Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 80
The Inverse Laplace Transform

So, we have been looking at the Laplace transform. We have looked at what the Laplace transform is, we have looked at certain properties of the Laplace transform itself. We looked at certain special types of functions for which the Laplace transform can be evaluated using the properties, and how that can be exploited to find the Laplace transform of you know composite functions right in which involve various standard types.

Now, in this lecture we will look at what is called the inverse Laplace transform right. So, it is important to be able to go back right. So, we know how to take a function which is defined for positive times, and go to its Laplace transform; oftentimes you know the other direction is also important right. So, typically we are imagining you know taking a complicated operation in time domain as electrical engineers would like to call it. And when you take a Laplace transform, you have reduced the complexity of the problem.

Typically, we are interested in looking at something like an ordinary differential equation which is a comparatively harder problem, and then when you go to the Laplace transform domain it becomes an algebraic problem. And then you can solve for the unknown in this domain where it is an algebraic equation, solving an algebraic equation is an easier problem compared to a differential equation.

But then once you have solved this in the Laplace transform domain, you have to come back right to the original time domain which is where you want the answer, so then you will have to carry out an inverse Laplace transform right. So, that is the subject for this lecture. We look at how to find out the inverse Laplace transform for certain standard functional forms ok.

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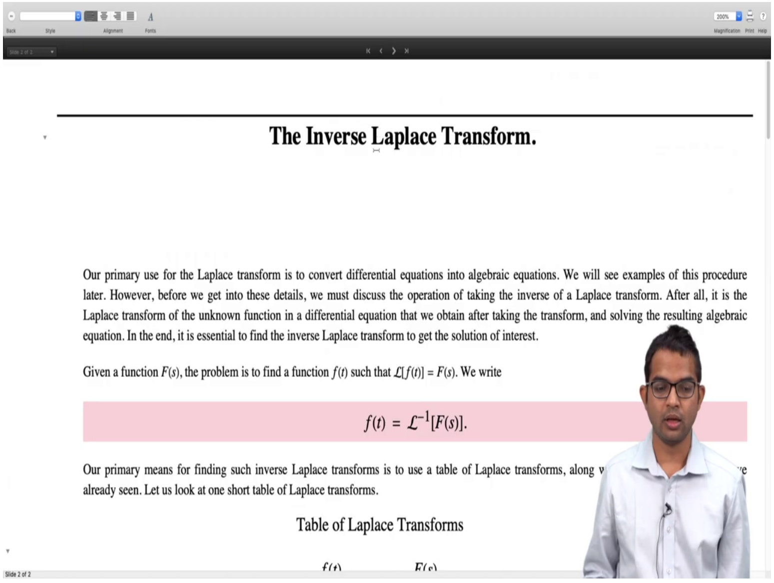

So, like we said you know, the Laplace transform finds its application in converting a relatively hard problem into a relatively easier problem, for example, converting a differential equation into an algebraic equation. You solve it and then it is really a complete solution only if you can come back to your original domain which is you know the time domain if you wish. And so that involves taking this thing called the inverse Laplace transform.

So, the problem of the inverse Laplace transform is the following. So, if you are given a function F of s ah. So, the problem is to find a function f of t such that the Laplace transform of f of t is F of s right. So, we write formally as f of t is equal to L inverse of F of s. That is the problem which we want to work out right.

So, the method we take here is actually you know a sort of brute force approach which is we just form a table of standard types, and then we just look up from the table and see if we can you know bring our the function that is given to us into one of the forms here, and then exploit some of the properties of Laplace transform to be able to work out the inverse Laplace transform right.

So, we do not have a formal approach here, but rather it is like a you know look up approach. You find a table in which an extensive collection of you know Laplace transforms of various functions which has been made up, but we will look at a very simple table based mostly on our own direct evaluation of some of them, but also there are other types which we have put in.

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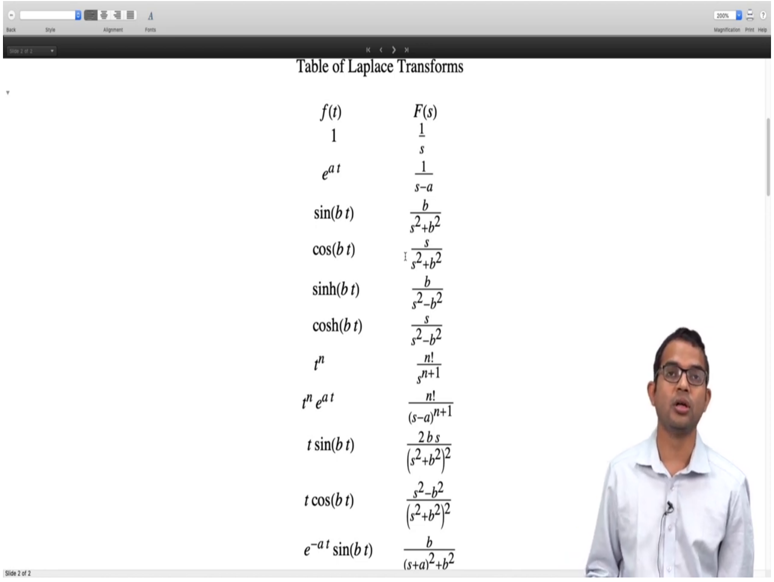

So, let us look at this table of Laplace transforms which I have collected here. So, f of t if it is 1, we know that it is just 1 over s. And e to the a t we have seen it is like it is like a translation in the Laplace domain, so it is 1 1 over s minus a; sin of b t we have seen b over s square plus b square; and cosine of b t it is b over cosine of b t is s over s squared plus b squared where a sin of b t is b over s square plus b squared.

Likewise, we can also evaluate the hyperbolic sine and hyperbolic cosine right. So, you see that it looks somewhat similar except that some sin involving the b squared part will change.

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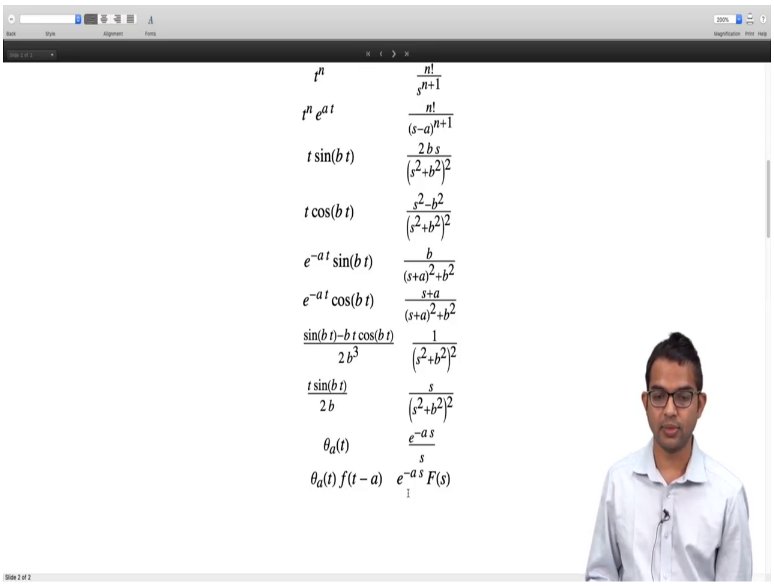

We worked out how to compute the Laplace transform of t to the n it is just n factorial divided by s to the n plus 1. Then t to the n times e to the a t so it is like doing a translation in the Laplace space. Then t times sin of b t we can work out it is 2 b s divided by square plus b square the whole squared.

Likewise for t times cosine of b t And then we can also work out you know e to the minus a t times sin of b t this also is like a translation in in the s space e to the minus a t cosine of b t, but it is convenient to put them all down in a table right because this the forms which appear on the right hand side are standard forms. And then when you have this form, you should be able to work backwards quickly.

And then I also collected sin of b t minus b t cosine of b t so divided by 2 b cube the Laplace transform of this is of this form 1 over s squared plus b square the whole square right. So, you see that on the right-hand side, you get all these standard algebraic forms right; oftentimes this is what we work with, and then these exponentials also come in. Exponential you can try to tweak it, and you know introduce special kinds of exponentials of this kind using the Heaviside step function.

So, t times sin of b t divided by 2 b this is very similar to you know this is one of these earlier ones. So, we already saw that 2 b t times sin of b t, so 2 b s over s squared plus b squared of the whole squared. So, indeed t times sin of b t divided by 2 b will have the Laplace transform s over s square plus b squared whole square.

Then we saw that theta a of t has this Laplace transform e to the minus a s divided by s. And we also saw this property that if you have a translation of this function in time domain, it gives you e to the minus a s in the Laplace domain times F of s right.

So, all these properties combined with some other properties also which we will discuss are very useful for you know working out the inverse Laplace transform. So, the technique is simply to take your function whose inverse Laplace transform you need to find and bring it into one of these standard forms. And then that is it, and then look it up and write down the answer further time domain function.

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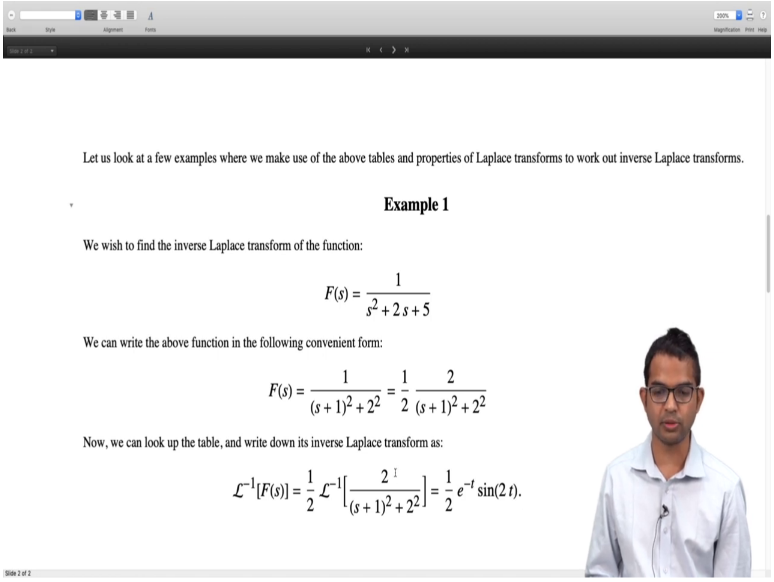

So, let us look at a few examples. So, this is best understood with the help of examples. So, F of s, suppose it is given to be this kind of a quadratic form 1 over a quadratic form, so you must massage it into one of the forms here.

So, we immediately see that you can write this s square plus 2 s plus 5 as s plus 1 the whole square plus 2 square which comes in the denominator, and then you might as well write it as 1 over 2 times 2, 2 divided by s plus 1 whole square plus 2 square to directly mimic of standard form here.

And then we look up the table and we see that, in fact, the inverse Laplace transform of this function is half you know which we have pulled out times the inverse Laplace transform of this function which is nothing but e to the minus t times sin of 2 t. So, e to the minus 1 t comes from this s plus 1 whole square it is like a shift in this Laplace domain. So, you get this e to the minus t, and then sin of 2 t you know the 2 corresponds to this 2 here and this 2 right. So, you have to just be careful that you bring it exactly into the right form and work backwards. So, that is example 1.

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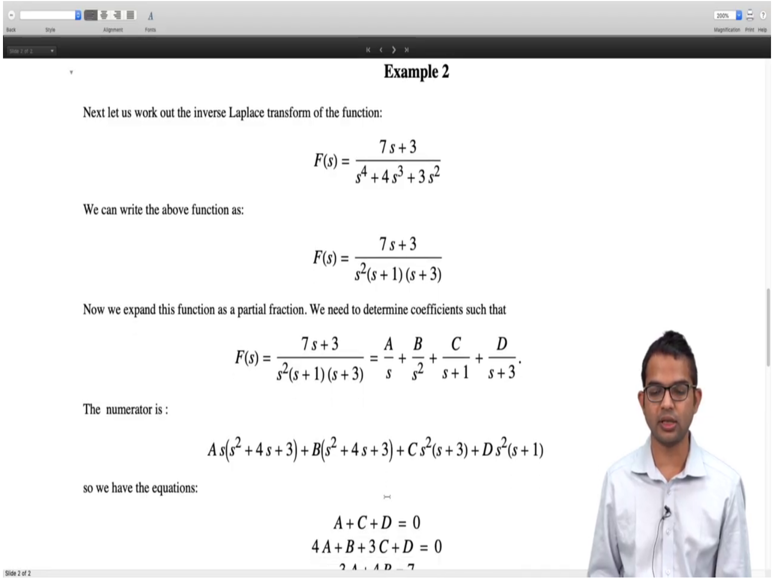

Let us look at another example. So, suppose you have a again a you know some algebraic expression in the denominator you see that you have s power 4 plus 4 s q plus 3 square, but this s squared is something that you can pull out. And so you can rewrite this as 7 s plus 3 divided by s square times s plus 1 times s plus 3. And then it is, it is in this complicated form, but we should write it as a partial fraction, so that we can bring it into one of the standard forms.

So, the partial fraction you know technique tells us that you should be able to write this as A over s you know look at all the factors, so you have s squared. So, you must account for both A over s term and B over s squared term, and C over s plus 1 and D over s plus 1, where A, B, and C, and D are coefficients to be determined.

So, can we find coefficients A, B, C and D? Such that you know these two expressions, the left hand side and the right hand side agree for all values of x of s. And in order for this to happen, we should match you know term by term s cube term will vanish, so that will give us A plus C plus D equal to 0.

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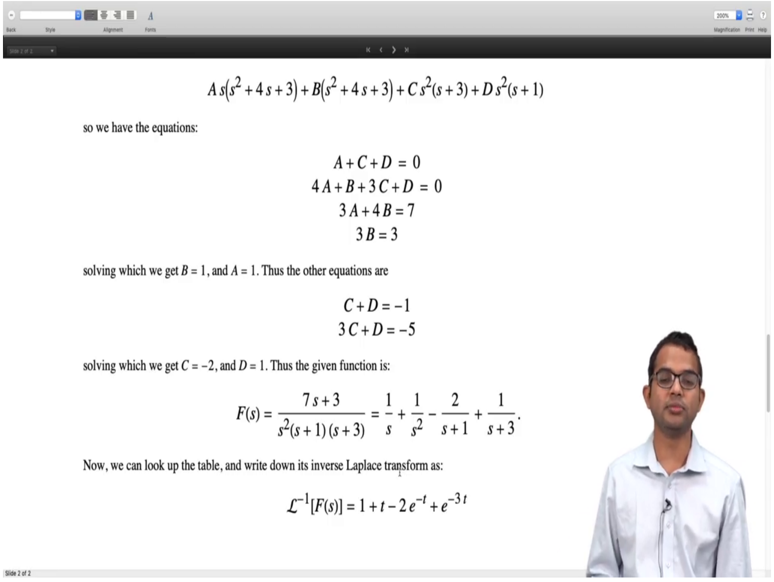

Then this quadratic part will also vanish because in the numerator you have just 7 s plus 3. So, 4 A plus B plus 3 C plus D, you can cross check this must go to 0. Then you have 3 A plus 4 B equal to seven 3 B equal to 3, solving which you immediately get B equal to 1.

Therefore if you plug backwards right from the bottom up if you go you get a also is equal to 1 right. So, 7 minus 4 divided by 3 that is A is 1. And so you have these 2 equations in 2 unknowns C plus D is equal to minus one and 3 C plus D equal to minus 5. So, you can solve for it and then you just simply get C equal to minus 2, and D equal to 1.

So, finally, F of s can be written as 1 over s plus 1 over s squared minus 2 over s plus 1 plus 1 over s plus 3, you should cross check this all right. So, typically doing this partial fraction expansion, entails some human error. And so it is always a good idea to cross check it after the final step to see if indeed we have got it right. And once we have checked this, it is just simply a matter of looking up from the table and writing down the answer.

So, the inverse Laplace transform of this function now 1 over s will give you 1 over s squared will give a t 1 plus t minus you know this these two will just give us these exponential so minus 2 times e to the minus t plus e to the minus 3 t, done, very simple. Once we bring it into the right form, it is just simply a matter of looking up the answer from the table.

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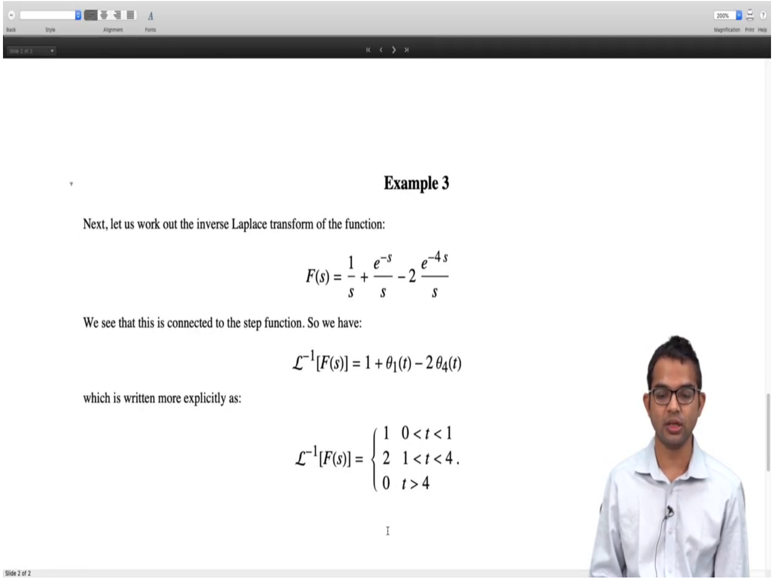

So, let us look at example 3. So, now, suppose we want to work out the inverse Laplace transform of 1 over s plus e to the minus s over s minus 2 times e to the minus 4 s over s. So, now we see that this is connected to the Heaviside step function right. So, 1 over s of course, will just be 1, but e to the minus s over s the inverse Laplace transform is theta 1 of t and minus 2 times inverse Laplace transform e to the minus 4 s over s is just minus 2 times theta 4 of t, done, that is the answer.

And of course, it may be convenient you know in some applications to write this more explicitly in this form. So, the function takes the value 1 from 0 to 1, and it becomes 2 from 1 to 4 right until this starts to kick in and once this also kicks in it just drops down to 0. It is like it involves 2 steps right. So, this is also something that you should cross check.

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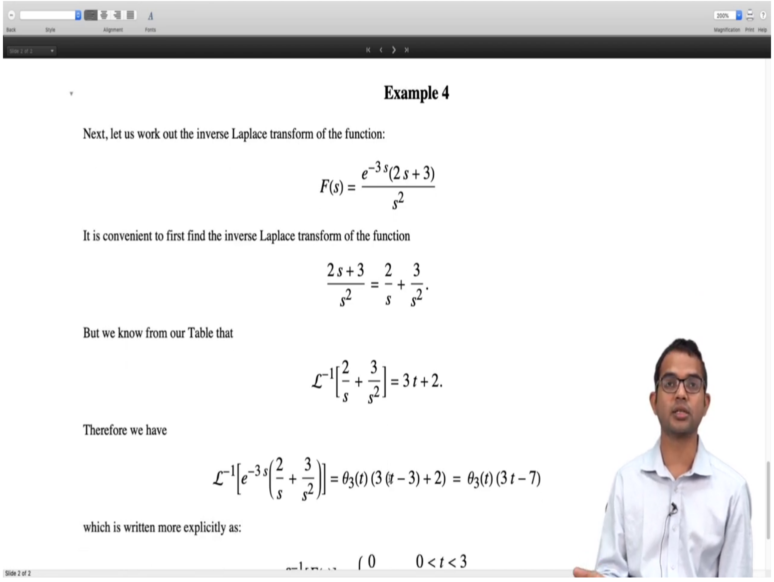

So, let us look at another example. Suppose we are interested in finding the inverse Laplace transform of this function e to the minus 3 s times 2 s plus 3 divided by s squared. So, whenever we have a factor like e to the minus 3 s, we should immediately see that maybe that can be incorporated in terms of a you know a step function of some kind. So, let us just work out the Laplace inverse Laplace transform of the other part 2 s plus 3 divided by s squared.

So, we have and so we can rewrite this 2 s plus 3 over s squared as 2 divided by s plus 3 divided by s square right each of these we know how to find the inverse Laplace transform of. So, from the table, we can say that the inverse Laplace transform of 2 by s plus 3 s 3 by s squared is 3 t plus 2 t plus 2 3 t plus 2. And once we have this, we simply use the property that this type of a factor e to the minus 3 s simply means that you have to tag along a theta 3 of t, but you have to be careful it is not just simply 3 t plus 2 right.

If you think of this as a function f of t, you must find t theta 3 of t times f of t minus 3 right. So, the function itself must be shifted by the appropriate amount. So, then you get the final answer as theta 3 of t times 3 t minus 7. You can explicitly check that indeed if you take the Laplace transform of this function, you get back the original function you started with.

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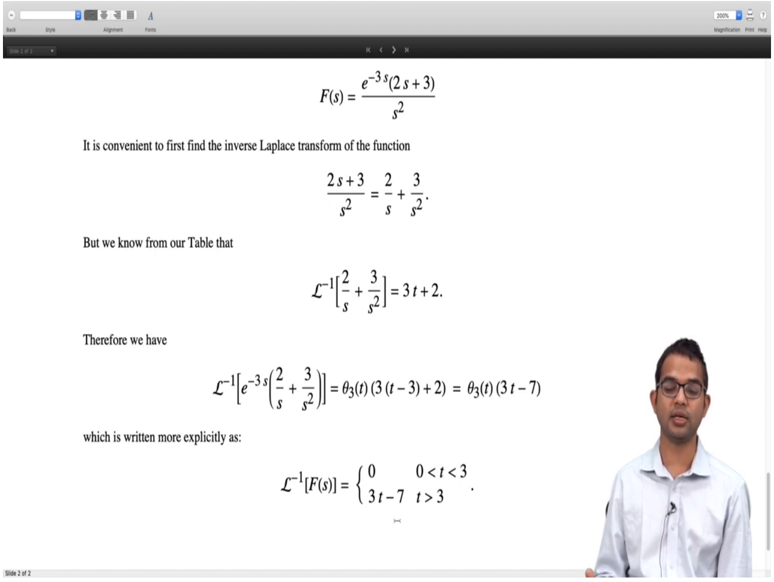

And if you want you can also rewrite this inverse Laplace transform as you know it takes the value 0 from 0 to 3, but it becomes 3 t minus 7 starting from t greater than 3 right ok. So, using these properties and with the aid of the table, there are many other functions whose inverse Laplace transform can be worked out, many of these algebraic functions are what we will care about.

So, it is useful to play with these kinds of examples. So, that when we do the you know the problem of solving ordinary differential equations using Laplace transforms these inverse transforms are essential to be computed. So, if you are very comfortable with these techniques, they will come in handy when we look at some of these applications ok. That is all for this lecture.

Thank you.