Mathematical Methods 1

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Ordinary Differential Equations
Lecture - 86
Green's function method: Boundary value problem

So, we have looked at how the Green's function method plays out with you know the aid of a simple example involving a harmonic oscillator with an external driving term. So, the idea is to look at the external driving term as you know being made up of many impulses and find the response of the system to you know, an impulse and then stitch together the solutions of all these impulses and we get the general solution for the given problem right.

So, in this lecture, we will look at another application of the Green's function method you know applied to a boundary value problem ok.

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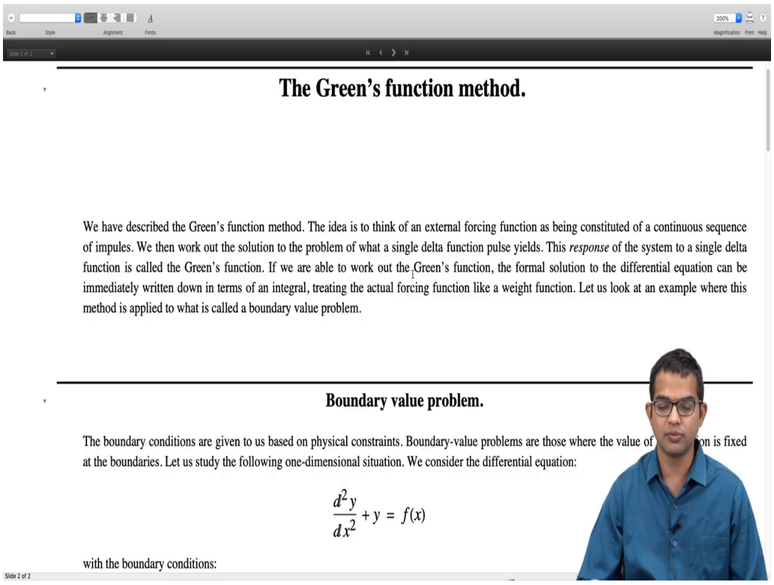

So, a boundary value problem is one where you know, the boundary conditions are given at the boundaries right. So, in an initial value problem typically like, you know get interested in some dynamics of some particle.

So, you specify its initial conditions, where you know where it is at time t equal to 0 and what is its speed at time t equal to 0 and then, how does it evolve as a function of time. But in a boundary value problem, you may think of something like a stretched string. So, you know at the two ends, the string is going to be fixed.

So, its displacement is 0 at the two ends of your string or you may have some you know in general, if you have a partial differential equation type set setup, you may be interested in the temperature profile of some rod of iron. For example, you may specify its temperatures at the boundaries and then you have to solve some partial differential equation to work out what its temperature profile is throughout the structure right.

So, boundary value problems are an important class of problems. And so, we will look at a one-dimensional version of it right, where the Green's function provides a very nice way of solving this kind of a problem right. So, it illustrates you know what a boundary value problem will look like in 1D, but also tells us how to use the Green's function method to solve such a problem.

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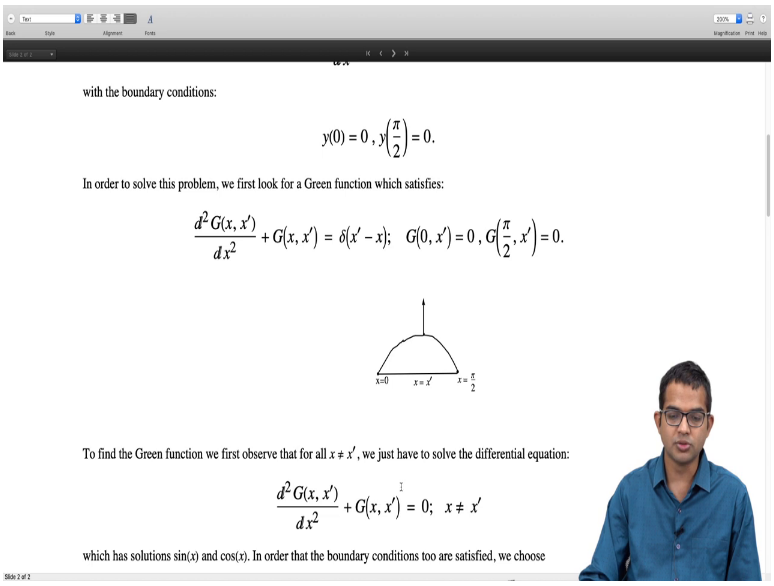

So, we consider this very simple problem. So, we already looked at you know what happens to the harmonic oscillator which is subject to an external force - the forced harmonic oscillator, but here it makes more sense to think of this as like a stretched string, which is attached at the two ends. So, the two ends, we take it to be one at x equal to 0 and the other one is at x equal to pi by 2, pi by 2 is just taken because, you will see that it will simplify the equation, you could have taken it to be at x equal to l right.

But you will see that pi by 2 makes you know things look clean and neat. So, at pi by 2, it is fixed and at x equals to 0, it is fixed and so, you ask what is y as a function of x.

You know given some arbitrary external force of illusion of effect. So, in order to solve this we look for a Green's function, which satisfies not only the differential equation with an impulse being applied at x prime right, I mean clearly x prime must lie between 0 and pi by 2 right.

So, x prime and also you must demand that this function the Green's function G of 0 comma x prime is 0 and G of pi by 2 comma x prime is also 0. So, but in general, you can provide your boundary conditions in many, many more complicated ways, you can provide you know derivative information at the boundaries, you can provide a mixture, you know all kinds of situations come up right.

So, but they all in fact, can be tackled within the Green function approach. Whatever boundary condition you provide for your function itself will get transferred to the Green's function itself. So, let us look at this concrete example, and when we see how this works out, we can use it to solve other kinds of other similar problems as well. So, here is a picture which illustrates the situation that we are looking at.

So, there is a string you know which is 0 at x equal to 0 and at x equal to pi by 2 and so, we imagine that you know there is this impulse function which is being provided at x equal to x prime. Everywhere else, it is just you know there is some value of this function right; it has to be evaluated right. So, the question is what happens to this function, if there is this you know under the action of this kind of an impulse right. I mean I have just exaggerated this and it is a very sort of freehand drawing.

So, it is just a sketch of what the nature of this function might be. We will find out what it will be when we solve this problem, but you know this is that sort of schematic picture we want to keep in mind. Now, to find this Green's function, we first observe that for all x naught equal to x prime, we just have to solve this difference, it is a very simple differential equation. We know we know how to solve this differential equation, we have solved it.

You know perhaps this is the first differential equation we all solved right in a maybe in high school or something. This is the harmonic oscillator problem and we know that the solution is just given in terms of sin x and cosine x and in an arbitrary combination of sin x and cosine x will be a solution to this differential equation

But now, we also have this boundary condition which we have to take care of and the boundary condition here is you know that at x equal to 0 your function must be 0 and x at x equal to pi by 2 also your function must be 0.

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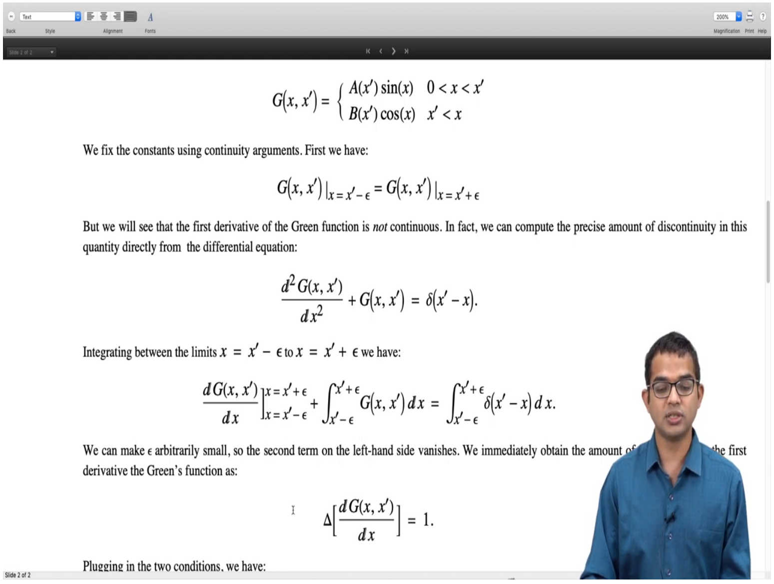

So, in this case, you know we choose sin x near the boundary where you know x equal to 0 and cos x near the boundary where x goes to pi by 2. So, we say that you know from 0 less than x less than x prime.

So, you know x prime is where the impulses from 0 to x prime, you take it to be A of x prime times sin of x and from x prime all the way up to you know if x is greater than x prime, then you take this function to be B of x prime times cosine of x. Of course, x will go all the way up to pi by 2 and so, we fixed the constants using continuity arguments.

So, you know this function itself must be continuous. So, the value of this function G of x comma x prime, you know slightly to the left of x prime must be the same as the value of this function slightly to the right of x prime, but the derivative it turns out is not going to be continuous right. So, this is the kind of problem which you might have worked out in quantum mechanics, when you look at the delta function potential.

The Schrodinger equation if you are solving for the delta function potential, then you demand that your wave function is continuous, but it turns out that the derivative of this function will not be continuous right. Because that is inherent in the differential equation itself, in fact, you can compute the precise amount by which there is a discontinuity right.

So, in order to do that, we will just go back to the differential equation. So, this is the differential equation we have. So, if you integrate you know from x prime x equal to x prime minus epsilon to x prime plus epsilon so, the first term will just become a first derivative, second derivative will become first derivative.

Now, it has to be evaluated between x prime minus epsilon to x prime plus epsilon and then, we have this integral of this function G of x comma x prime going from x prime minus epsilon to x prime plus epsilon and then the right-hand side is again this integral involving delta function. But the delta function integral will just give us 1 right, because that is the property of the delta function right, no matter how small this epsilon is, we imagine epsilon being arbitrarily small.

So, in fact, the second term will go to 0 right. So, this is an ordinary function. Only if you have a delta function will this kind of an integral yield a non-zero value. So, the right-hand side will become 1, the left hand side second term will actually vanish. Because, we are envisaging a scenario where epsilon becomes arbitrarily small and then the first term is actually, what were after it is the magnitude of the jump in your first derivative.

So, the first derivative has a discontinuity at x prime and the value of the amount of this discontinuity is exactly 1 right. So, this is something that we have to put into our, you know guess for G of x comma x prime or rather to when you want to evaluate the constants in our function in our solution.

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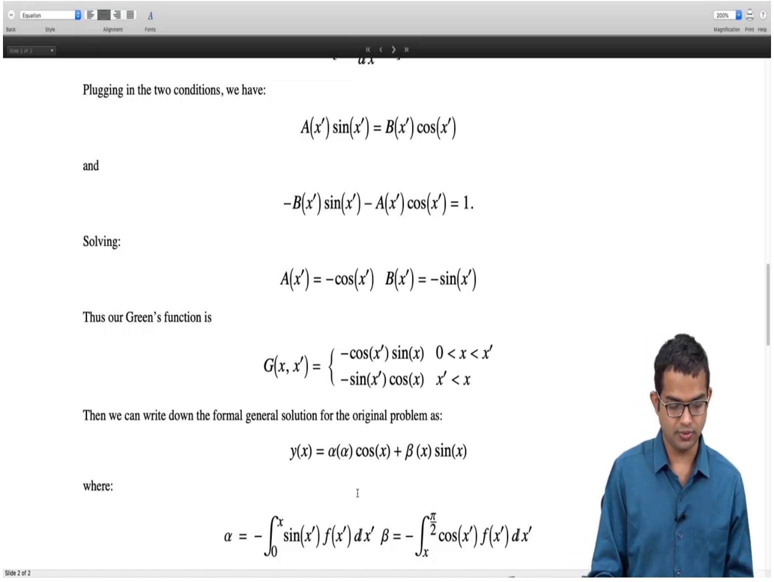

So, we have A of x prime times sin of x prime equal to B of x prime times cosine of x prime, this comes from the continuity of the function itself. So, which comes from physical arguments and the second condition is that the derivative of this function has this jump. So, which means minus B of x prime times sin of x prime minus A of x prime times cosine of x prime, you can work this out, you know take this as your you know Green's function and take its derivative and the jump is given by this function that must be equal to 1.

So, we have two equations and two unknowns. So, we can solve for both of them and you can check that in fact, A of x prime is nothing but minus cosine of x prime and B of x prime is minus sin of x prime right, this will work out for these two you know linear equations. Therefore, our Green function is simply given by minus cosine of x prime times sin of x, if x is less than x prime and it is minus sin of x prime times cosine of x, if x is greater than x prime.

So, we can write down the formal general solution in this very nice form right. So, it is just alpha of x so, it is alpha of x, it is alpha of x times cosine of x plus beta of x times sin of x, where alpha is given by this integral minus 0 to ; minus 0 to x sin of x prime f of x prime d x prime, beta is minus x 2 pi by 2 cosine of x prime f of x prime d x prime right.

How do you get this? I mean you just get it because, you know you have to take this Green's function multiplied by f of x prime and integrate over x prime right and then, but since, this Green function has these two parts, you have to integrate x prime separately from 0 to x is 1 and then x to pi by 2 is the another right, both when you work this out carefully you get, you know this kind of a solution y of x is equal to some alpha of x times cosine of x plus beta of x times sin of x.

So, this above is a very suggestive form right. In fact, it is something that we have seen a while back and so, this is the comment I want to make right. So, this is the method of variation of parameters, we discussed a while back. If you remember, you know we have this differential equation, you know we have this differential equation so, the method of variation of parameters tells us that you find the solution for just the homogeneous part.

So, in this case, the solution to the homogeneous part is just sin of x and cosine of x. So, in general the solution is c 1 sin of x plus c 2 cosine of x right. So, but in fact, what we get in the end is the answer for the inhomogeneous equation is also just something of this kind, but these constants c 1 and c 2 have been elevated to functions alpha of x and beta of x.

And though we use the Green's function technique we have also managed to show that you know it is really the same form, for this the solution for the differential equation and then we have worked out this alpha of x in terms of you know, some other physical arguments have gone into this. So, in some sense the Green's function is a little more, well motivated, but ultimately, we recover the same solution which we have seen in the past.

So, in order to make this connection even more concrete, it is useful to recall the method of variation of parameters.

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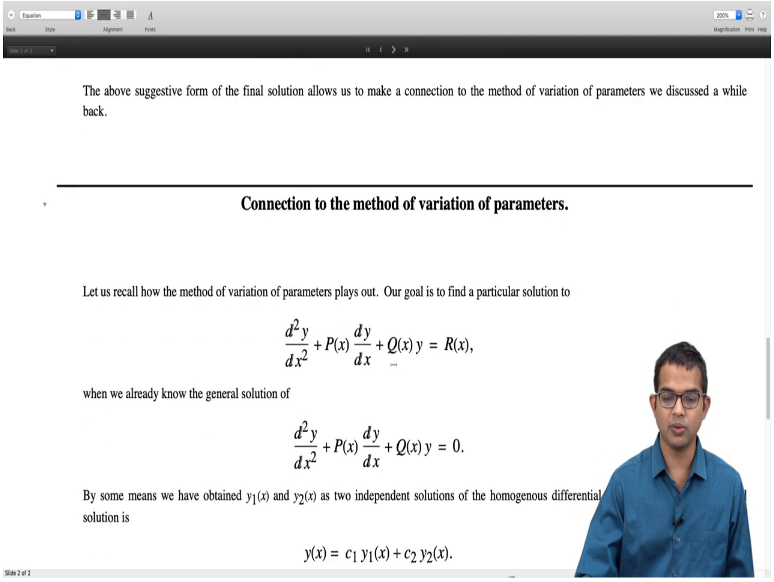

So, we have this connection to the method of variation of parameters. What is the method of variation of parameters? So, we are given some differential equation, let us look at a second order differential equation of this kind d squared y by d x squared plus P of x d y by d x plus Q of x y is equal to R of x right.

So, if we can somehow by some means, we have found the solution for the homogeneous differential equation right, which itself in general is a hard problem by the way right. So, there is no a priori given solution for even the homogeneous differential equation, because you have these P of x's and Q of x's sitting in here right.

But suppose, we assume that we have by some means obtained y 1 and y 2 right. So, then the method of variation of parameters we saw was like a systematic way to work out the solution for the inhomogeneous differential equation.

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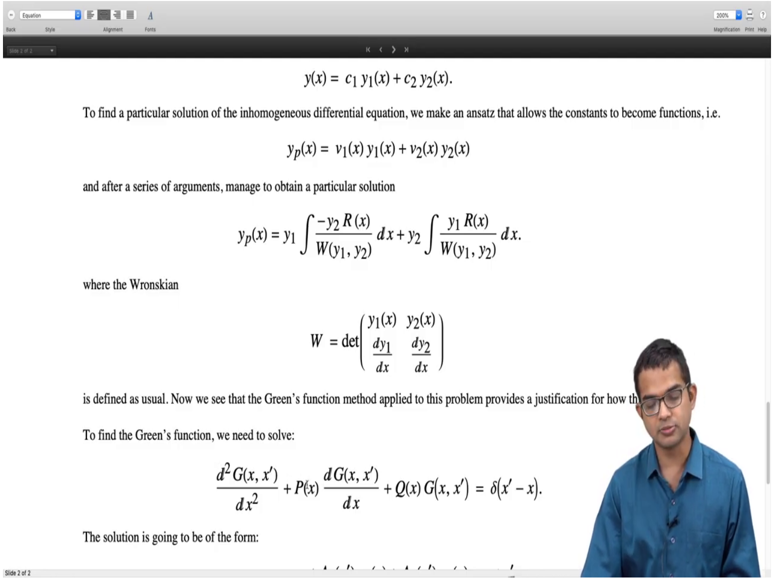

So, what we showed was that you know, you start with this kind of a general solution for the homogeneous differential equation and then you elevate c 1 and c 2 to functions v 1 and v 1 of x and v 2 of x and then we have a string of arguments. And then we managed to show that the final answer is a particular solution - it is given in terms of the Wronskian of these functions.

So, you have a y 1 times this integral involving minus y 2 times R divided by Wronskian d x and then likewise, you get another you know v 2 also has this integral right. So, this is where Wronskian is given by this determinant right. So, this is something that we showed many lectures ago. Now, we will just quickly point out that the Green's function method would also lead to an answer of this kind right.

So, the details we will have to work out right, if you have to give all the details, it is better to do it sort of you know problem by problem by basis rather than keep it completely general, but let us indicate what the general approach is right. So, if we were to do this problem by the Green's function method, we would look for the Green's function of this kind.

So, we have to find d squared by d x squared acting on G of x comma x prime plus P of x times first derivative of G plus Q times G is equal to delta of x prime minus x and the solution is going to be of this form right. So, it is something very similar to the concrete problem we worked out. So, you will have something like.

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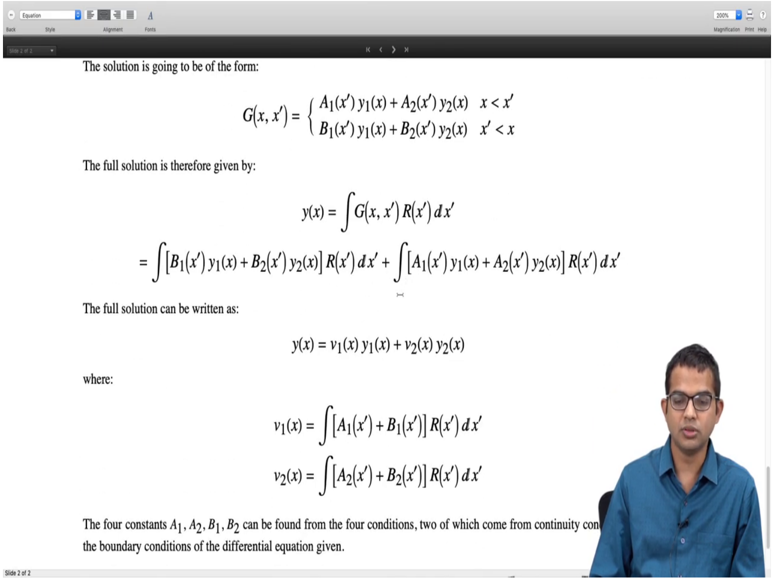

So, now, y 1 and y 2 are functions which are given. So, when x is less than x prime or x is greater than x prime, you will have y 1 of x and y 2 of x as your solutions. Now, it is just that you may have different coefficients associated with them. In our case, we saw that only one of them operated in the left region and another operated in the right region, because of the boundary conditions.

But here, we have not specified any boundary conditions. If you specify boundary conditions, you know maybe these equations will simplify. But in general, you will have some A 1 of x prime y 1 plus A 2 of x prime y 2 for x less than x prime and then there is another B 1 of x prime y 1 plus B 2 of x prime y 2 for x greater than x prime.

And therefore, the full solution is going to be given by y of x is integral so, you have to take the response to the impulse, and then multiply by R of x prime, which is the weight and then integrate right. So, that is the philosophy of the Green's function. And so, in this case, G of x prime again you will have to break it into two parts for you know, this is an integral over x prime so, if x prime is less than x, you will have to do this

B 1 of x prime y 1 plus B 2 of x prime y 2 and then the entire thing has to be multiplied by R of x prime, then you have to do d x prime. So, I am not providing the limits of this integration explicitly, because you know all the boundary conditions have not been given. So, basically you will have an integral of this kind and likewise, we will have another integral involving A

1 of x prime y 1 plus A 2 of x prime times y 2, the whole thing multiplied by R of x prime d x prime.

So, the full solution can be written as you know exactly in the form that was given by you know this method of variation of parameters, we will have y of x is v 1 of x times y 1 plus v 2 of x times y 2 where v 1 will be this integral A 1 of x prime plus B 1 of x prime the whole thing multiplied by R of x prime d x.

So, if you see there was this you know, the final solution also had is R of x sitting here and then there is this stuff involving you know which we showed it to be exactly minus y 2 by W right. So, here, we have you know A 1 of x prime plus B 1 of x prime and so, you know with some work, it is possible to show that these two coefficients will also be you know exactly in this form right.

So, that comes from following these boundary conditions. You have to match the boundary conditions at the point x prime right so that will give us two conditions. So, the two functions you know function to the left of x prime and right of x prime should agree at the point x prime and the derivative will have a jump which can be calculated and it will be given in terms of all this you know this original differential equation.

So, we will not work out all the details and also there will be two more boundary conditions, which come from the ends. So, I am just going to leave it at this and tell you that we have this nice connection between the variation of parameters method, which would seem like a you know heuristic approach, but which you know which works in the end right.

As long as you find a particular solution, which is it does not matter by what means you do, we know that the theory tells us that is the correct solution. So, but it was somewhat of a you know magic trick in some sense, that is variation of parameter method works, but here, this Green's function method is a little more you know well-grounded physically, you think of this you know right-hand side has made up of a bunch of pulses, you find the response to an impulse, and then you stitch together solutions of all of these and then compute the final answer.

But it turns out that the final answer has exactly this form, it is y of x times is equal to v 1 of x times y 1 plus v 2 of x times y 2 of x and then we have seen that in fact, those v 1 and v 2 can also be worked out and you can write it in terms of minus y 2 divided by W times R of x d x integral right. So, here we have R of x prime d x prime is there, but you can also show that this A 1 plus B 1 will be of that precise form which we will not explicitly do ok.

So, that is all for this lecture. Hopefully, we got a you know fairly you know a bird's eye view of what the Green's function method is, but with the aid of these concrete examples, we have also seen not only the theory of it, but we have also seen how to actually, apply it practically to a couple of examples and that is all for this lecture.