Mathematical Methods 1
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Linear Algebra
Lecture - 08
Applications of the Cauchy-Schwartz inequality

So, we have seen what the Cauchy Schwartz inequality is. We will look at some applications of this and also you know see how the notion of the scalar product is quite general. And you know I will give you one example of a scenario where this is you know it is defined in a way which is perhaps unusual which you have not seen perhaps, ok.

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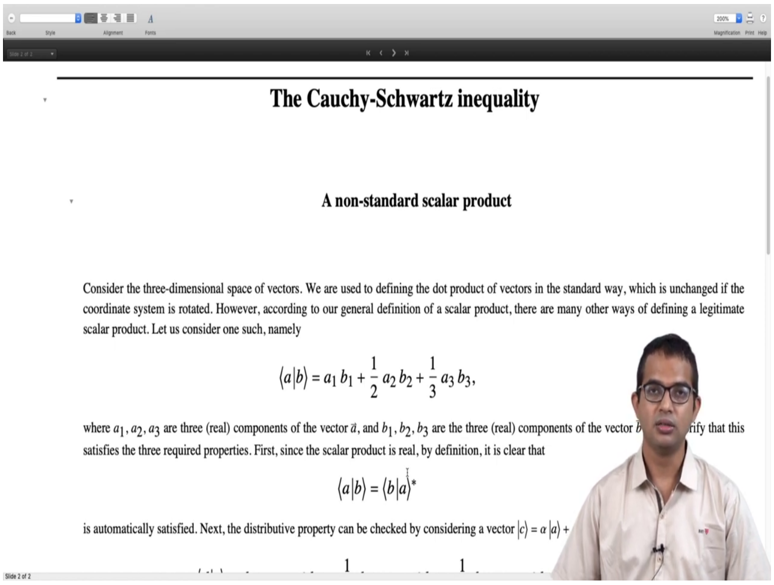

So, for a you know for a function to classify as a scalar product, right it is a function of it takes any two vectors and gives out a complex number, right and it must happen in such a way that certain properties are satisfied, right. We have already stated what those properties are, right. You know provided these properties are satisfied you could have many different ways in which an acceptable scalar product can be defined.

So, now suppose we define for you know three-dimensional space of vectors, we consider this three-dimensional space of vectors, the standard way in which we would define a dot product is you know just take the components along the x direction, along the y direction and along the z direction and then you know take the products and add them up, right. That is a valid dot product definition.

It turns out that you can have a much more general definition, but this quantity is not going to be invariant when you rotate your coordinate axis. Whereas the kind of dot product that were used to where you are think of it as just a 1 b 1 plus a 2 b 2 plus a 3 b 3, it's going to appear the same no matter you know if you do a rotation if you do certain transformations to your axis, it is going to remain the same. Whereas, this is going to take on a different form if you are in a different basis so to speak, right.

So, I mean all of these concepts are familiar and I am using them loosely, but we will also define these things a bit more carefully as we go along. But, for now the point is that I am going to define my inner product between two vectors in this manner a 1 b 1 plus half times a 2 b 2 plus one-third a 3 b 3, where a 1, a 2, a 3 are 3 real components of this vector a and b 1, b 2, b 3 are the 3 real components of the vector b, right.

So, what are the properties that we have to verify? Right. First of all you know there is this requirement that the inner product of a with b and the inner product of b with a must be complex conjugates of each other, which is clearly true here because we are just dealing with real numbers. So, the inner product of a b is actually equal to the inner product of b a, because these are all just it is not a not a generic complex number, but a real number. So, this is automatically satisfied.

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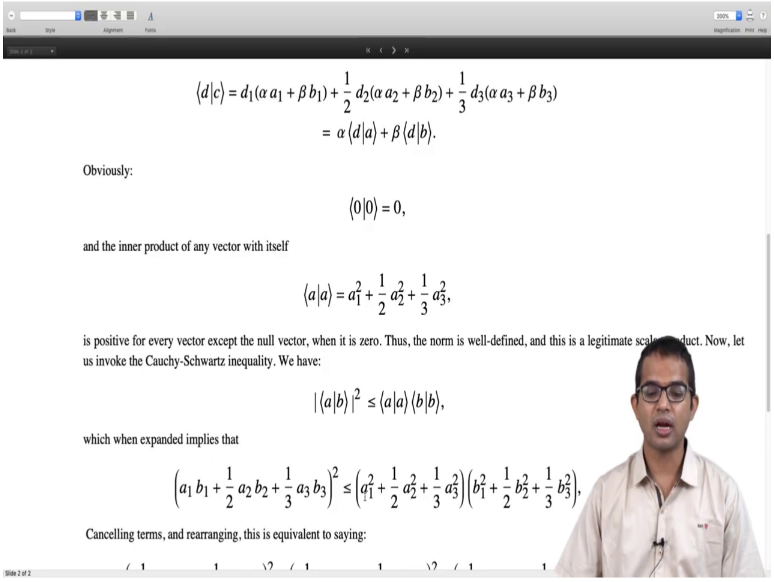

So, the next is the distributive property, right. If you take a vector like c which is a linear combination of some two vectors a and b, and then if you take the inner product of this with respect to some d, right you bring in this bar vector d from the left hand side. So, then you have these components of d, d 1, d 2 and d 3.

So, if I invoke this definition d 1, then the vector c itself will now have the components alpha a 1 plus beta b 1, then alpha a 2 plus beta b 2 and alpha a 3 plus beta b 3, right. So, then if I bring you just invoke this definition I will have d 1 times this stuff plus half times d 2 times this the you know second component and one-third time d 3 times the third component.

And now we can rearrange all this and quickly convince ourselves that indeed this is the same as alpha times the inner product of d with a plus beta times the inner product of b with b, right. So, the third property of course is if you take the null vector and take an inner product of the null vector with itself you are going to get 0, right.

And the inner product of any vector with itself, right, is a 1 squared plus half a 2 squared plus one-third a 3 squared, all these components are real numbers, so the inner product of any vector with itself will give you necessarily a non-negative number. It is going to be positive unless the vector is a null vector then it will be 0, ok.

So, all the three required properties are met and therefore, this is a legitimate scalar product, right. There are other ways, you can play with it, you can come up with your own definition of an inner product and see if these three properties are satisfied. Then what would be the Cauchy Schwartz inequality?

You are going to look at how the Cauchy Schwartz inequality plays out with this definition, right. So, the Cauchy Schwartz inequality just relies on the fact that there is a well-defined inner product.

Does not matter what the precisely how you define it, as long as you have an inner product defined you can go ahead and apply the Cauchy Schwartz inequality, which says that if you have two vectors a and b inner product of a b, a and b the modulus of this squared is going to be less than or equal to the product of the inner products of each of these vectors, right.

So, in this case we will just you know invoke this general c as inequality and apply it to this inner product. So, then we see you get a 1 b 1 explicitly expanding it out a 1 b 1 plus half a 2 b 2 plus half one-third a 3 b 3, the whole squared is must be less than or equal to a 1 squared plus half a 2 squared plus one-third a 3 squared times b 1 squared plus half b 2 squared plus one-third b 3 squared, alright.

So, let us see if. So, this is like a very general sort of result, right. It does not matter what you know a 1 b 1, a 2 a 1, a 2, a 3, b 1, b 2, b 3 are. They just have to be real numbers and this inequality must hold, right. It is a complicated looking inequality. So, let us expand these terms out and see if we can derive the same result independently, alright.

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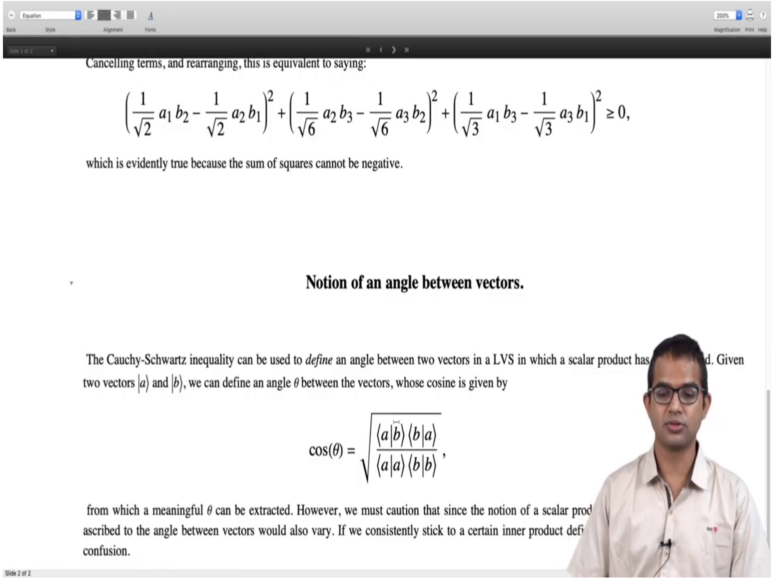

So, cancelling terms and rearranging we have. So, I mean you just expand on both sides, so you have you know some the sum of 3 times the whole squared will give you a square plus square plus square and then 2 times the product of the first 2 times terms plus 2 times the product to the second 2 terms and plus 2 times the product of the first and the third term. That is what will appear on the left hand side.

And then on this you will see all the square terms will immediately cancel with these diagonal terms if you wish. So, if I multiply a 1 squared with b 1 squared that is going to cancel with a 1 squared b 1 squared. And likewise, if I multiply half a 2 squared with half b 2 squared that is going to cancel with this half a 2 b 2 the whole squared and then the third term also cancels and so on, right. And then you will be left with if you carefully rearrange all terms, right. So, this is left as an exercise for you all to check.

You can actually rearrange all these terms to bring everything; well I will move everything to the right hand side and rewrite this as the sum of 3 squares. There is 1 by root 2 times a 1 b 2 minus 1 over root 2 a 2 b 1 whole squared plus 1 over root 6 a 2 b 3 minus 1 over square root 6 a 3 b 2 the whole squared plus 1 by root 3 a 1 b 3 minus 1 over root 3 a 3 b 1 the whole squared.

Now this, so the claim is that this sum of 3 squares must always be greater than or equal to 0, right which is definitely true because we are dealing with real numbers and the square of a real number cannot be negative, the sum of squares of real numbers also can never be negative, right. So, the case where they all become 0 is of course, when you can verify it will be when you know when they are the null vector; I mean you the case when it they become 0 is when sorry not when they become the null vector, but rather when they are parallel, right.

So, this is like one of the when the two vectors are really pointing on the same direction, so then you have on the left hand side you will have a mod squared and all the right hand side also you will have a the whole squared which is the same as mod square because it is going to be real number. So, the extreme case appears when the two vectors are parallel to each other. You know you can also think about this geometrically. There is no question of dropping a perpendicular and so on. And if the two vectors are parallel then this equality will hold, ok.

So, this was a quick exercise in seeing how you know you can have a non standard type of scalar product defined and how the Cauchy Schwartz inequality plays out in this context. And, I just want to quickly make one more point, right and it will be part of this lecture is this notion of an angle which can be defined between any two abstract vectors, right. So, this also can be obtained from the Cauchy can be defined based on the Cauchy Schwartz inequality, right.

So, the Cauchy Schwartz inequality guarantees, right, that this quantity on the left hand side is less than or equal to the right hand side, so you can bring this whole object on the right hand side to the denominator in the left hand side. And then, you can think of this overall object you know using the analogy from Euclidean vectors three-dimensional vectors.

You can go ahead and define an angle theta between any two arbitrary vectors as the square root of cosine, you can define this cosine squared theta to be this ratio and then cos theta is the square root of this. And since this quantity is always going to be necessarily going to be less than you know the modulus of this is less than 1, so there is a meaningful theta associated with you know with this quantity always, right. So it is yeah.

So, one must use some caution because the notion of inner product itself is not a it is not a rigid notion, right. As we have seen you there are different ways of defining a scalar product, but given you know a definition for a scalar product there is going to be a well-defined angle between any two vectors, right.

You have a space of vectors, you can take any two vectors, find the inner product between them according to one definition. And then the angle between them is fixed, and that is now all the angles between vectors will change to a different number, but the qualitative aspects are all going to be the same if you use a different inner product definition, ok.

Thank you. That is all for this lecture.