Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 83

The Dirac Delta Function

So, we have seen how using Laplace transforms, we can solve for you know a fairly broad class of differential equations in a very clever and you know elegant systematic way right. So, you know typically these kinds of differential equations had some forcing term on the right hand side.

We had like a sin of omega t or you know cosine omega t, sometimes a square wave may be in there. But there are also certain kinds of forcing terms which are very impulsive in nature. They operate for a very, very tiny amount of time, and they have a certain you know impulsive effect associated with them. And these two are of great importance in physical applications right.

So, these kinds of scenarios are modeled with the help of what is called a Dirac Delta function. So, the subject of this lecture is to discuss the Dirac Delta Function and look at some of its properties.

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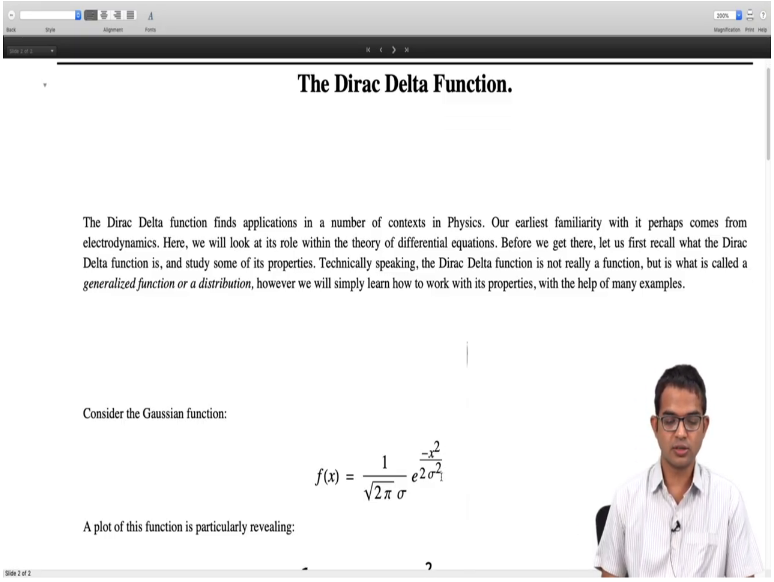

So, the Dirac Delta function is you know technically speaking you know not quite a function, it is you know thought of more like a generalized it is a distribution right; it is thought of more like a distribution or sometimes it is called a generalized function. So, you know our approach here is to learn how to work with it, something that perhaps what we have already seen chiefly in the context of electrodynamics right.

So, but it is a topic which is somewhat confusing until we have learnt to use it, you know with some practice it is something that we can work with, and it is somewhat subtle. So, let us look at intuitively what a Dirac Delta function is, and more importantly what it does. So, the way our approach will be to learn to work with the Dirac Delta function. So, it is useful to start with a Gaussian function right.

So, this is something that we are all familiar with. So, f of x is equal to 1 over square root of 2 pi sigma e to the minus x squared divided by 2 sigma squared.

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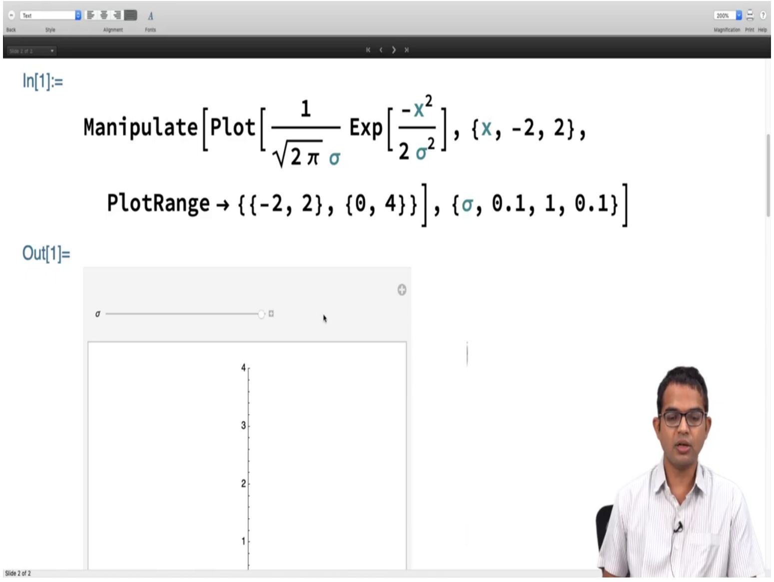

And let us plot this function. If I plot this function, I have this parameter sigma. And so if we, if I plot it, I start with you know let us say I start with some very high value of sigma.

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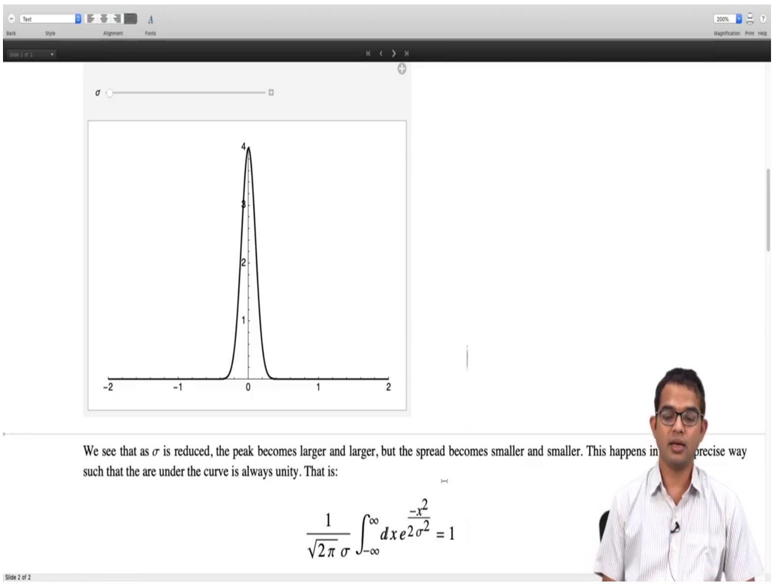

So, I find that this function is well spread out and its peak is small, and you know it has it spans over a you know large region where it you know in the scale that we are looking at, it has its it has a significant value right. But as I keep on decreasing sigma right, so here I am going to decrease sigma, so then you will see that the peak goes higher and higher and higher, and its width begins to shrink.

So, and in fact, when I reduce sigma to a very small value right, its peak has become very large and its width is very tiny right. So, in fact, sigma itself is a measure of its width right, so as you might know from a discussion of Gaussian functions.

But, what is interesting is, what happens to this function? As you keep on shrinking it further and further and further, you take the limit of sigma going to 0 right, so then this function becomes a weird object. It is what is called Dirac Delta function right.

So, you know one thing which is very interesting here is no matter what value of sigma you choose, we have defined our function in such a way that the area under the curve or this integral you know minus infinity to plus infinity this function dx is always unity right.

So, what you know, whatever area is in there is unity, it may be squished into a very, very tiny region or it may be spread out over a broader region, but the total area is the same. So, even when you take this you know the limit of sigma becomes very small to go into 0, even then it is still supposed to admit an area of unity.

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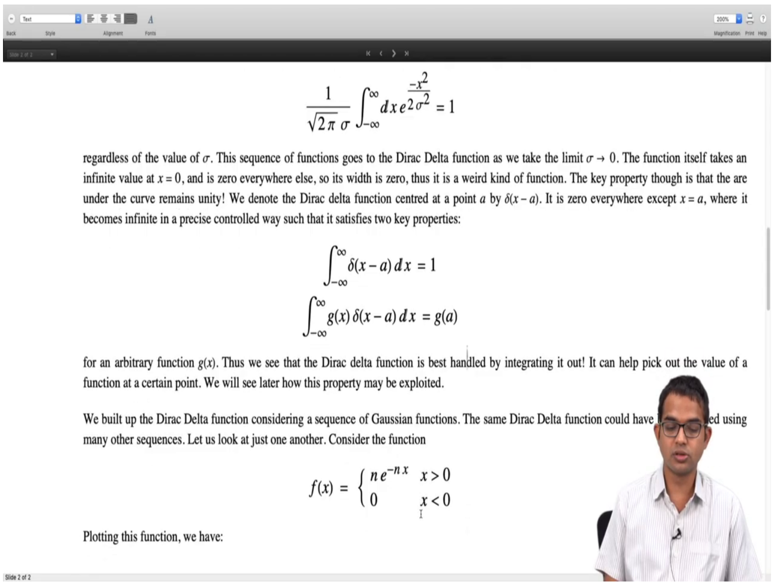

And so the way the function manages to do this is by becoming infinity you know at exactly x equals 0, but it is 0 everywhere else. It is a weird kind of function, so that is why you know mathematicians rather do not call it a function, because it takes a value of infinity at one point and it is 0 everywhere else.

But the key point is that the area under this curve is also guaranteed to be unity. And so there are two key properties which are of vital importance right. So, both of these are integral properties, one is like what we just said if you find the area under this curve you know we call this function as delta of x minus a right.

Physicists, loosely call it you know a function, but it is you know these ideas go back to Dirac. And so it is called the Dirac Delta function. And minus infinity to plus infinity if you integrate this function, you know in the entire span from minus infinity to plus infinity you get unity.

So, and this is true in fact if you take even a tiny you know range as long as the point a is also included. You could go from minus 1 to plus 1, for example, or you know even a very very small region as long as the point x equal to a is included in this case right. So, we, I mean we started with a discussion of this Gaussian being peaked about the origin, but this could be made to peak around any point a.

So, this is the idea of a delta function which is infinity at one point and 0 everywhere else, and which integrates to 1. Also if you take any function g of x and multiply by this function delta of x minus a and integrate, so this will peak out for you the value of this function at precisely that point, and it basically kills all the information about this function everywhere else, but it will give you just the information about just one point.

So, you can also think of the delta function as you know something which when integrated out, will extract for you the value of a function at just one point right. So, this is a very useful you know way of constructing a function: you can think of a function being made up of values at every point.

So, you can think of putting a delta function at any given point and pulling out the information. And then you can have a train of such delta functions and pull out values at various points and so on right. So, these are techniques which will probably return at a later time.

So, the key point here is that you know Dirac Delta function has these properties, these are fundamental properties. And operationally the best thing to do with a Dirac Delta function is to integrate it right. So, whenever you have a Dirac Delta function in any application, always look for a way to integrate it out or multiply with some function and integrate it out, so that you can use these properties that you know the safest way to treat a delta function is to integrate it right.

So, we built up this Dirac Delta function using a sequence of Gaussian functions. But there are many different ways of building the Dirac Delta function, and they all give you the same right. For example, we could have instead of having ah you know Gaussians, we could have had rectangles whose width keeps on squishing, but whose height increases you know such that the product of the width times the height is always unity right.

So, that the area is maintained, and then you can imagine keeping on squishing such that you take the limit and that is the delta function that is also the same delta function, or you could have started with a triangle right or some other shape which you can squish such that the areas held constant.

So, let us look at just one another example of you know a sequence of functions which gives you a you know the same Dirac Delta function. So, let us look at a function like this f of x is equal n times e to the minus n x which is defined for x greater than 0, but it is 0 for x less than 0, this also will go to the Dirac Delta function as you make n very large.

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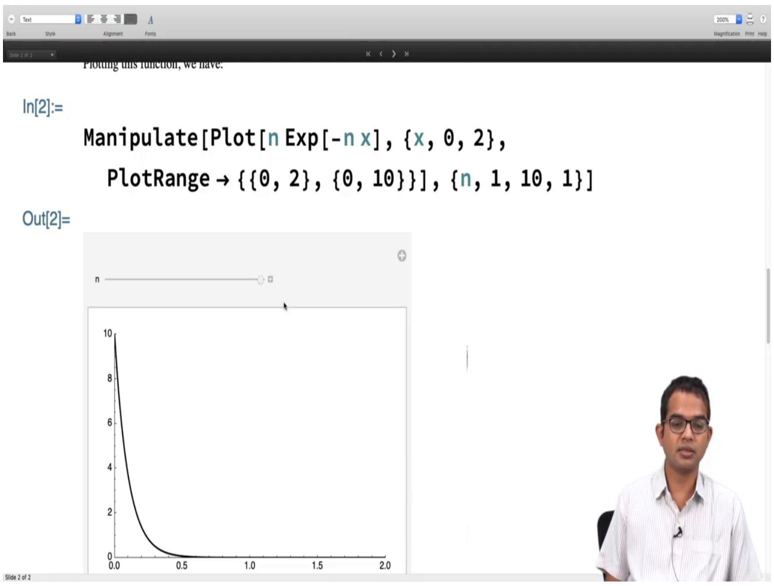

So, let us plot it out. So, if I have n if n is a very small value, then it is a you know this function of course, it is exponentially decaying, but it is you know it is it has a large span all the way up to 2 it has not really vanished. But as you increase, and you see that its peak becomes larger and larger. And for all practical purposes, it is 0 for you know larger values of x, you know x equal to 1 already you see that hardly it has any representation.

And then you see that as you increase and further and further and further, its value at x equals to 0 is becoming larger and larger. And in the limit and going to infinity you know one can show that this function also becomes the Dirac Delta function; meaning that it is going to be infinite at x equals 0, and 0 everywhere else right.

So, this is just to illustrate that you know one can think of the Dirac Delta function in many different ways. Sometimes it is more convenient to use a different representation of the Dirac Delta function when we are working with you know doing algebra of which involves Dirac Delta functions. Often we may have to work with a sequence you know, find out the limit, and then take the appropriate limit and calculate certain properties and so on.

So, in this discussion, let us look at how one can find the Laplace transform of the Dirac Delta function right so where we directly exploit these key properties. There are other properties which we should discuss, perhaps in a later discussion, but here we will try to explore just these integral properties to work out the Laplace transform of the Dirac Delta function. And we will make use of that to solve certain you know differential equations.

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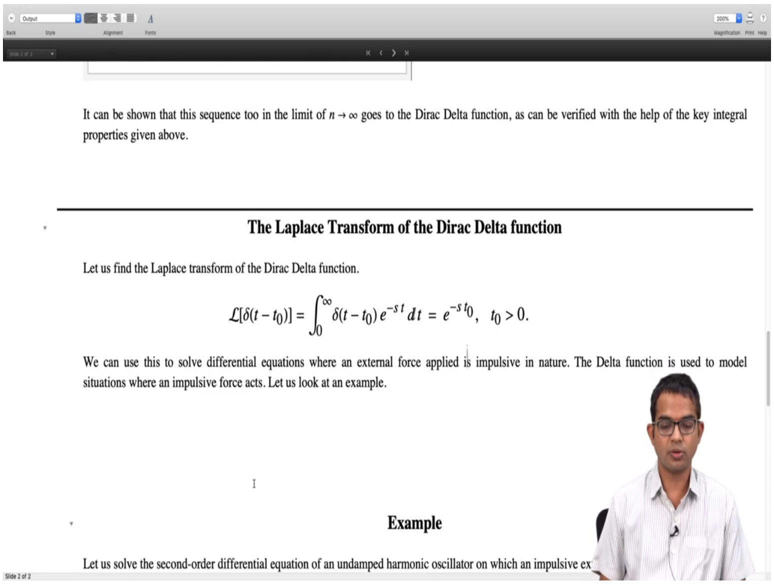

Suppose, you have an impulse function right. So, you know, oftentimes you think of you know turning on some I do not know some voltage, for example, in some electrical circuit or you know you are applying a force which operates for an infinitesimally small amount of time it provides an impulse, and then it is gone right. So, that is modeled with the Dirac Delta function.

So, if I take the Laplace transform of delta of t minus t naught, the Dirac Delta function is localized at t naught. So, I have to do integral 0 to infinity delta of t t minus t naught e to the minus s t dt, and then this is something we know how to do right this is a property of the Dirac Delta function is if you multiply you know by any function it is going to peak out the value of the function at that point provided that t naught lie is greater than 0.

It is within this integral right. So, the limits must cover this point; otherwise, it will be 0 right. So, we get e to the minus s t naught. So, by the way this is a property which we can in fact derive right. So, you can start with some function g of x. And then in place of delta of x minus a, you can plug in this value you know this function do the integral, and then take the limit, and then you can show that indeed this property holds right.

So, but for our purposes we will just accept this as a property right, and maybe that will be that can be homework you can explicitly check that this holds. And this will hold you know regardless of which sequence of functions that becomes the Dirac Delta function you use – whether you use triangles you know rectangles, or if you use you know these exponential curves, or if you use Gaussians, it does not matter ok.

So, the main message here is the Laplace transform of this Dirac Delta function delta of t minus t naught is simply e to the minus s t naught. So, it is an exponential decay in this s. So, this holds only if t naught is greater than 0.

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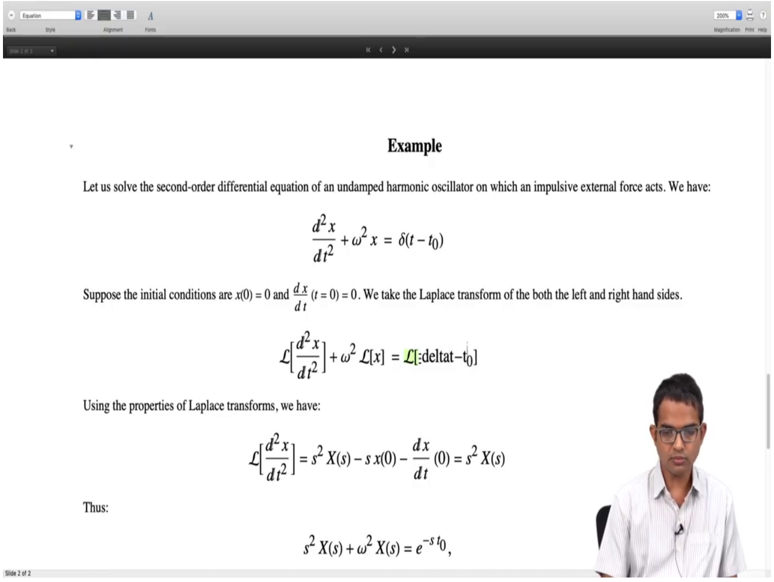

Now, let us look at what happens if you know this was the external force that applies to some harmonic oscillator-like problem, say. Suppose we consider the undamped harmonic oscillator d squared x by dt squared plus omega squared x, and you apply this impulsive external force at some time t naught which is greater than 0 right.

And we are also given the initial conditions - it is at rest at time t equal to 0 right. So, there is no speed associated with it. So, we take the Laplace transform throughout. And we know that the Laplace transform of the second order derivative is given by s squared times X of s minus s times x of 0 minus dx by dt at time t equal to 0. Both the last two terms are 0 because of the initial conditions here. So, we simply get s squared X of s.

And so the Laplace transform gives us an algebraic equation in this X of s. So, I have s squared X of s plus omega squared X of s is equal to Laplace transform of so this should be a delta function. So, I should say delta. So, I have a delta function on the right hand side.

So, this is something that I can immediately solve for and I have the answer X of s is e to the minus s t naught divided by s squared plus omega squared right. So, this is a function for which I know how to compute the inverse Laplace transform. So, we have seen that you know if you have a factor like e to the minus s t naught, it is like doing a shift in time domain right.

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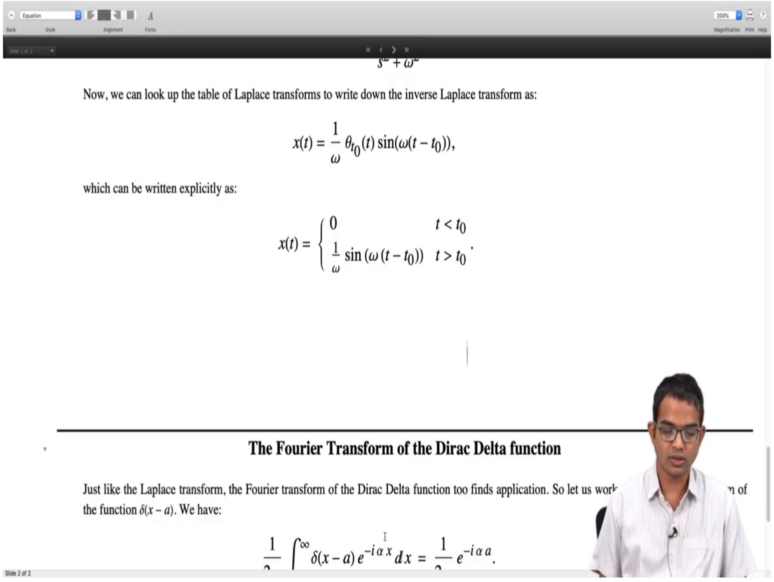

So, it is a, so it gives us this theta t t naught of t and this function itself is 1 over omega inverse Laplace transform of 1 over s squared plus omega squared is just simply 1 over omega times sin of you know because of the shift involved we have omega of t minus t naught, and which can be written more explicitly as x of t is 0 if t is less than t naught. And it is 1 over omega times sin of omega times t minus t naught if t is greater than t naught right.

So, basically your system is at rest, there is nothing happening to it. Of course, it is physically reasonable that x of t will remain 0 all the way up to t naught. When there is this impulse and the moment this impulse has acted and stopped, it is going to you know get these oscillations in play, and they will last right that is what is going on right. So, physically it is very reasonable what is given by this solution.

So, before we end we will finally look at just one more you know topic which is that the Dirac Delta function and the Fourier transform of the Dirac Delta function is also useful. Just like we looked at the Laplace transform of the Dirac Delta function, it is also useful to study the Fourier transform of the Dirac Delta function. It finds application in all kinds of contexts.

So, if you do the Fourier transform, so in our convention we have a 1 over 2 pi times integral minus infinity to plus infinity delta of x minus a times e to the minus i alpha x dx right. So, we have seen that this is supposed to peak out just the value of this function e to the minus I alpha x at x equal to a. So, we simply get 1 over 2 pi times e to the minus i alpha a.

Now formally we can use this since this is the Fourier transform of the function delta of x minus a. So, delta of x minus a is the inverse Fourier transform of this function one over 2 pi times e to the power minus alpha. So, formally we have this inverse Fourier transform relation.

So, delta of x minus a must be equal to 1 over 2 pi integral minus infinity to plus infinity e to the i you know e to the i alpha times x minus a right. So, we have this one of these factors comes from just doing the inverse Fourier transform, and other is this function itself.

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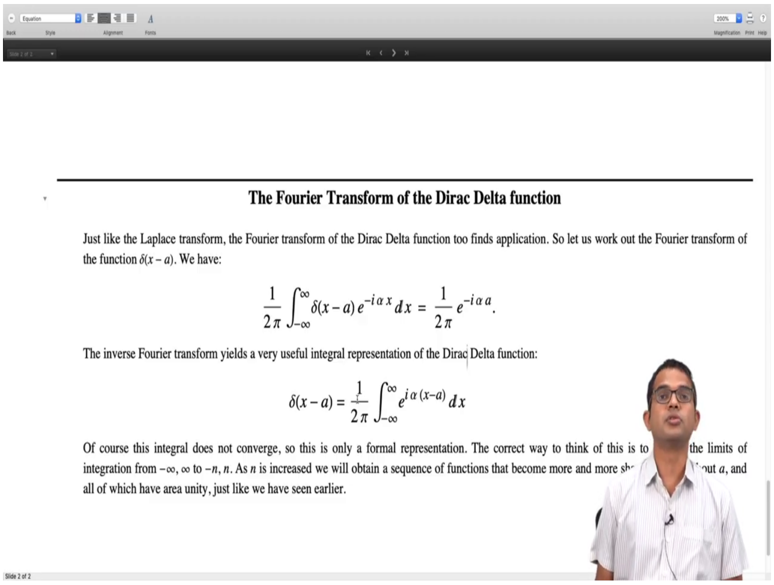

But you see that this integral on the right hand side is not a convergent integral right. So, it is in fact, just a formal representation, but it is often a very useful representation right. So, in place of delta of x minus a, one can just put in this integral. And then many simplifications you know may arise you know depending upon the context.

So, the way to think about this is you know you should replace these limits minus infinity to plus infinity instead of that you put in some minus n to plus n, and you know think of taking n to be larger and larger right. So, you will then construct a sequence of functions which are you know more and more sharply peaked about x equals a, and such that the area under the curve is always taken to be unity right.

So, when you think of it in that way, then it is similar to what we had earlier, we had a sequence of Gaussians or a sequence of triangles or rectangles or exponentially following functions or you know something like this. They are all you know completely equivalent ways of constructing the you know of the final object in the limit is the same.

So, this is a formal representation because as such it does not converge, but we know what it means. So, delta of x minus a has this formal integral representation right which came out of taking the Fourier transform, and the inverse Fourier transform of the Dirac Delta function ok.

That is all for this lecture there are more properties of Dirac Delta function which we will look at later on, but that is all for this lecture.

Thank you.