Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Ordinary Differential Equations
Lecture - 82
Solving ODEs using Laplace transforms

So, we have spent a few lectures you know talking about Laplace transforms looking at many of their properties, looking at you know operations which help us compute Laplace transforms you know there are you know also techniques primarily using a table to go from the Laplace transform you know back to the original function or taking the inverse Laplace transform right. So, we have all the machinery all set.

So, in this lecture, we will look at how to use these Laplace transforms to solve ordinary differential equations with constant coefficients right. So, particularly this class of problems which are called initial value problems you know in a very elegant and you know clean manner that is what Laplace transforms gives us right. So, it's a very powerful and beautiful technique. And so this lecture is going to look at how to do this with the help of several examples.

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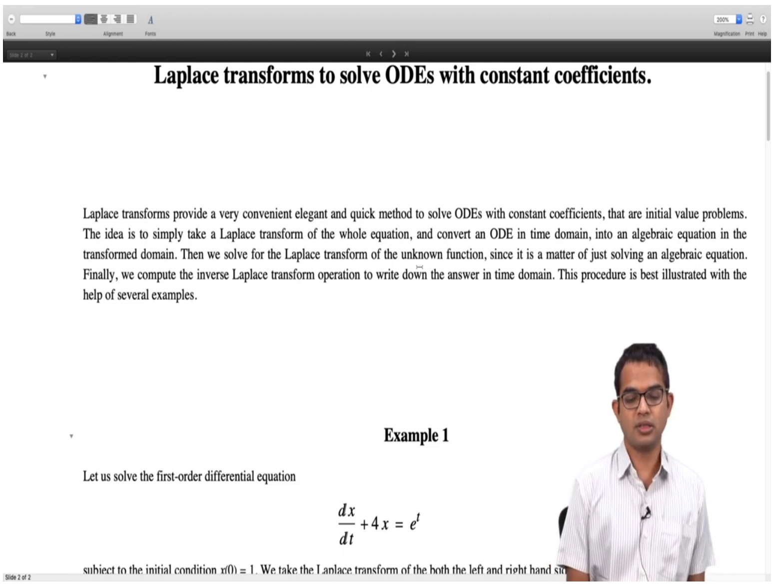

So, the idea is to simply take the Laplace transform. You have a differential equation if you take the Laplace transform you know and use the properties of Laplace transforms, we have said that you know the differential equation will get converted into an algebraic equation. So, what is an unknown in you know in the original domain which is a hard problem relatively because it is a differential equation becomes you know a case of finding an unknown from an algebraic equation which is much easier.

So, then you solve for it solve for the algebraic equation and then you have managed to find the Laplace transform of the function that you are after, and then we have to use our table and our you know bag of tricks to find the inverse Laplace transform and get the final answer right. So, this is the philosophy of this technique. And you know the initial conditions are all inbuilt when we are taking the Laplace transform; the initial condition also goes into it already.

So, the final answer that we get is indeed just the final answer. It is not like you have a general solution and a particular solution, it comes in built right, so that is what the Laplace transforms are able to do is to give us the final readymade answer. So, let us look at you know examples, the best way to understand this technique is to look at many examples.

So, let us start with a simple example. So, we have this first order differential equation. So, dx by dt plus 4x is equal to there is a forcing term; it is an inhomogeneous first order differential equation. And so we know how to solve this right. So, we know how to solve first order differential equations whether they are homogeneous or inhomogeneous, but let us solve it using Laplace transforms.

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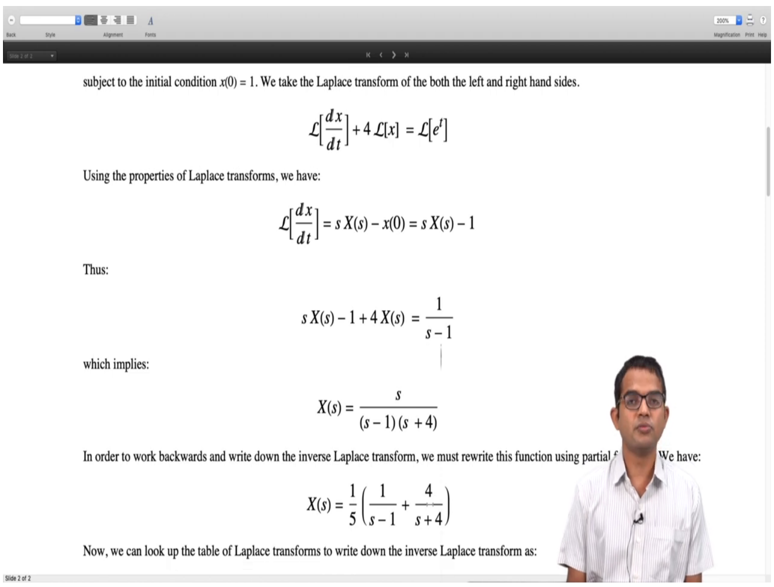

So, and we are also given the initial condition at x at time t equal to 0, the position x is just 1. So, what do we do? We just take the Laplace transform throughout, and we have Laplace so linearity tells us that the left hand side becomes the Laplace transform of dx by dt plus 4 times the Laplace transform of x is equal to the Laplace transform of e to the t.

But we have seen that the Laplace transform of the derivative will is just s times the Laplace transform of the function itself minus the value of this function at 0 right. So, this is a key property which will get used when you are working with differential equations. So, we should know this relationship between the Laplace transform of various orders of derivatives, and the Laplace transform of the function itself right. We have seen that this idea can be generalized to higher orders as well.

So, this we should know well. So, here it becomes s times X of s minus x of 0, but x of 0 is given to be 1. So, we simply write down this as s times X of s minus 1, and l of x is just x of s, and the Laplace transform of e to the t we know is just 1 over s minus 1 right. So, this is you know one of the Laplace transforms we worked out right at the beginning.

So, this is or we can look up from the table or even if you have forgotten we can work it out directly, it is not very difficult right from first principles this can be worked out. So, s times X of s minus 1 plus 4 times X of s is equal to 1 over s minus 1. So, now, you see that what was the differential equation in time domain has been converted into an algebraic equation now you know a function of x of s.

So, we can go ahead and solve for X of s. So, we collect all these terms we get s plus four times X of s is equal to 1 over s minus 1 plus 1. So, 1 over s minus 1 plus 1 will give us s divided by s minus 1, then we bring down this s plus 4. So, X of s is s divided by s minus 1 times s plus 4 right.

So, all that is left is now to work backwards and work out the inverse Laplace transform of this function X of s, and we will be done with the initial conditions everything is inbuilt. So, the way to do this we recall is using partial fractions. So, we write it as you know some alpha divided by s minus 1 plus beta divided by s plus 4, and then we have to match the coefficients. And so in fact, you can work out in this case that alpha is 1 over 5 and beta is 4 over 5 right.

So, once you have got the answer, you should explicitly calculate this and check whether you have got it right. So, here indeed we have s plus 4 plus 4 over. So, we have this minus sign. So, this 4 and this minus 4 will cancel. So, the constant part is going away, and then we get s plus 4, so it will get a 5 s. So, 5 will be canceled. So, indeed it all works out.

So, we have X of s in this manner from which we can look up from the table and work out the inverse Laplace transform here which is just 1 over 5 times e to the t plus 4 e to the minus 4 t. So, this is the final answer.

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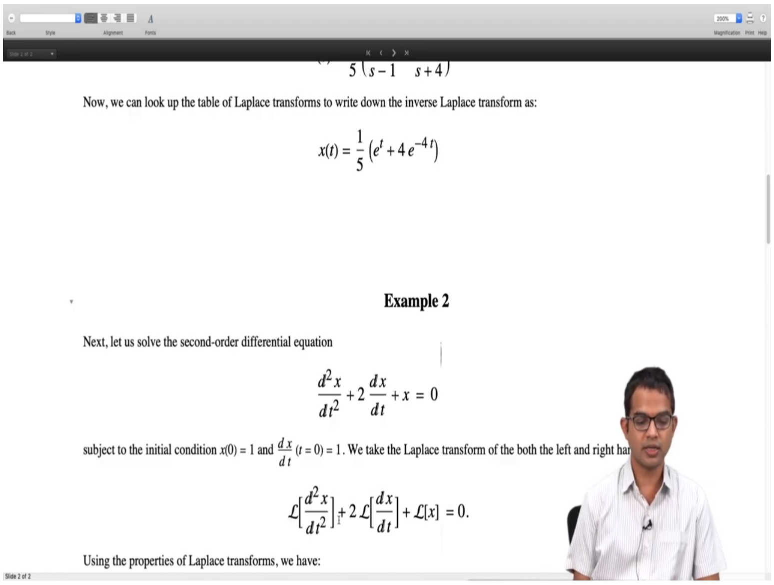

So, if you spend a moment looking at this equation, we see that e to the t what is e to the t. If you plug back e to the t in this original equation, so we see e to the t will give us you know the first derivative is just e to the t. So, e to the t plus 4 e to the t you know divided by 5 is important. So, we have also 1 over 5 times e to the t. So, it is 1 over 5 plus 4 over 5 which is just 1 e to the t.

So, in fact, so e to the t is the particular solution e to the t over 5. Whereas, e to the minus 4 t you can check is the solution was the homogeneous equation that is the complementary function e to the minus 4 t will give us when you take a derivative you get minus 4 e to the 4 minus 4 t, then you have plus 4 e to the minus 4 t, you can add them you will get 0.

So, you can tag along whatever factor you want, but here it turns out that the correct factor to tag is 4 by 5 because that is how you get the initial condition. You can also check that if you put t equal to 0, you get 1 plus 4 by 5 which is just one x of 0 is 1. So, indeed it all works out ok.

So, let us look at another example which is a little more complicated. Suppose, we look at the second order differential equation, but here I am taking it to be a homogeneous differential equation. We know how to solve this, but let us do it using Laplace transforms. So, we have the initial condition x of 0 equal to 1 is given to us and the derivative. So, we have to have two conditions now because the second order differential equation dx by dt at t equal to 0 is given to be 1.

Now, when we take the Laplace transform on both sides you know the right hand side is just 0. So, this expression involving Laplace transform linearity has been invoked.

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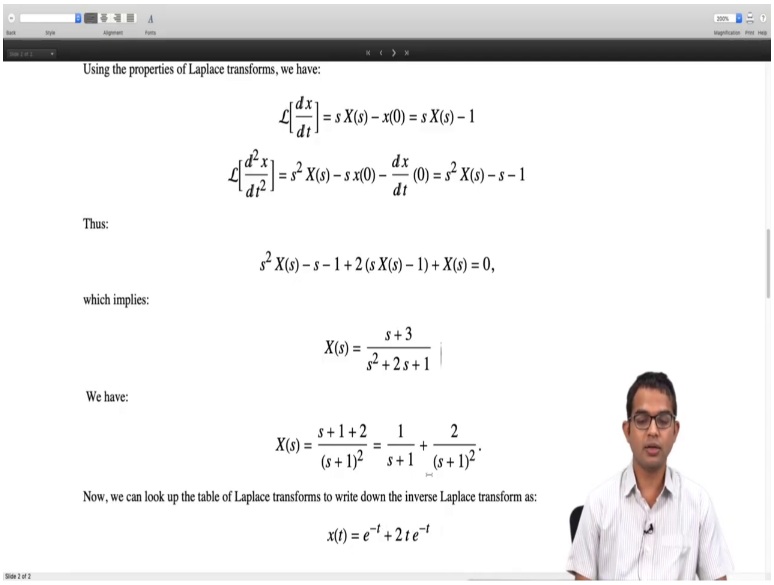

Now, once again we can recall the properties of Laplace transforms of derivatives of these functions. So, the Laplace transform of dx by dt will give us s times x of X s minus x of 0 which in this case will just go to s times X of s minus 1. And when we take the Laplace transform of the secondary order derivative d squared x by dt squared, we get s squared X of s minus s times x of 0 minus d x by d t of 0 at time t equals 0 so which is s squared X of s minus s in this case minus 1 right.

So, we just go ahead and plug these two expressions back into the original one to get an algebraic equation. Now, we have s squared X of s minus s minus 1 plus 2 times s X of s minus 1 plus X of s equal to 0. So, it's a homogeneous equation. So, you know it is not difficult to solve this. So, we have x of s is s plus 3 right which so you get s plus 1, and then you also get this plus 2 which goes to the other side, so we have s plus 3 divided by s squared plus 2 s plus 1.

So, once again we have you know the denominator is s plus 1 the whole square. So, we can rewrite the numerator as s plus 1 plus 2. So, then we have 1 over s plus 1 plus 2 over s plus 1 the whole square right. So, this is an easy problem in terms of finding the partial fraction representation. So, in general, we will have to work with this method of undetermined coefficients. You know, write it in a certain form, and match the coefficients and so on right, match the factors involving various powers.

Now, this is easy. We can, you know, do it directly without a recourse to the formal technique. So, once we have this form, you can immediately write down the answer. We can do the inverse Laplace transform. It is just e to the minus t plus 2 t times e to the minus t right. So, we recall that you know s plus 1 will give me just 1 over s plus 1 is just e to the minus t.

But when you have a factor like this, when you have a square of this, there is going to be a factor of t which comes in right. So, this is also a property which we have seen right. So, and there is a factor of 2 as well which comes from here. And so this is the answer. So, it is always a good practice to take your final answer, and check you know plug it into our original differential equation, and see that it works out, and also check the initial conditions.

So, x of 0 for example, here is if I put t equal to 0, I get 1 which is indeed ok. And then you can also check that the first derivative at t equal to 0 is also 1. So, indeed this is the correct answer. So, it is an elegant technique right. So, it works. I mean there is some difficulty involved sometimes with finding the inverse Laplace transform.

There is some technique involved, but the point is that this is you know with the aid of the table of Laplace transforms you know there is a streamlined approach available. And the benefit is that the final answer you get is truly final; the initial conditions are already inbuilt in there.

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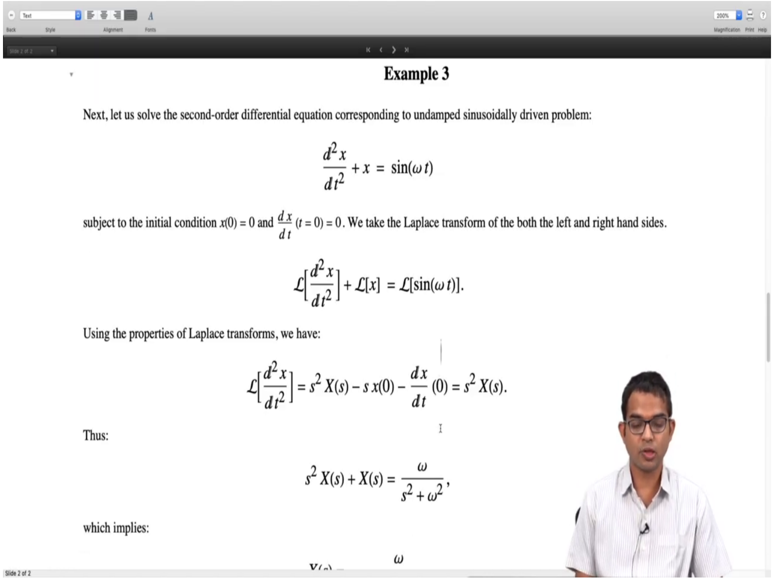

Let us look at another example which is where you have an inhomogeneous term. So, this is in fact a problem which we have solved already. So, suppose you are driving a system sinusoidally you know there is an external force being applied it is a sinusoidal external force which from frequency omega you know and your system is an undamped harmonic oscillator on the left hand side you just have d squared x by dt square plus x.

So, now we are also given these initial conditions which is very convenient, the particle at x equal at time t equal to 0 is at 0, and the speed is also 0 at time t equal to 0. So, this makes the Laplace transform very you know simple to evaluate. So, if you take the Laplace transform on both sides, we get the standard equation using linearity.

And now in this case the Laplace transform of the second order derivative which is s squared X of s minus s times x of 0 minus d x by d t of 0, but these two terms the last two terms will just go to 0 because x of 0 is given to be 0. And the speed also at time t equal to 0 has been given to be 0 so, we simply have s square X of s, very easy.

And then we have. So, the algebraic equation now becomes s square x of x plus X of s equal to the Laplace transform of sin omega t. So, you can work it out or look up from the table is just omega over s squared plus omega square.

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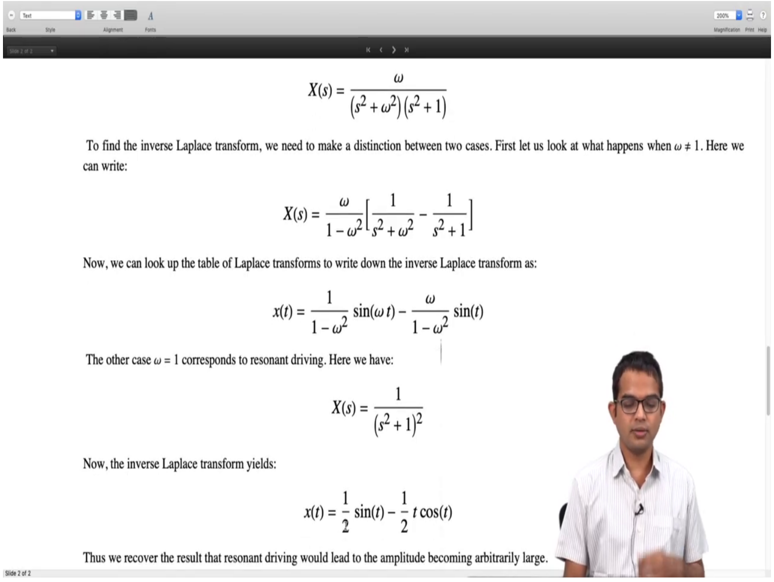

So, therefore, X of s is omega divided by s squared plus omega squared divided by s squared plus 1. Now, in order to find the inverse Laplace transform, we have to tread with caution here right. So, there we have to make two separate cases. If omega is 1 and omega is not equal to 1. So, let us first look at what happens if omega is not equal to 1. If omega is not equal to 1, then we have this legitimate way of writing this partial fraction expansion.

You see this factor of omega divided by 1 minus omega square comes in and if omega were one this we would not be allowed to do this right. So, you can check that. So, here I have s squared plus 1 minus s squared plus omega squared, so I get 1 minus omega square which will cancel with this right. And indeed this is exactly this expression provided omega is not 1 right.

So, let us see what happens the omega is 1 a little bit later. But if omega is not 1, this is the partial fraction expansion and we can look up the inverse Laplace transform using the table. So, I get you know this omega divided by s squared plus omega squared will give me sin omega t. So, I have sine omega t divided by 1 minus omega squared minus omega over 1 minus omega squared times sine of t, after all 1 over s squared plus 1 is the Laplace transform of sin of t. So, it is done.

In fact, you can go back and check that you know one of these is going to be the particular solution and the other is a complementary solution for your original differential equation. So, you can immediately see that sin t is a solution to the homogeneous differential equation. So, that is the complementary part. So, you can tag along whatever factor you want here, but this is a particular solution which has to be carefully worked out.

And, but the initial conditions that we have will also force this factor along with sin of t to be just precisely omega over 1 minus omega squared as you can check that the initial conditions are satisfied. And you can also check explicitly that sine of t is the complementary function sine omega t is the yeah divided by 1 minus omega squared is the particular solution, all of this has to be checked explicitly.

So, now let us look at what happens when omega equals 1. So, this is the case of resonant driving. So, we have an undamped resonant resonantly driven harmonic oscillator. If omega equal to 1, so let us go back to X of s omega equal to 1, we have simply one over s squared plus 1 the whole square. So, it is a different way of you know working out the inverse Laplace transform.

So, this is the function for which we have to work out the inverse Laplace transform. So, we have seen that when you have something like this, you can look up from the table right. So, or you can work it out by some you know other means. So, I am giving you the answer look, look it up from the table we have this type of function is also covered in the table. So, x of t here in this case is simply given by half times sin of t minus half times t times cosine of t.

You can also check that this holds out by taking the Laplace transform of this function. So, now you see that in fact you know both sin t and cos cos t appear right. So, but there is a factor of t right now if you recall from many lectures ago or when we were looking at you know the harmonic oscillator, the damped harmonic oscillator, but which is driven at resonance.

So, resonance here means that you know another way of thinking about this is this coefficient you know on the this omega it matches exactly with you cannot blindly take sin t and cos t as you know as an on sorts for your particular solution because sin t is in fact, a solution of your homogeneous differential equation.

So, in fact, the prescription given is to multiply by t and so that is what you know is coming out automatically with this method right. So, there is a t times cosine of t which comes in factors that have to be arranged properly in order to match with the initial conditions right.

But the theory tells us that you know you take sin of t, sin of t is just the complementary part, and then to find the particular solution here you have to multiply by t in a suitable way.

And here in fact, this t is what contributes to your amplitude becoming arbitrarily large right. So, this is also something that we have seen. When you drive your system at resonance and if there is no damping what do you expect, so the amplitude will have to keep on increasing right although there is a sinusoidal dependence that becomes sort of not so important for very large times. It is this factor of t which will dominate and so in fact, your amplitude becomes very very large right as we have seen ok.

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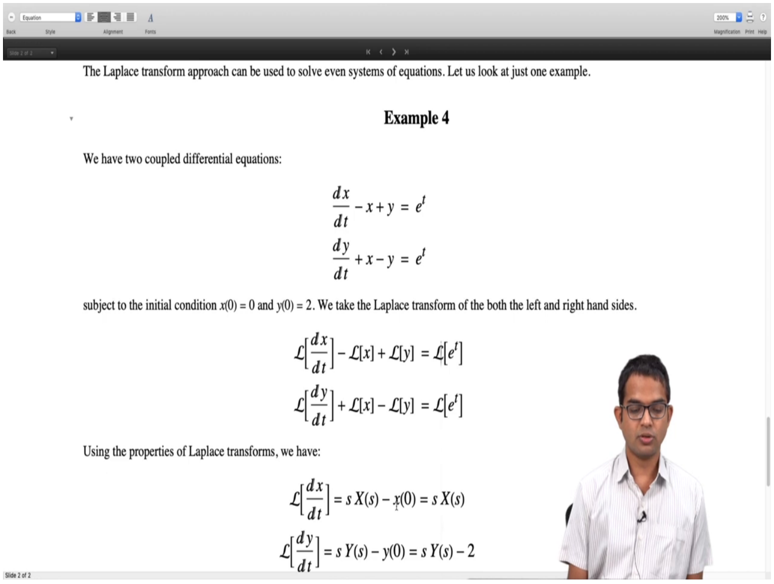

So, let us look at you know just one more example. And here we will see the power of the Laplace transform approach to solve even a system of equations right. So far we have not studied systems of equations. So, you know where you have coupled differential equations. You have more than one unknown variable you know dependent on the same time variable for example, but they are all coupled in some complicated way.

And then you have to use not only your theory of differential equations, but also use some linear algebra in such a context right. So, here instead of you getting into the theory of linear equations right, so linear equations and differential equations. So, let us just work it out using Laplace transform. So, with the aid of just one example right. So, suppose we have these two coupled differential equations. We have dx by dt minus x plus y is equal to e to the t and dy by dt plus x minus y is equal to e to the t.

So, you can see that there are forcing functions which are applied on the right hand side. And the left hand side is you know there is the rate of change of x also it depends both on x and y. And likewise the rate of change of y also depends on both x and y right. So, it's not just a differential equation, but it's a coupled differential equation and there are two of these.

So, you can write this as a matrix equation and work with the properties of the matrix matrices, and also the differential equations involved and so on. And there is a way of theory right of how to solve the technique involved. But let us look at how this can be solved using Laplace transform, and it is a quick and elegant approach. So, the initial conditions are also given x of 0 is 0, y of 0 is 2.

So, what we do is we simply take the Laplace transform of both of these equations. So, we have algebraic linear equations that is what it will give us. So, L of x d x by d t minus L of x plus L of y is equal to L of e to the t. And likewise L of d y by d t plus L of x minus Laplace transform y is equal to Laplace transform of e to the t.

So, again we use the fact that the Laplace transform of the derivative you know becomes s times X of s minus x of 0 x of 0 is 0. So, it is just s times x of S. But the Laplace transform of dy by dt becomes s times Y of s minus 2, because y of 0 is given to be 2.

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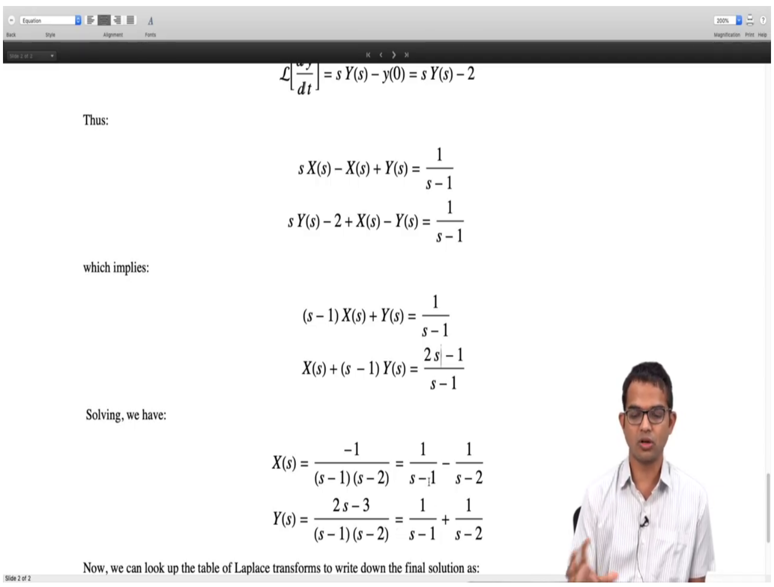

If you plug these back in into these two equations, we have s times X of s minus X of s plus Y of s is equal to 1 over s minus 1 and s times Y of s minus 2 plus X of s minus Y of s is equal to 1 over s minus 1. So, we have two equations and two unknowns. One is as s minus 1 times X of s plus Y of s is equal to 1 over s minus 1 and X of s plus s minus 1 times Y of s is equal to 2 s minus 1 over s minus 1.

It is very easy to solve this. These are two linear equations and two unknowns. And you can immediately write down the solution for this X of s will turn out to be just minus 1 over s minus 1 times this minus 2 which you know writing in partial fractions is just 1 over s minus 1 minus 1 over s minus 2.

And Y of s also you can solve for and this will turn out to be 1 over s minus 1 plus 1 over s minus 2 as you can verify right. So, now the only thing that remains is to work out the inverse Laplace transform of each of these functions which in this case is very easy because we know how to do 1 over s minus 1 1 over s minus 2, there all of the same class.

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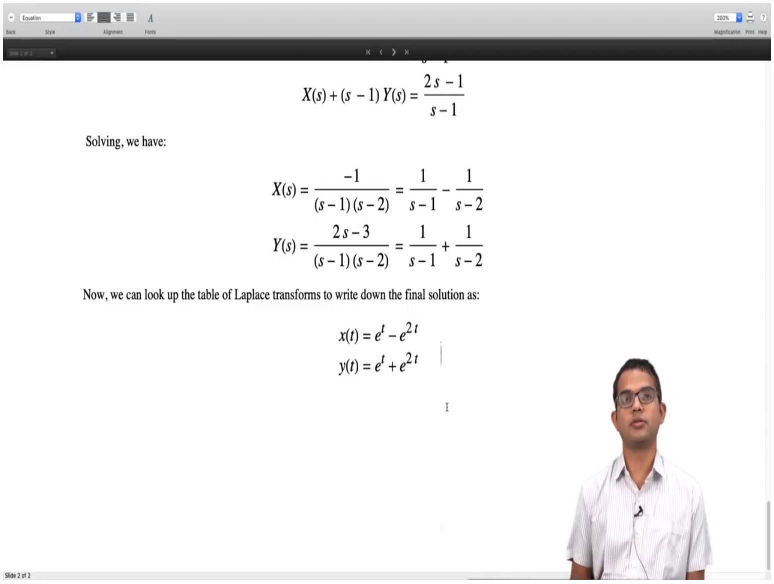

And the answer is immediately written down. So, the answer is for x of t it is e to the t minus e to the 2 t, and y of t is e to the t plus e to the 2 t. So, you should check that indeed this is the correct solution. The way to do that is to plug these two solutions back into your original coupled differential equations, and verify that they all hold on the one hand. But also x of 0 is 0 and y of 0 is 2 that is something we can verify.

If x of 0 is you know e to the 0 is 1 minus e to the 2, 2, 0 is that is also 1. So, x of t, x of 0 is 0, and y of 0 is also immediately seen to be 2. So, the initial conditions hold out. And you can also cross check that these are indeed the solutions of this coupled you know set of differential equations ok.

So, that is all for this lecture. We looked at a bunch of examples involving Laplace transforms and how they can be used to solve differential equations.

Thank you.