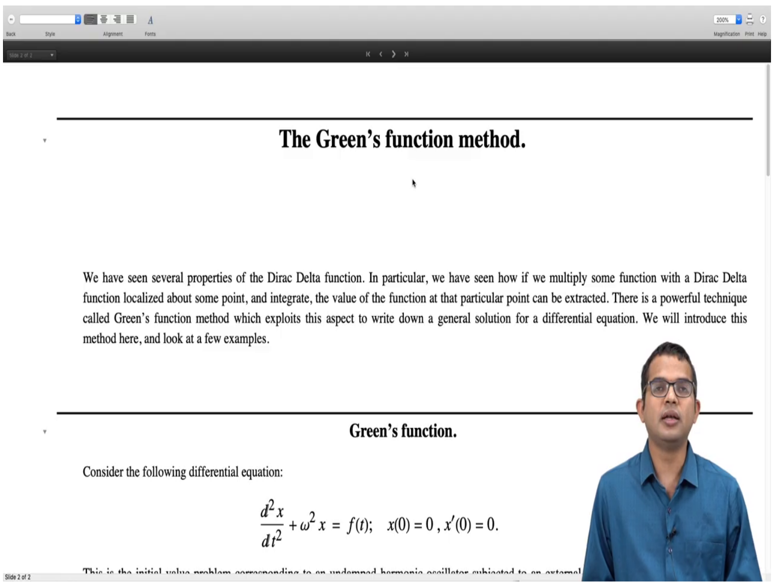
Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Ordinary Differential Equations
Lecture - 85
Ordinary Differential Equations

So, we have looked at many properties of delta functions. And so in this lecture, we will apply some of these properties and to something called the Green's function method right. So, it is a powerful method for solving ordinary differential equations, but it also appears when we are working with partial differential equations in higher dimensional variance of the problem and so on right.

So, varying levels of sophistication are possible. We will look at some very simple applications of the you know Green's function method. And we will use this as a you know method to firm up our understanding or you know how to use delta function – properties of delta functions on the one hand, but also to firm up our understanding of solving ordinary differential equations ok.

(Refer Slide Time: 01:15)



So, the idea is that you know the Dirac delta function can pull out for you the value of a function at a point right. We have seen how if you take a function multiplied by a Dirac function localized at some point and integrate, it can pull out for you the value of the function at that particular point right.

So, now, if you have some forcing function, we have seen that you know there is a homogeneous differential equation, and then there we can look at you know in general in homogeneous differential equations of where there is a forcing function involved. So, the core idea behind the Green's function method is to argue that you know you can think of this forcing function as being constituted of a continuous sequence of impulses right.

So, if you have a function f of t which is your forcing function, you can say that you know this forcing function is made up of lots of delta functions which all are you know because it is a continuous sequence. They get integrated to form f of t. And then we make use of the principle of superposition to add up the solutions of each of these tiny impulses.

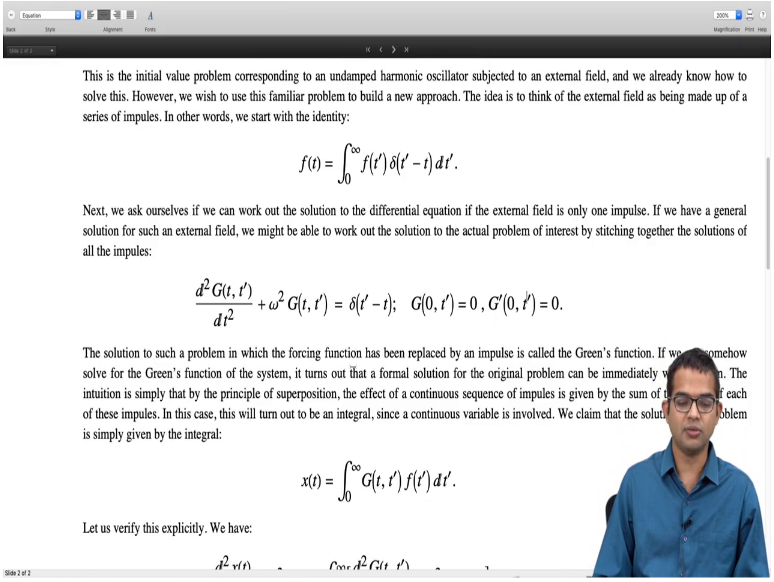
We look at, we find the solution for some impulse and arbitrary impulse, and then we find the solution by adding up the solutions for all of these which will turn out to be an integral. So, it is possible to write down a formal integral solution for you know differential equations.

So, this is best explained with the aid of an example. We will consider a very simple example which we are already familiar with and know how to solve it. So, this is the initial value problem corresponding to the harmonic oscillator which is undamped, but which is forced. So, there is an external field applied to it. So, d square x by dt square plus omega square x is equal to f of t. So, you imagine that it is at rest at time t equal to 0, so x of 0 is 0 x prime of 0 is also 0.

So, the idea here of the Green's function method is to think of this f of t as you know being made up of lots of these delta function impulses, but with weights f of t prime. So, you think of the function as actually a bunch of weights. So, from 0 to infinity all the way up to infinity, you have f of t times delta of t prime minus t dt prime. So, this is like an identity of the delta function.

But when you look at this equation, now we see that we can actually use this to think of you know any function in general as presenting weights. You know f of t gives you t prime has information about weights corresponding to impulses at every point you know where the functions values are defined.

(Refer Slide Time: 04:02)



So, now we exploit the principle of superposition. Suppose we work out the solution not for this original differential equation as it is, but for this differential equation where we have only a delta of t prime minus t sitting on the right hand side. So, instead of having the full function f of t which is a hard problem in general so instead of that we will just put in a delta function there.

So, it is just one of these impulses and argue that if we can find the solution for this, we will call the unknown here as G of t comma t prime because the impulse is applied at the point t prime right. So, the derivative of course, it must be emphasized, is taken with respect to t alone, t prime is your variable which you are introducing right.

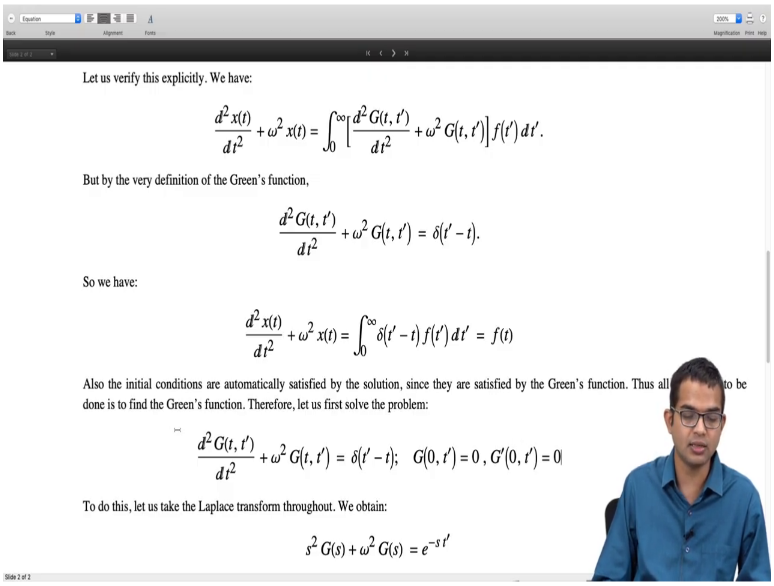
And so then we have this unknown G of t comma t prime and you know these boundary conditions still hold. Now, these boundary conditions will I mean it is perhaps good to you know transfer it to explicitly to G right, so that is what we intend to do which is to say that G of you know at time t equal to 0 t prime t prime is a 0, and G prime of t prime 0 comma t prime is also 0. So, that is the initial value problem that we have.

So, the solution to such a problem if by some means we are able to find such a G which not only satisfies this differential equation, but also these boundary conditions, then this solution is called the Green's function of the particular problem that we are interested in. It is like a response of your function to an impulse which is applied to it. The forcing function is reduced to just an impulse. And we claim that the solution to this problem is simply given by this integral.

So, the idea is that you treat your overall forcing function as being made up of an integral of these impulses, and therefore, the solution must be an integral of the responses which are which is the Green's function right. So, if you put just delta of t prime minus t, you get G of t comma t prime. So, if you put you know you multiply by f of t prime, and then integrate 0 to infinity that is the actual response.

So, that is the input. So, the output also must be just integral from 0 to infinity G of t comma t prime which is times f of t prime. So, instead of delta of t, t prime minus t which is input the output is G of t comma t prime, then you have to do this integral right. So, it seems very reasonable. And in fact, we can check that this works out. The way to verify this is to explicitly just plug this back into the differential equation.

(Refer Slide Time: 06:58)



So, we have to do d squared by dt squared or acting on x of t plus omega square you know acting on x of t which you know in this case will be you know this integral is over t prime right. So, the differentiation on the other hand is over t. So, you have to be careful. And but I mean it's clear that there is no problem with you know bringing in so the differentiating with under the integral sign.

So, you get this d square by dt square plus omega square G of t comma t prime multiplied by f of t prime. But then when we look at this stuff in the brackets, we see that you know this is nothing but delta of t prime minus t because that is how we obtain G we obtain G by solving for this differential equation. So, indeed this stuff inside the brackets can be replaced by a delta function.

But the delta function, when you multiply it by f of t prime and you integrate, so you get f of t right. So, it is a delta of t prime minus t times f of t prime d prime which is nothing but the integral is going to give you f of t right which is what we want after all. And ultimately we want our x of t to satisfy this property that d square by dt square x of t plus omega square x of t is equal to f of t, and we are done right.

And automatically the initial condition x of 0 is going to be 0 because G itself is 0. So, when you integrate this, you know like every one of these coefficients is 0. So, if you sum all these coefficients or integrate this you know the function it also is going to be 0. And likewise the derivative also is going to be 0 at t equal to 0. So, automatically the boundary conditions are satisfied, and therefore, we are done right.

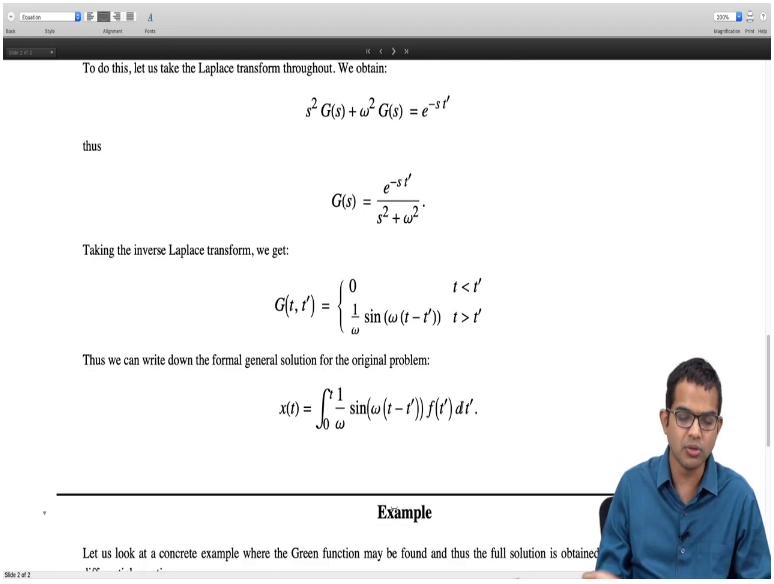
So, the only thing that remains is you know to be able to actually work out this Green's function which let us actually let us do this for this particular problem it is easy to do right. So, we can use Laplace transforms to carry this out. So, we have you know we need to solve for this differential equation with this boundary conditions which is you know it is better to write these boundary conditions as in terms of G, instead of x.

So, we have these boundary conditions. So, we take the Laplace transform throughout. So, the second derivative is going to give us s square. And since the boundary conditions are very convenient for us, you know no other terms come in when you take the second the Laplace transform of the second derivative is simply s square times the Laplace transform of the function itself.

So, which I am calling you know just capital G right maybe you know maybe there is better notation for this or we could have just come called it Laplace transform of G you know this script l or something, but it is ok from the context I hope there will be no confusion.

So, s square times the Laplace transform of the Green's function G of s plus omega square times you know the other again the Laplace transform G of s is equal to Laplace transform of the delta function we have already worked this out, it is e to the minus s t prime.

(Refer Slide Time: 10:19)



So, therefore, you know we can solve for G of s, we can solve for G of s it is e to the minus s t prime divided by s square plus omega square. And then all that needs to be done is to work out the inverse Laplace transform which we also know how to do for something like this.

When you have some function for which the inverse Laplace transform is known 1 over s square plus omega square, we know it is just sin omega t. But then if you want to do for e to the minus s t prime, we know that you have to just multiply by a you know step function.

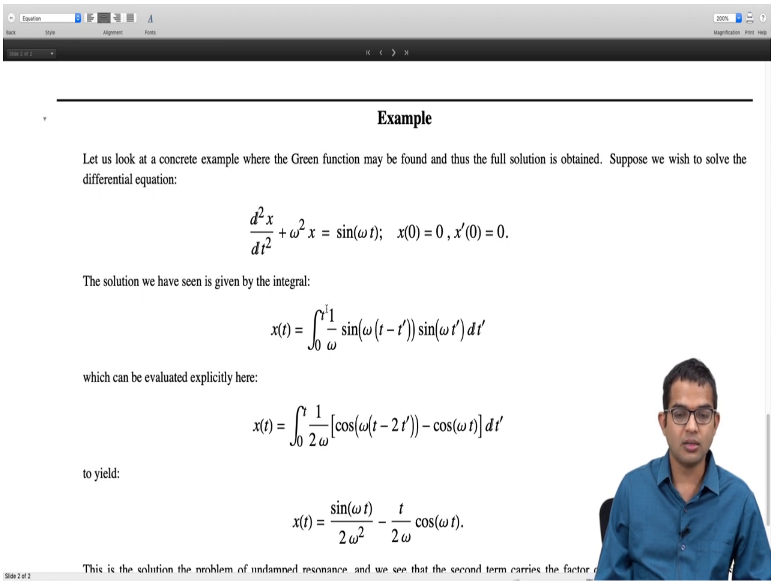
So, in this case, you get 1 over omega times sin omega t minus t prime if t greater than t prime; and it is 0 if t is less than t prime right. So, this is the Green's function. And a moment thought reveals that in fact, this is quite a reasonable answer right, I mean you have you know nothing is happening to your system until t prime up to so it better be 0 the response is 0 because you have not put up the system it is at rest you know starting from initial time t equal to 0 all the way up to t prime.

But at t prime there is some impulse given to it. And after that you know for t greater than t prime, of course, you will get some response, and that it turns out is given by the sin omega of t minus t prime the whole thing divided by omega. So, once we have this using the prescription that we gave, we can immediately write down the solution of the problem as you know this integral.

So, this integral we have seen will go from 0 to infinity. You know where t prime goes from 0 to infinity G of t comma t prime. So, in this case t prime you know t prime has to be if t prime is greater than t it is 0. So, t prime only will go up to t in this case. So, it is 0 to t 1 over omega sin of omega times t minus t prime for the whole thing multiplied by f of t prime d t prime right.

So, there is one concrete example if we work it out we will see this even more explicitly and in fact this is a problem we have or a very similar problem is something we have already solved.

(Refer Slide Time: 12:26)



Let us see what happens when you have a you know an undamped harmonic oscillator, and when you put your external forcing function is sin omega t x of 0 equal to 0 x prime of 0 equal to 0. So, it is exactly like the previous problem. Now, I have provided what this f of t is. If I take it to be sin omega t, then the solution is given by this integral.

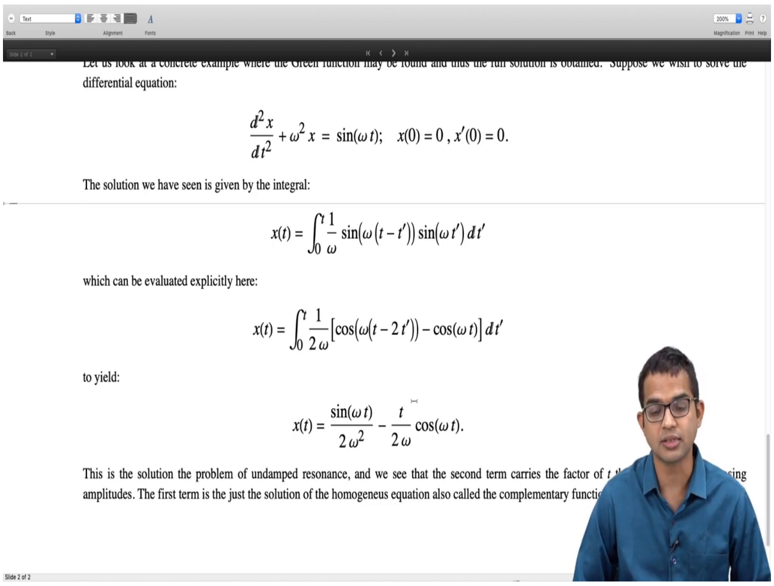
I have to work out this integral 0 to t 1 over omega sin omega times t minus t prime the whole thing multiplied by sin omega t prime dt prime which we can evaluate right. So, this is a trigonometric identity which we will use sin c sin d will be cos of c minus d right so minus cos of c plus d and then the whole thing divided by 2. So, if you work this out.

So, you have cosine of omega t omega times t minus 2 t prime will come in and minus cosine omega t. So, if you carry out this integral, you get two terms. The first of these terms will be sin omega t divided by 2 omega square. And the second one is minus t divided by 2 omega cosine of omega t right.

So, the first one you will recall is you know just the you know it is the natural frequency of this harmonic oscillator sin omega t. So, in fact, the same type of forcing function exists as the natural frequency. So, clearly the system is being driven at resonance. So, this sin of omega t itself cannot be a solution - a suitable onset for the particular solution you must or cosine omega t is not you will have to choose t times sin omega t and t times cosine omega t and then with appropriate coefficients.

And we seem to find that the only coefficient that comes out here is for cosine of omega t right. And so you get minus t over 2 omega cosine of omega t right.

(Refer Slide Time: 14:20)



So, this is the part which you know makes the systems amplitudes arbitrarily large because there is t sitting here, and you know although there is there are oscillations, but there is also going to be you know amplitudes becoming very very large right because your system is being you know operated at resonance right.

So, this is a concrete example of you know the general method which is a very powerful method and finds applications in all kinds of physical applications. You know ODEs, but also

PDEs higher dimensions and some of which perhaps we will also look at. But hopefully this is a you know a fairly simple introduction to a powerful technique the Green's function method. That is all for this lecture.

Thank you.