Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 84
Properties of the Dirac Delta function

So, we have introduced the Dirac delta function and we have seen some of its fundamental properties. So, in this lecture, we will go further and look at many useful properties of Dirac Delta function and you know learn to work with it, right.

So, a full-fledged study of Dirac delta function would require looking into the notion of generalized functions and you know there is quite some theory with it, but our philosophy will be to learn to work with Dirac delta function. And, in this you know approach we will look at many examples where how you know we learn how to play with it and you know develop a facility ok.

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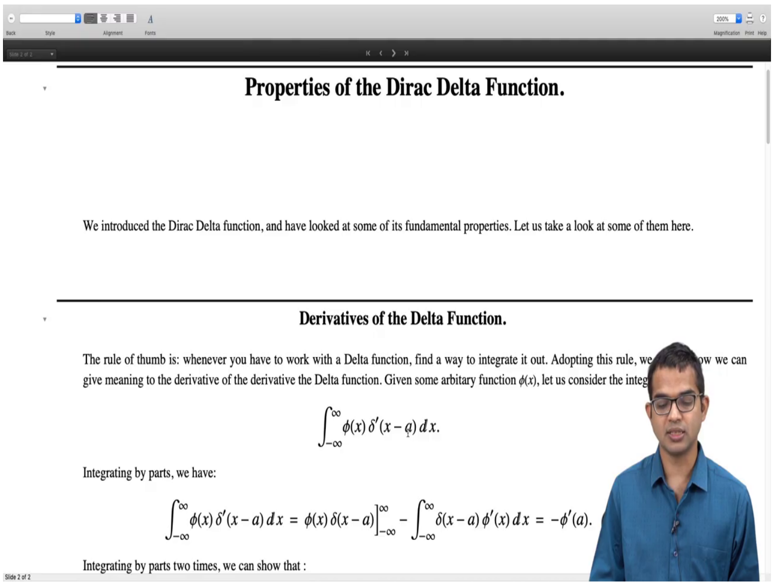

So, one is you know a useful property of the Dirac delta function is you know when you are working with derivatives of the delta function. So, we have seen that you know if there is a delta function the best thing you can do is to try and integrate it out, right as such it is a, it is a strange beast to just work with the function itself I mean technically speaking is perhaps not even a function, right.

But, I mean we understand it as an object which becomes arbitrary large at one point right it is this impulse kind of a thing and it is 0 everywhere else and you know the best you can do is you know integrate it out and that is true also with functions of you know the delta function like if you take the derivative of this function what does it do, right.

So, in order to understand this we look at what it does when it is integrated out. So, consider some test function you know phi of x and you know integrate with you know after multiplying it with this derivative. So, you have a delta function which is peaked about a point a. So, you take the derivative delta prime we define it like this and then let us see what happens if you multiply it by some test function and integrate.

And, so, we integrate by parts. So, you have u dv is given by uv minus v du that is the rule right by which integration by parts happens. So, you have u dv. So, uv will be just phi of x times delta of x minus a and minus infinity to plus infinity.

So, in these kinds of situations you take the test function to be such that it dies down at both plus infinity and minus infinity right. So, and so you know the value of this function at both plus infinity and minus infinity delta function is just 0.

So, this boundary term as it is called is 0 and then you are left with minus phi prime of a, right. So, you know there is just a delta function and the derivative has shifted to the test function and then you get minus phi prime of a.

So, we see that you know the derivative of the delta function is, you know, it is as if it will take a function and you know it can point out the value of the derivative of your function at up at a given point right. So, we can also go ahead and try to see what happens if you take the second derivative.

So, if you do phi of x del prime of x we can show that it will be pull out for you in fact, phi double prime of a, right. So, the delta function will pull out for you the value of a function at a point. The first derivative of delta function seems to pull out the value of the derivative of the function at a particular point, but there is a minus sign which comes in as well, second derivative the minus sign goes.

And, so in fact, you can show by successive integration by parts this general result that the nth derivative of the delta function you know can pull out for you the nth derivative of the function itself at that point, but there is also this factor minus 1 to the power n which you know gets tagged along, right. So, this is the result.

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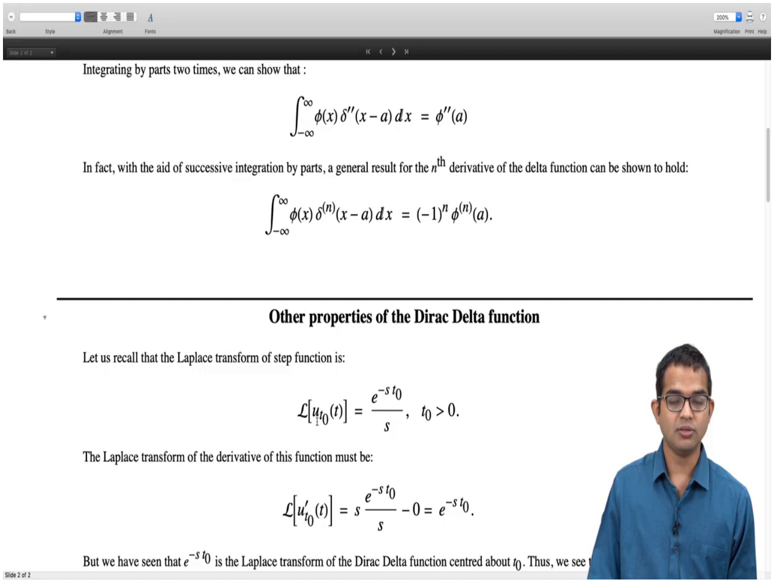

Now, there are other properties of the Dirac delta function which are useful. So, let us look at you know you know one such thing, but approach it from the point of view of the Laplace transform. We have seen the Laplace transform of a step function.

So, the Laplace transform of a step function is you know is given by e to the minus st naught divided by s, right. So, the step function you remember is something which is 0 all the way up to t naught and then it becomes one beyond that.

Now, we remember that you know the step function has this Laplace transform ah. So, the Laplace transform we also saw that the property of Laplace transform is that the Laplace transform of the derivative of a function you know is just going to be multiple or you know you take the Laplace transform of the function itself and multiply by s and you have to subtract the value of the function at 0 which in this case is just 0.

So, you get e to the minus st naught, right. So, we see that you know property from the properties of Laplace transforms the Laplace transform of the derivative of this function step function will be just e to the power minus st naught.

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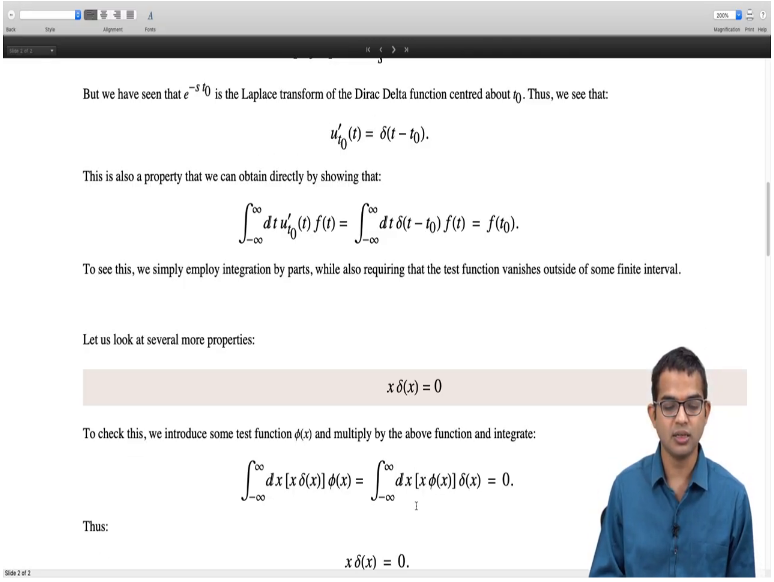

But, we have seen that in fact, e to the minus st naught is the Laplace transform of the Dirac delta function centered about t naught. So, thus you know we argue that in fact, the derivative of the step function is the delta function, right. So, this is in fact, one way in which one could have defined the delta function as the derivative of this you know there is a step function.

So, for all values t less than t naught of course, the function the derivative is 0 and for all values t greater than t naught the derivative is 0. But, exactly at that point t naught there is a jump whose magnitude is equal to 1 and so, that is why it is the Dirac delta function, right. So, which is also confirmed from this Laplace transform idea.

But, you could also directly work this out using you know integrating out this function. So, you take some test function, integrate minus infinity to plus infinity. Well, I mean in this case I have restricted it to be you know time going from 0 to you know time being positive and so on. But, in general you can have a step function which is also negative, right.

So, you can define maybe instead of t perhaps it is more you know standard to call the variable x in such a scenario, but the point is that you can have a you know delta function which is isolated at any point and you would have to have the step which is also you know corresponding to that point and you take the derivative of it. So, if you do a derivative like this with some test function f of t, so, you get you know you can show that this is the same as f of t naught, right.

So, to see this you have to just employ integration by parts and also require that the test function, of course, vanishes outside some finite interval. But, in any case it is intuitive that you know if you have a step function and if you take a derivative you know there is mess which is isolated at that one point and that mess is in fact, the same as the Dirac delta function which one can show you know with more sophisticated arguments, but that is what it is.

Let us look at some more properties, right. So, it is kind of a weird object, but you know we have to learn to work with it. So, if you take the delta function and multiply it by x right it turns out that it will just go to 0, right. So, you have a mess sitting at the origin and then you multiply it by you know x which is which goes to 0. So, where is the guarantee that if you multiply infinity by 0 why should it go to 0, right.

So, you know the argument for this is to take a test function and multiply and see what happens and integrate. So, we introduce this test function phi of x you know multiply x times delta of x times phi of x integrate from minus infinity to plus infinity, then we see that you can think of this as you know you can pull out this delta function to the right side and say that ok we are looking at you know the function x times phi of x.

So, what does this integral do? Right, we know that the job of this delta function is to just pick out the value of this function at x equal to 0, but the value of the function at x equal to 0 is just 0 because you have x sitting here. So, indeed x times delta of x is nothing but 0, right.

So, one way to show identities involving generalized functions is you know what does the generalized function do? You have some you know some stuff on the left-hand side on, some other stuff on the right-hand side if we have to establish their equality.

Then we must show that for any test function you know an arbitrary test function if you multiply with it and integrate from minus infinity to infinity, the value obtained by each of the terms one on the left-hand side on the other on the right-hand side must be the same. So, which in this case is true, right. So, it is the same as if you had multiplied with 0 and integrate it. So, therefore, x times delta x is equal to 0.

You could also try to prove this you know using your explicit construction of your delta function. You start with the Gaussian whose width keeps on shrinking and you know you multiply by x, evaluate this integral and then take the limit of this integral and then also you will see that this is indeed 0.

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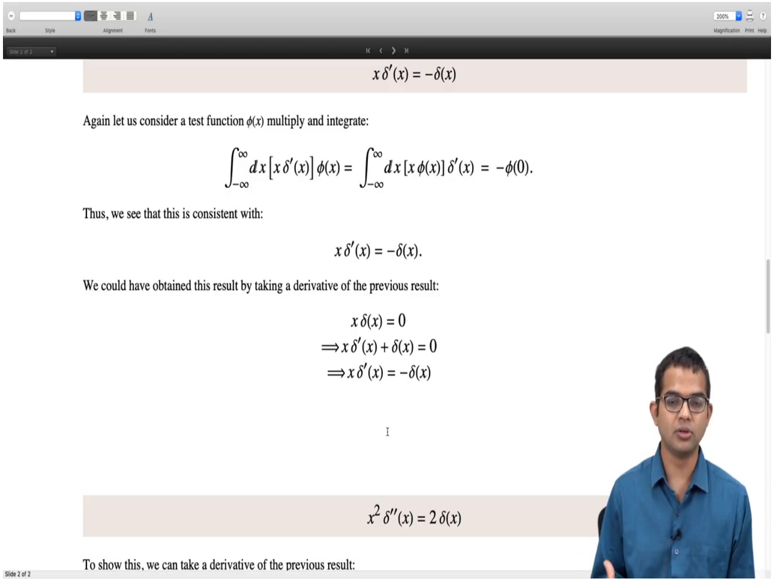

But, when you multiply delta prime of x with x it turns out that you will get a minus delta of x, right. How do we see this again? Take a test function, multiply and integrate. So, if we do this x times delta prime of x times phi of x which then we write it as x times phi of x delta prime of x, but we have seen that delta prime of x will you know extract for you the value of the derivative of this function at x equal to 0.

But, what is the derivative of this function at x equal to 0? right. So, you have on the one hand you know with a minus sign minus phi of x and you also have x times phi prime of x, but x times phi prime of x will just go to 0 because x is there right. So, you are left with only minus phi of 0, right.

So, this we say you see that this is consistent with the claim that x times delta prime of x is equal to minus delta of x, right. Once again this also could have been worked out using some you know like sequence of functions which are all converging to the delta function and then you take the derivative you know there are other ways of seeing this.

But this is a standard way is to take a test function integrate and check that you know both left-hand side and right-hand side, yield the same answer regardless of what the test function is right subject to certain you know conditions we generally assume for the test function like it dies off to 0 for at plus infinity and minus infinity.

We could have obtained this result by taking a derivative of the previous result also right. So, we could have directly got this. So, we have x times delta of x is equal to 0, right. So, therefore, x times if you take a derivative you have x delta prime of x plus the derivative of x is just one times delta x equal to 0. Therefore, x delta prime of x is equal to minus delta of x. So, this is another way of seeing the same result.

Then we have you know we are looking at a bunch of results. It is useful to get practice with you know working with these kinds of identities, x square double derivative of x it turns out to just 2 times delta of x, right. I mean I guess one can make a guess. You know that you know if you take x and multiply with the first derivative, it is going to have some kind of a delta function with it.

But likewise, if you take x squared and multiply by del delta double prime of x you are going to get again some kind of a delta. You have to work out the constants involved, right. Likewise, you can play, you can come up with your own functions, you can take the next order derivative, multiply by x cube and so on and check for yourself that you know if there is a delta function which comes and so on.

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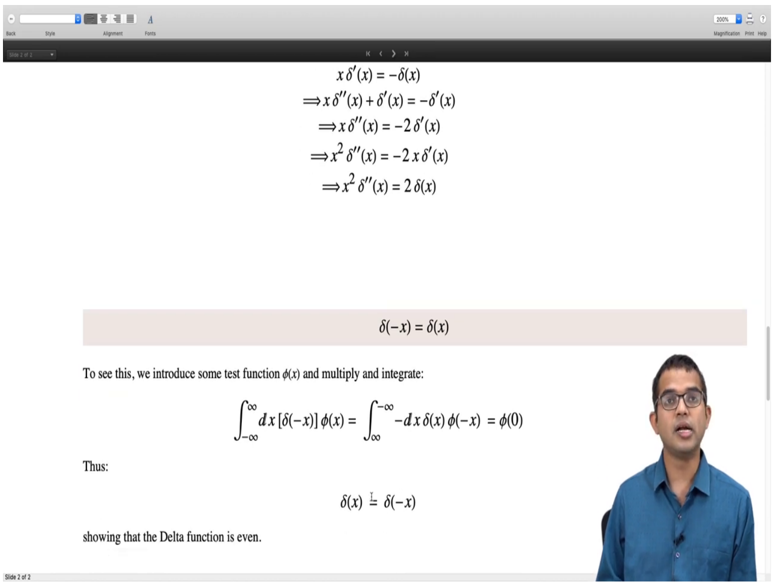

So, if you take a derivative of the previous result. So, you have x times delta prime of x is equal to minus delta of x taking derivatives on both sides we see x times delta double prime of x plus delta prime of x is minus delta prime of x. So, this delta prime of x and this minus delta prime of x if you couple them together you get a minus 2 delta prime of x on the right-hand side.

So, you have x delta double prime of x is minus 2 delta prime of x. So, x squared delta double prime of x is minus 2x delta prime of x, but we have already seen that x times delta prime of x is the same as delta of x with a minus sign. So, therefore, we get the result that x squared delta double prime of x is 2 delta x. You can continue like this.

You can take one more derivative and so on or you can also work it out from you know first principles which means you multiply by some test function and integrate throughout and see whether it holds. Also, I guess it is obvious that this function is delta of x is even, but it is also something that we can check with the help of a test function, you multiply delta of a minus x times phi of x.

And, then you make a change of variable. You send x to minus x. So, limits of integration go from plus infinity to minus infinity, then you get a minus dx then delta of x times phi of minus x. So, you have this minus sign which cancels with this you know the order of integral limits being changed.

So, if I change them back, then this minus sign goes away and then this is a standard integral involving a delta function and phi of minus x. So, I just get phi of 0 as the answer. So, delta of x is the same as delta of minus x showing that the delta function is even.

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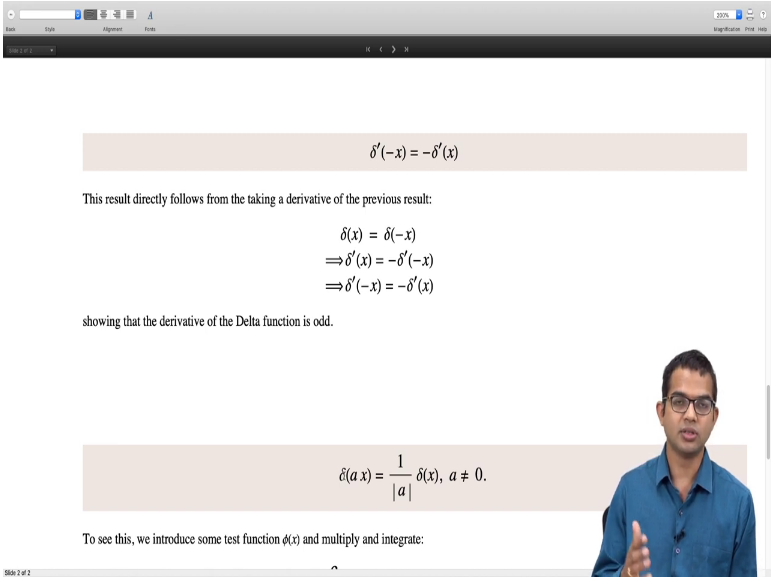

And, it turns out that the first derivative of the delta function is an odd function. So, delta prime of minus x is minus delta prime of x. So, once again you can either show it by first principles or we can use the previous result. So, we have delta of x is delta of minus x. So, delta prime of x, if you take derivatives on both sides so, you have minus delta prime of minus x in other words delta prime of minus x is minus delta of x, right.

So, the first derivative of the delta function is odd. You can also think of these kinds of problems as you know you know you start with a sequence of functions which will approach the delta function in the limit, take their derivatives and check that indeed that will also give you know this type of property.

And, now we come to a very important property which is that if you take the delta function of some constant times x, this it is the same as one over modulus of that constant times delta of x as long as the constant is not 0. If the constraint is 0, then I mean it is not really a function at all. We just have delta of 0 right and then we know that it blows up at that point.

So, eh when a is nonzero, if you are scaling your variable then your delta function itself also gets scaled by a factor which is one over modulus of a, alright.

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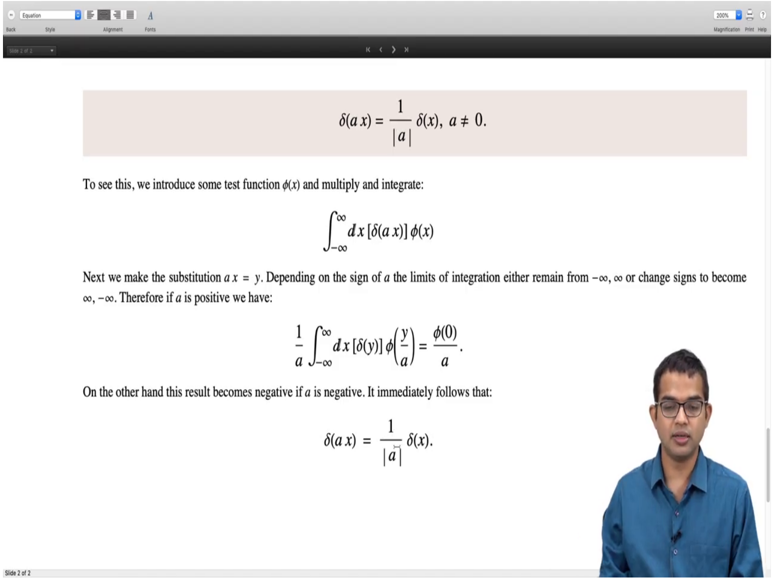

How do we see this? Get a test function and multiply and integrate. So, we have minus infinity to plus infinity dx delta of ax times phi of x. Now, we make a substitution ax is equal to y depending on the sign of a the limits of integration may remain minus infinity to plus infinity or they may change to infinity to minus infinity.

Now, therefore, if a is positive, then we will just get 1 over a in minus infinity to plus infinity dx delta by phi of y by a and then we have to just put in y equal to 0 ah. So, which will give us you know 1 1 over a times it should be phi of phi of phi of a phi of 0.

So, it is so, it is phi of 0 by a and here we would get it would be the negative of this if you put if a were negative right because you will get minus 1 over a, if you want to keep the same limits and then you will get the negative value if a were negative.

Therefore, a compact way of saying this is delta of ax should be equal to 1 over the modulus of a delta of x, correct. So, this is a very important result and in fact, the next result is very closely related to this result, right.

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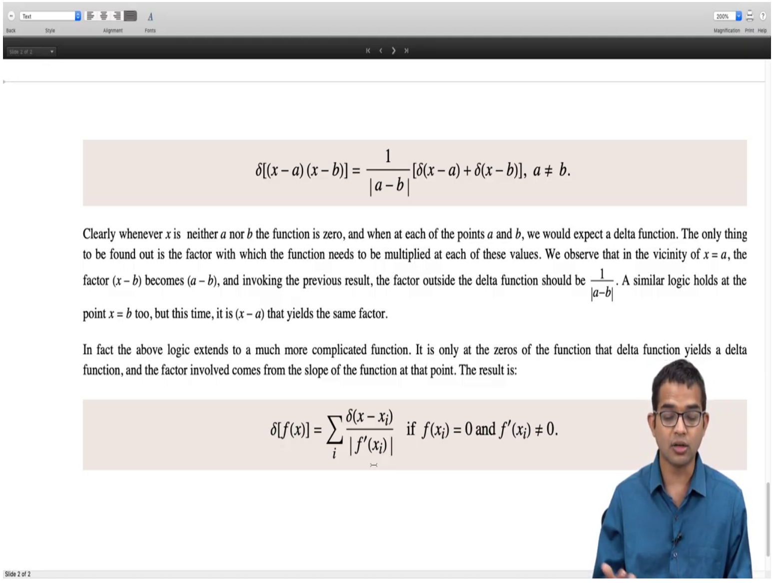

So, if you have some function which is sitting in here; so, in this case I am considering a quadratic function. So, if you think of this logically, right. So, you have delta of a function and basically this is going to be 0 for all values of you know x which are not you know roots of the function sitting in here because when the value of the function is 0, we know that there is going to be mess we have to find out what; that means, is.

But, and when does that happen that happens when x equal a or x equal to b, right. So, it is you know reasonable for us to expect that this function is the same as some constant c 1 times delta of x minus a plus some other constant c 2 times delta of x minus b and then we have to invoke some properties to work out that constant it turns out that that constant is actually nothing, but 1 over mod of a minus b right.

How do we argue for this? So, let us think of this as you know suppose you are very close to the point a, you are in the vicinity of a. So, you see that the mess is coming only from you know this term; whereas, the other one is going to take the value a minus b. You are very close to a, so, for all practical purposes this is just the constant in that vicinity and whose value is a minus b.

And, we have just seen that you know when you have a constant multiplying a delta function at that point that constant will just come out with the modulus sign. So, you have one over mod of a minus b. And, likewise when you are at b the second part you know x minus b is the one which is, you know, where the mess is and then the other one is the factor b minus a.

So, you just pull out that factor and you know it comes in the denominator as we have seen and, but you also have to take the modulus, right. So, it is basically the same argument as the previous problem, but you have to be careful now. In fact, there are two points at which this delta function exists.

So, if you have a quadratic function in there you know in general there are two roots and each of these roots are going to give you a delta function sitting there and you can evaluate what the coefficients are. And, using the argument which I have just given and that follows from the previous problem. So, go over this argument carefully and understand it well.

And, once we have understood this well in fact, this extends to some arbitrary function you do not have to look at just you know a function of this kind - a quadratic function, but an arbitrary function with lots of roots. So, as long as you know the slope at the root is not 0, this is valid right.

If the slope at that point is 0, then you know you cannot, you know it is going to be a mess it is not you cannot just divide by 0, so, that will not work out. But, if you have a function whose roots are such that f prime of x i is nonzero, then you will get this is the result that holds delta of f of x is summation over i delta of x minus x i, right.

So, it is clear that whenever you are at any of these 0s you are going to get a delta function. So, you have a sum of all these delta functions and then the coefficients they have to be worked out. And, the coefficients are in the vicinity of this function of this 0, the function is going to be like f of x is equal to delta you know x minus x i times you know f prime of x i right.

So, this is something like a Taylor expansion of your function at that point and then you have to. I mean then we invoke the result from the previous you know this result. So, you get a modulus 1 over modulus of f prime of x i at that point, right. So, this is a general result.

And, you know these kinds of properties are useful and you know they appear in many contexts some applications we will see and in the applications particularly applied to differential equations, right. So, there is a technique called you know you know the green function approach which we will discuss.

But the point is that a facility or a familiarity with all these properties of delta functions is of great importance and they appear in all kinds of context electrodynamics for example, you know other fields also have these kinds of delta functions you know.

Whenever you are modeling any kind of impulse behavior delta functions appear. So, these properties must be well understood and you know with practice we will know how to work with them right. That is all for this lecture.

Thank you.