to choose your energy E in such a way, if your energy were such that at some point this series will terminate.

And then if we cleverly chose one of these coefficients a naught or a 1 to be 0 right. So, we have an odd sequence and an even sequence, if you choose a naught to be 0 then you are only left with a 1 a 3 so on and let us say so if a naught is 0, a 2 0, a 4 is 0 all of these are 0. But on the other hand let us say you start with a 1 and then it goes to a 3 and then it goes to a 5 and then there is a certain n at which lets say that you know a n plus 2 is going to go to 0.

If by some means you can arrange that right by choosing your E appropriately, then you are guaranteed that you will get a solution which does not become you know which does not go to this type of a solution right. Let us see what happens if we do this. So, how do we force this power series to truncate beyond a particular n right, so we choose our energy E to be 2n plus 1.

So, we see that here if there exists a j which is equal to n we will call it n when j becomes n. So, you have 2n plus 1 minus E. So, if n 2n plus 1 is actually equal to E then we will get a n plus 2 is 0, once you get n plus 2 0 everybody after that is also 0 right. We choose one of these coefficients either a naught or a 1 to be 0 and we also choose our energy E to be 2n plus 1 right, such that n can take values 0 1 2 and so on.

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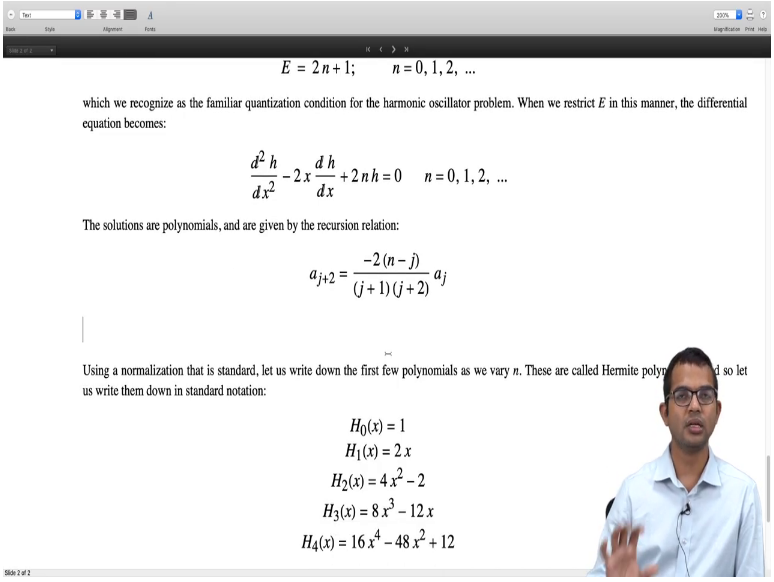

Which we already see that you know this is how the quantization conditions of r a for energies of the harmonic oscillator works out right. So, this is some very obscure way of obtaining this quantization condition which comes from you know putting in these physical requirements on the solution of a differential equation which we have obtained by power series methods right.

So, if you do the operator you know algebra method you have another way of obtaining you know this quantization condition. But from the differential equation point of view it appears because of this demand that you know the solutions obtained for this differential equation be meaningful right.

In particular we do not like solutions which become very large for large x and so there is normalization that is the requirement which forces you know energy is to take only certain values right. So, when we restrict in this manner, the differential equation also changes. So, we see that we started with the d squared h by dx squared minus 2x dh by dx then we had for this differential equation we had plus E minus 1 times h equal to 0.

But E minus 1 will just become 2n right in this unit. So, the differential equation just becomes plus 2nh equal to 0 right and the solutions when you do this you know are polynomials right. So, a j plus 2 so which we fix n to be 0 1 2 and so on and then we get these polynomials in. So, in other words these power series do not go all the way to infinity they are polynomials.

So, definitely there is no difficulty with convergence and so on right. So, for this particular very precise choice that we make, they become polynomials which are extremely well behaved functions and they are you know meaningful for all values of x and so on and they have exactly the properties that we demand for the wave function right.

So, they will yield properties for the wave function which is connected very closely to h of x. So, if we can find h we can also find psi which is just you know with the help of a factor. Now, so here what I have done is so recursion relation also has changed right. So, in place of E I have put n plus 1. So, the recursion relation here in place of you know E I have put n plus 1.

So, 1 will cancel and then I will have with just 2 times j minus n or which can be written as minus 2 times n minus j divided by j plus 1 times j plus 2 you know this whole stuff times a j. So, this is the solution for this differential equation and here the solution is a polynomial solution for this right.

So, the set of polynomials which are obtained you know for different values of n right. So, this is a well known set of polynomials. So, these go by the name of Hermite polynomials and they have some you know convention which gives them certain special factors right. So, I have just put in you know those factors involving some factorial n and you can look this up right.

So, if you use these Hermite polynomials right which goes back to this differential equation and whose recursion relation is this and you use a suitable you know set of factors which depends on n in a very special way I am not giving you that formula here. But probably there will be more extensive discussion about orthogonal polynomials and so on in the next part of this course right.

So, in any case this is like from a differential equation point of view using the power series method, we see that for the harmonic oscillator problem we will get Hermite polynomials with are which are you know the power series truncates beyond a certain point and it becomes in fact a set of polynomials. The first few polynomials are just 1 2x 4x squared minus 2, 8x cube minus 12x, 16x to the 4 minus 14x 48x squared plus 12 and so on right.

So, these are all you know polynomials which you can look up from some tables. For example, or you know you will see many properties of this you know in a discussion on orthogonal polynomials. But the key point here is that you know these are alternately odd and even polynomials right; which is what you expect, because you have one of these either a naught or a 1 to be 0 and the other one is non-zero.

And then it truncates at a certain finite value and it keeps on increasing. So, the first one has just one term, second also has only one term and third one will have to yeah I mean H 2 has 2 H 3 has again 2 and H 4 has 3 and so on right. So, this has to do with how you know it the series truncates at the n-th position.

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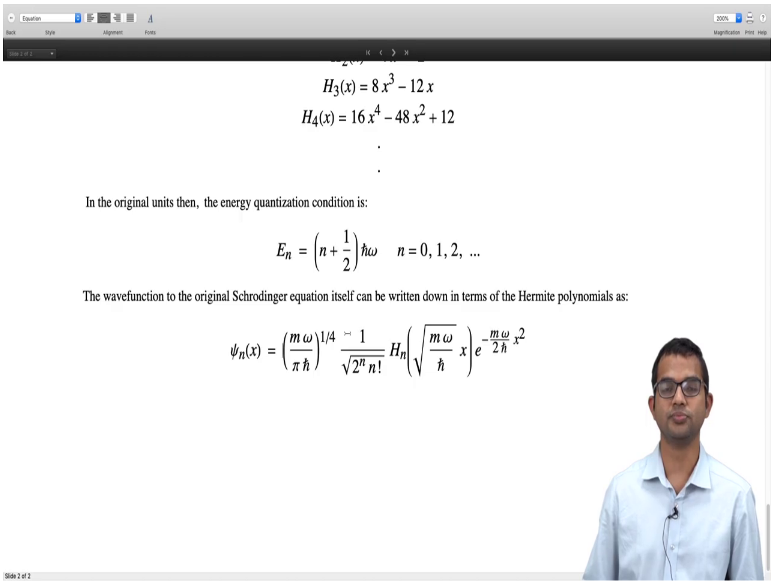

So, in the original units so the quantization condition is the form in which we have seen it you know most of us have seen this relation E n is equal n plus half h cross omega right. So, we are told in more elementary courses that you know there is a quantization which happens for the harmonic oscillator and this is the formula for it and that n can take values 0, 1, 2 and so on.

So, the ground state energy is just half h cross omega and so on. And the wave function of the original Schrodinger equation itself also can be written down in terms of these Hermite polynomials. So, you see there are these factors of 1 over square root of 2 to the n n factorial you know this comes in because of the convention that is followed for Hermite polynomials.

And then we have plugged in the original you know units therefore you have all these factors of m omega divided by phi h cross the whole to the power 1 fourth. And the polynomial Hn is not of just x, but you also have this factor that you have to use because you have this original units square root of m omega divided by h cross and then you have E to the minus m omega by 2 h cross times x squared.

So, in this lecture we managed to work out the solution for the Schrodinger equation, or the differential equation coming from the Schrodinger equation of the one dimensional quantum harmonic oscillator problem. And we use this you know problem to illustrate how the power series method works out that is all for this lecture.

Thank you.