Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 79
Step functions, Translations, and Periodic functions

So, we have been looking at Laplace transforms. We have first of all looked at what the definition is then we also studied some of its properties. And, how these properties can be used to evaluate you know Laplace transforms of you know fairly complicated functions using some of these principles.

So, in this lecture we will look at some special class of functions right, which are you know whose Laplace transforms we will work out. And we will see how knowing this, it will allow us to evaluate Laplace transforms of other functions. You know more complicated or you know composite functions which make use of you know the properties of these special kinds of function right, you know that is the subject matter for this lecture ok.

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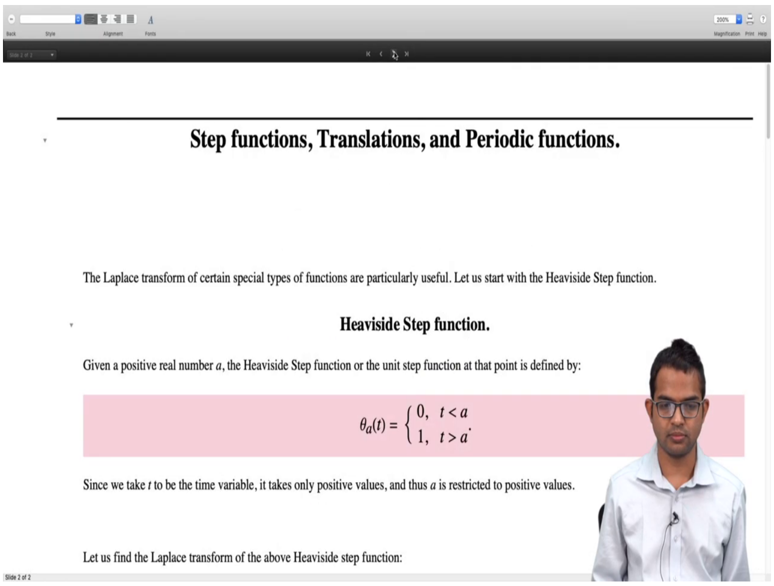

So, we start with something called the Heaviside step function right, it's a very useful function it appears, in physics it appears, in various applications, in engineering electrical engineers uses all the time. And so, we imagine a scenario where you know something is turned on at a certain point right. So, up till a certain time the signal is absent and at that particular time, there is like a switch right you turn it on right its (Refer Time: 01:46) like an off on button.

And so, in the context of Laplace transforms we are looking at functions which are defined only for positive times. So, you imagine that this type of a switch is turned on at some time, which is you know represented by this parameter a. And, so we take a to be a positive value. Now, at this moment you imagine that this function will be 1, beyond that point and it is 0 up to that point it's a discontinuous function.

But, it is very useful as you will see and it's also possible to find its Laplace transform.

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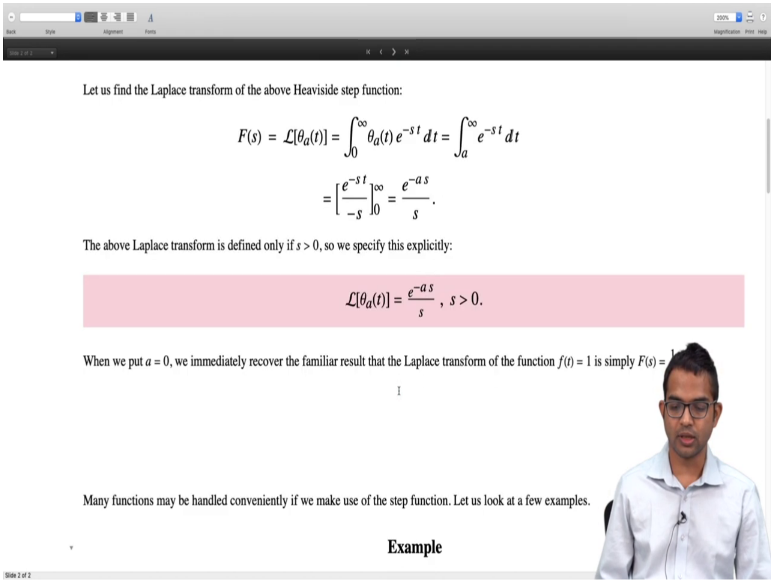

So, let us work out the Laplace transform of this Heaviside step function, it is simply given by you know 0 to infinity this theta function e to the minus s t. So, when you have an integral of you know some discontinuous function. So, we go from 0 to that particular point, where the discontinuities and then again integrate from a to infinity. And in this case 0 to a it is just 0, because the value of the function is 0 in this interval.

And, then from a to infinity it is just 1. So, you get integral a to infinity e to the minus s t which is easy enough to integrate and so, you just have e to the minus s t divided by minus s from 0 to infinity. So, we end up with the result that the Laplace transform of this Heaviside step function is simply e to the minus a s divided by s right. So, we can quickly check that if you put a equal to well I mean it is understood that s is greater than 0 right.

So, we have seen that it's worth writing down explicitly that s is taken to be greater than 0. Otherwise it will not be a convergent integral ok. So, it's good to put it down explicitly. Now, let us check that when you take a to 0 right. So, then the function itself is like saying that your function is 1 for t greater than 0.

And anyway you do not care about what happens for t less than 0. So, for all practical purposes you can just say that the function is 1. So, we know that the Laplace transform of the function f of t equal to 1 is simply 1 over s right and that is what this reduces to in any case. If you put a equal 0 you get 1 over s so, which is nice because you recover a result which we have already seen.

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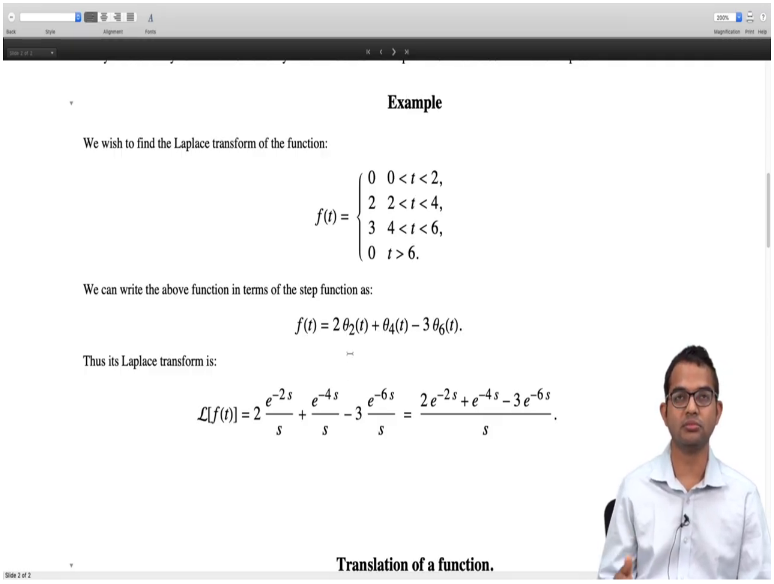

So, now let us look at a few examples. So, let us look at one example where this is made use of directly. So, the point is that this Heaviside step function appears in many contexts. And sometimes it appears in a more complicated form. So, we have worked out the bare form right. So, you can multiply it by some factor, you can add you know and so on like let us look at this example.

So, you have f of t is 0 from 0 to two and then it becomes 2 from 2 to 4 and stays at 2 from 2 to 4, and then it's 3 from 4 to 6 and then it drops back to 0 beyond t equals 6 right. So, a function like this you can imagine has actually more than one step right. The magnitude of the jump is also something that you can play with, you know and then you can have a switch in the positive direction, negative direction and so on right.

So, you can have a fairly complicated step function one can imagine. So, the key point is that you can rewrite this type of function in terms of multiple such Heaviside step functions. So, first of all we note that from 0 to it is just 0. So, all the action happens at t equal 2 and the size of the jump is 2. So, we write it down as 2 times theta 2 of t plus, then the next instant where something happens is at t equal to 4. And so, then we see that the size of the jump there is 1 right it goes from 2 to 3.

So, we just put theta 4 of t and so, if you are in doubt you can check what happens you know at a time slightly greater than 4. So, then you see that this is going to give us 1 and this is going to give us 2. So, it's going to be 3 and that is what the value should be between 4 and 6. And beyond time t greater than 6 for t time greater than 6, we want it to go back to 0. So, this 2 plus 3 will of course, add 2 plus 1 will give you 3 and so, you must subtract 3 so that you can bring it back to 0 alright.

So, for t less than 6 of course, this does not operate so, it does not matter and for t greater than 0 this minus 3 theta 6 of t will ensure that it is going to go to 0. So, this is a very compact, nice way of expressing the same function. And, now we use the linearity of the Laplace transform and the fact that we already know the Laplace transforms of each of these functions, we have worked it out above.

And using that we have the Laplace transform of this function is just 2 times e to the minus 2 s divided by s plus e times minus 4 s divided by s minus 3 times e to the minus 6 s over s, which can be written simply as 2 e to the minus 2 s plus e to the minus 4 s minus 3 times e to the minus 6 s the whole thing divided by s. So, that is our answer for this problem.

You can cook up your own you know functions involving many such steps right, write it in terms of these you know, many Heaviside step functions appropriately and then work out the Laplace transform right. So, this is the game you can play. So, now, let us look at the second type of you know scenario, which is useful and where we can work out the Laplace transform of.

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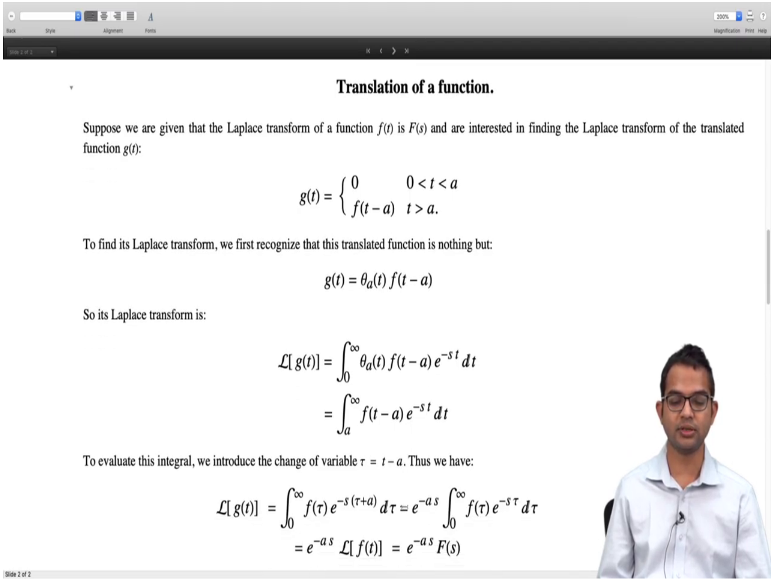

Suppose, we are given that the Laplace transform of some function f of t is F of s. And, we are interested in finding the Laplace transform of a translation of this function. So, your f of t is given, but you want to just shift this f of t to some you know future value of time.

So, in other words you are considering a function g of t which you know which is going to say which is just said to be 0 from 0 to a, a is of course, a positive quantity. And, then the function entire function whatever information was in f of t is still retained, but it is shifted by this amount it goes to f of t minus a. So, if you have a scenario like this, then we can find its Laplace transform and to do this we use this trick that in fact, g of t can be written in this you know compact form as this theta a of t times f of t minus a.

Because, we have just seen that theta is anyway going to be 0 all the way from 0 to a. So, it's going to give you the same function. And once we have this so, to find the Laplace transform, we have to invoke the definition 0 to infinity theta or a of t times f of t minus a times e to the minus s t d t. And then theta a anyway is 0 from 0 to a. So, it becomes just a to infinity f of t minus a times e to the minus s t d t.

And, now we do a change of variable so, it's convenient to define t minus s tau. So, thus we have the Laplace transform of g of t is equal to integral 0 to infinity f of tau e to the minus s times tau plus the whole thing multiplied by d tau.

So, we plot this e to the minus a s, and then now is this integral 0 to infinity f of tau e to the minus s tau d tau. But, immediately we recognize this as the Laplace transform of the function f itself. So, we get e to the minus a s times the Laplace transform of the function itself.

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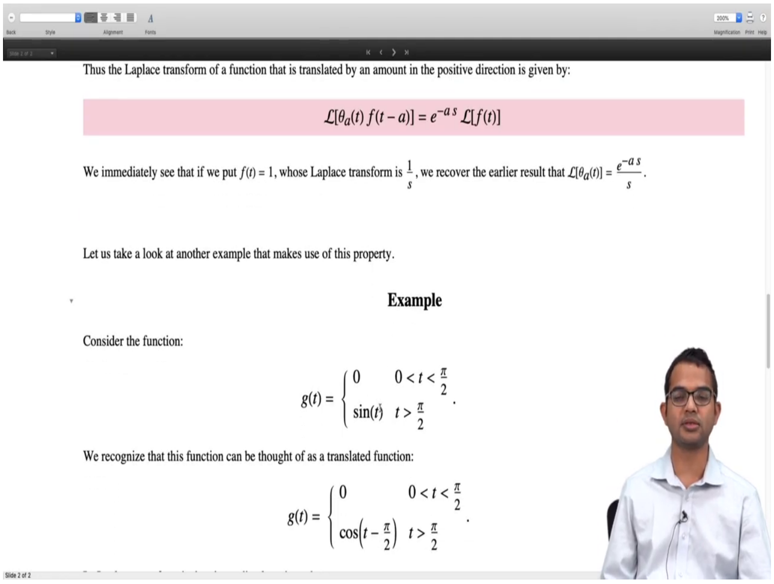

So, the Laplace transform of the translated function is simply given by e to the minus a s times the Laplace transform of the function itself right. So, basically the information contained in f of t is still there and in addition, we also have this you know stuff which comes from this shift or the translation is just this factor e to the minus a s right, it gets tagged along in the Laplace transform..

So, immediately we see that this like a sanity check we can do is, if you put f of t to be just 1 we already know that its Laplace transform is 1 over s. So, we recover the earlier result that you know if you put f of t is 1. So, you just get Laplace transform of theta of a is e to the minus a s times Laplace transform of 1 which is just 1 over s.

So, the Laplace transform of theta a of t is equal to e to the minus a s over s which is a result which we already obtained right. So, it is a sanity check. So, indeed it's all consistent. So, let us look at an example where this property is exploited. So, suppose we have a function like this g of t is 0 from 0 to pi by 2, and its sin of t for t is greater than pi by 2.

So, now we make this observation that in fact, sin of t is the same of as cosine of pi by 2 minus t, but cosine of plus theta and cosine of minus theta are the same so in fact, it is more convenient to write it here as cosine of t minus pi by 2. So, now, we immediately see that when you put it in this form so in fact, it is a translated function right its 0 from 0 to pi by 2, but then you have this you know this form of t minus pi by 2 is also exactly you know it matches with this a.

So, it is exactly in this form its f of t minus a for t greater than a and a is of course, pi by 2.

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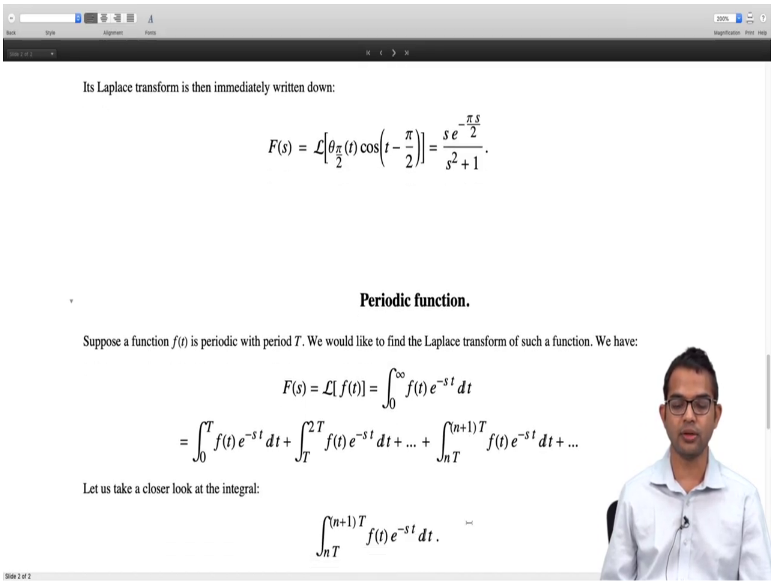

So, its Laplace transform is something that we can immediately write down right, using the property that we already have. So, F of s is Laplace transform of theta pi by 2 of t times cosine of t minus pi by 2 which is the same as e to the minus a s. So, in this case e to the minus pi by 2 times s times the Laplace transform of this function itself right f of t so, which in this case is s over s squared plus 1 right cosine of t.

So, we just have this factor e to the minus pi s by 2 times s over s squared plus 1, you know as simple as that we have to be careful that you have the correct function here, it should be f of t minus a for t greater than a ok. So, let us look at the third type of function which we want to discuss in this lecture, which is suppose you have a function f of t and which is periodic with period t and we want to find its Laplace transform.

So, we have F of s is you know by definition its from 0 to infinity f of t times e to the minus s t d t, but this time interval from 0 to infinity can itself be divided into you know intervals of length T. So, 0 to T T to T to 2 T 3 2 T 2 3 T so, on in general n T to n plus 1 T, and then we introduce the substitution. So, we look at the generic integral involved here.

So, it is an infinite sum. It's an infinite series of you know summing over many of these integrals, but the point is that all these integrals are connected right. So, in order to see that let us look at just one typical integral and T to n plus 1 T f of t to the minus s t d t.

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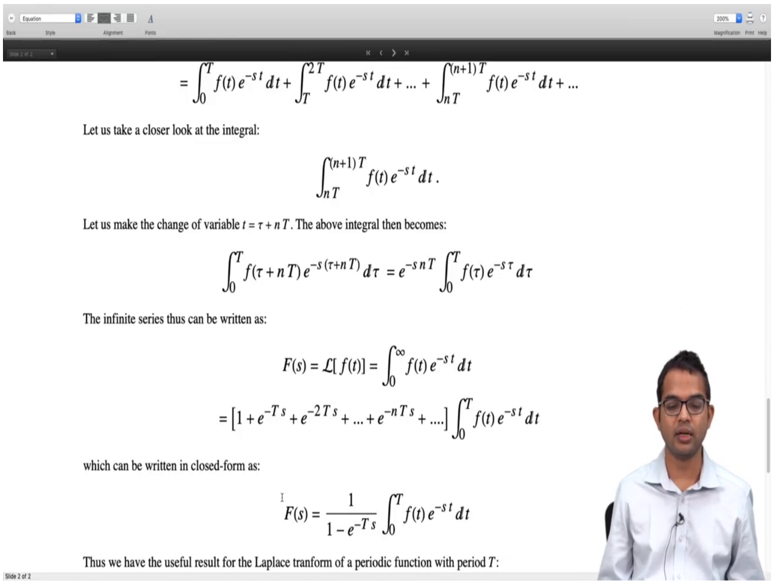

So, if we make the substitution you know if you introduce a change of variable t is equal to tau plus n T then, we see that tau will go from 0 to T. So, in place of f of t, we write f of tau plus n T and then in place of e to the minus s t we write e to the minus s times tau plus n T. And we have d t in fact, its d tau right d t and d tau are the same.

So, its more correct to write it as d tau. So, let us correct this. So, we have d tau and once again here also we have d tau ok. So in fact so, we notice that first of all f of tau plus n T is the same as f of tau. So, it and this factor e to the minus s n T can be pulled out. And, then we have left with this integral 0 to T f of tau e to the minus s tau d tau, which is nothing, but the Laplace transform of this function itself.

So, we have F of s is Laplace transform of ok. So, this integral 0 to T is not quite the Laplace transform, it just a its just a common integral that you get for every one of these terms. So, we will pull out this common integral. So, the Laplace transform is this integral from 0 to infinity f of t e to the minus s t d t. But, then we have you know these various factors that we can tag along to each of these, but all of them have the same factor which is this integral 0 to T f of t e to the minus s t dt.

So, it's like doing a Laplace transform, but within just one period right. So, the integral is carried out in just one period and then you have this infinite series, which is something which we can sum it's a familiar series.

And, each of these factors are you know less than 1 so, you can it's a convergent series. And, then we will get a closed form expression for this F of s is nothing, but 1 over 1 minus e to the minus T s times this integral 0 to T, within just one period f of t e to the minus s t dt.

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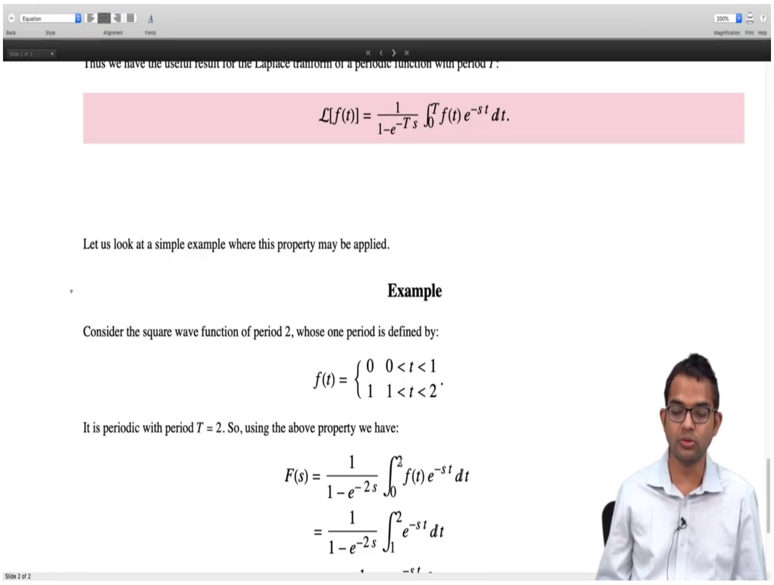

So, thus we have this very useful result. So, the Laplace transform of this periodic function with period T is simply given by 1 over 1 minus e to the minus T s times integral 0 to T f of t to the minus s t dt. So, let us quickly look at one example where this can be applied. So, suppose we have a square wave function. So, we have seen how to integrate how to take the Laplace transform of the cosine function of the sin function.

But, suppose we look at the square wave right, it starts at t equal to 0 and it remains 0 from 0 to 1 and then it becomes 1 between 1 and 2. And, then again it comes back to 0 and then it goes back to 1 and so on its periodic with period T equal to 2. So, using the property above we have the Laplace transform of this function is just simply 1 over 1 minus e to the minus 2 s times integral 0 to T f of t e to the minus s t dt.

But, f of t is non zero only in the interval 1 to 2. So, this integral from 0 to 2 2 is reduced to an integral from 1 to 2 and we can go ahead and evaluate.

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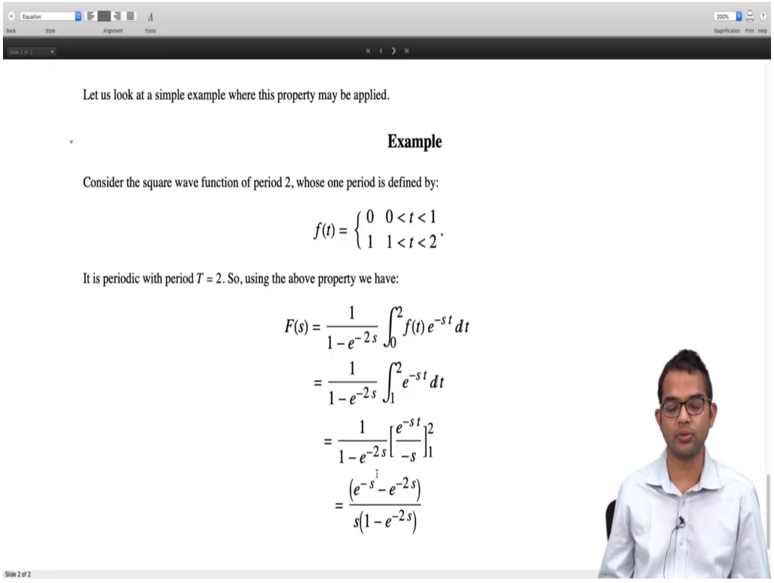

So, we have 1 over 1 minus e to the minus 2 s times you know this stuff e to the minus s t divided by minus s evaluated between 1 and 2. So, plugging in for T equal to 2 and T equal to 1 here and subtracting, we have e to the minus s minus e to the minus 2 s divided by s times 1 minus e to the minus 2 s right so, which is the Laplace transform of the square wave right.

So, you can compare this with the Laplace transform of the sinusoidal periodic function. In fact, you can try to use this method to compute the Laplace transform of the sine function or the cosine function. But, in the end the labor involved will not be significantly reduced.

If you use this technique, because you still have to do this integral involving you know cosine of t times e to the minus t or sin of t times e to the minus s t and so, probably some kind of integration of parts will anyway have to be carried out. But, here you see that for the square wave you have this you know nice simplification, and then you have this final answer ok. So, that is all for this lecture.

Thank you.