**Mathematical Methods 1**
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**Ordinary Differential Equations**
**Lecture - 78**

# Basic Properties of Laplace Transforms

So, we have defined what a Laplace Transform is and based on the you know this first principle you know the definition of the Laplace transform which will we manage to compute Laplace transforms of several standard functions. In this lecture we will look at certain properties of Laplace transforms and how you know these properties can be exploited to work out.

The Laplace transforms of you know other functions which may be more complicated or in some time in some cases you know there are we will show how Laplace transforms of functions which could be worked out directly from first principles may also be obtained using these properties of Laplace transforms. That is the content for this lecture ok.

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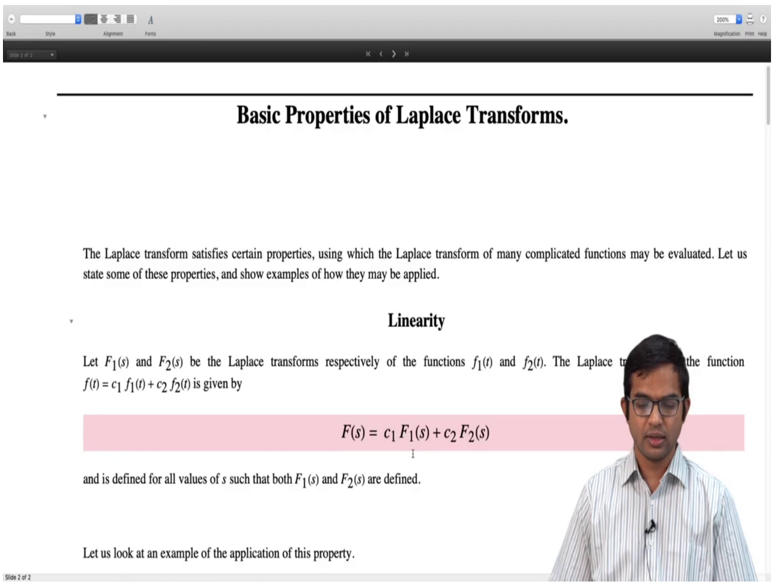


So, first is linearity right so what does Laplace transform do? it takes a function and gives you another function right. So, it is a function of s where the value of s is going to be restricted as we have seen several examples. Now given 2 functions f 1 of t and f 2 of t and both of them have Laplace transforms F1 of s and F 2 of s then you know the Laplace transform is a linear operation. So, if you take the Laplace transform of some linear combination of these 2 functions c 1 f 1 of t plus c 2 f 2 of t, then the Laplace transform of this linear combination of functions is in fact simply given by c 1 F1 of s plus c 2 F 2 of s that is c 1 times Laplace transform of the first function plus c 2 times the Laplace transform of the second function right.

So, the Laplace transform of a linear combination of functions is the linear combination of the Laplace transform of the functions right. So, this is the property of linearity right. So of course, you have to ensure that you know now the convergence is guaranteed only you know for those values of s for which both f 1 and f 2 are well defined right.

So, in general this will extend so you can club together many such functions add them all up and then s is going to be restricted by you know the most restrictive of all of them right. So, that is how linearity plays out.

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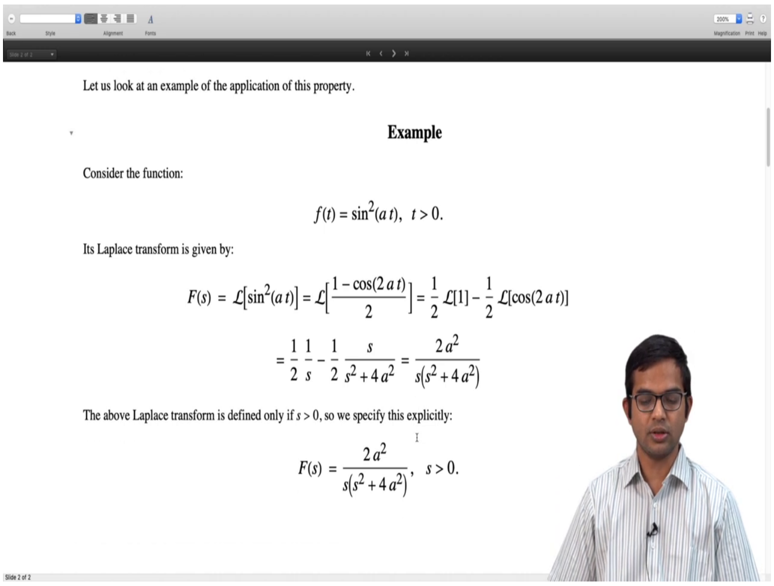


So, let us look at an example where we can apply this property. So, we have this function f of t is equal to sine square of at, t greater than 0. Now suppose we want to find it is Laplace transform, so we see that F of s is Laplace transform of sine squared of at, but sine squared of at can be written as 1 minus cosine of 2 at by 2. Now so you can think of this half times the function 1 minus half times another function cosine of 2 at. So, using this linearity property I can write this down as half times Laplace transform of 1 minus half times Laplace transform of cosine a 2 at.

But Laplace transform 1 is something which we already know - it is just 1 over 2 times 1 over s it gives you and then minus one over a half times we also know the Laplace transform of cosine of 2 at it is just s over s squared plus 4 a squared. So, if I club these 2 together then I get one over s I can pull out ah.

So, I can write it as s times s squared plus 4 a squared in the denominator and then I will get s squared plus 4 s squared minus s squared. So, that will be 4 a squared will cancel with one of the 2's and so I get 2 a squared divided by s times s squared plus 4 a squared right. So, it is some simple algebra.

So, the point is that using this apparently very naive property we will we managed to work out the Laplace transform of a function using the Laplace transform of a function which we already know. So, the answer is for this problem it is 2 a squared by s times s squared plus 4 a squared with the condition that s should be greater than 0.

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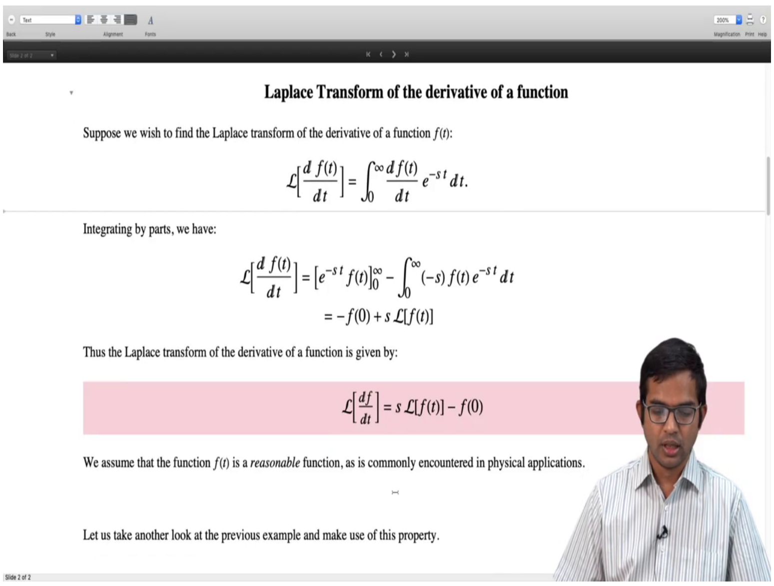


Now let us look at another property of Laplace transform, if you take the Laplace transform of the derivative of a function right. So, suppose you know the Laplace transform of some function f of t and you take the Laplace transform you are interested in the Laplace term of the transform of the derivative of this function right.

So, in other words you want to know integral 0 to infinity df by dt times e to the minus s t d t. So, let us integrate this by parts. So, we now have the natural you know function to treat as v. So, in the terminology of integration by parts you have something like u d v.

So, here u is going to become e to the minus s t and d v will be df by dt right, so it is standard integration by parts. So, you have e to the minus s t times f integral from 0 to infinity minus v d u you have to do integral 0 to infinity minus s f of t e to the minus s t right. So, this comes from taking a derivative of this function e to the minus s t. So, you get a minus s and so it is convenient to write it like here.

And so now you know only one of these boundary terms will contribute. So, at t equal to infinity so we assume right, so here I have certainly made the assumption that you know f of t is I have said that it is a reasonable function. So, it is the kind of function which will die down sufficiently fast. So, that at t equal to infinity this is going to be you can take it to be 0 and at t equal to 0 you just have minus f of 0 right. So, this e to the minus s t will give you 1, so you have minus f of 0.

So, plus s times the Laplace transform of f of t. So, the key point is that the Laplace transform of the derivative of the function is simply given by s times the Laplace transform of the function itself minus f of0 right. So, the value of the function at the starting point matters for this when you are taking the derivative or you know Laplace transform the derivative.

So, like I said without going into any details or you know ah doing a careful study of the properties of this function f of t, I am just saying that we assume f of t to be a reasonable function where this works out right. So, which is justification for you know these kinds of um operations will be just based on how you know it works out for us right. So, we will just take functions which are reasonable and work it out and see that it works that is all. So, we do not go into the nitty gritty of proving these statements and so on right.

So, that would be for a more advanced or a more abstract course which is outside the scope of our present attempt right. So, let us learn to use these properties ok.

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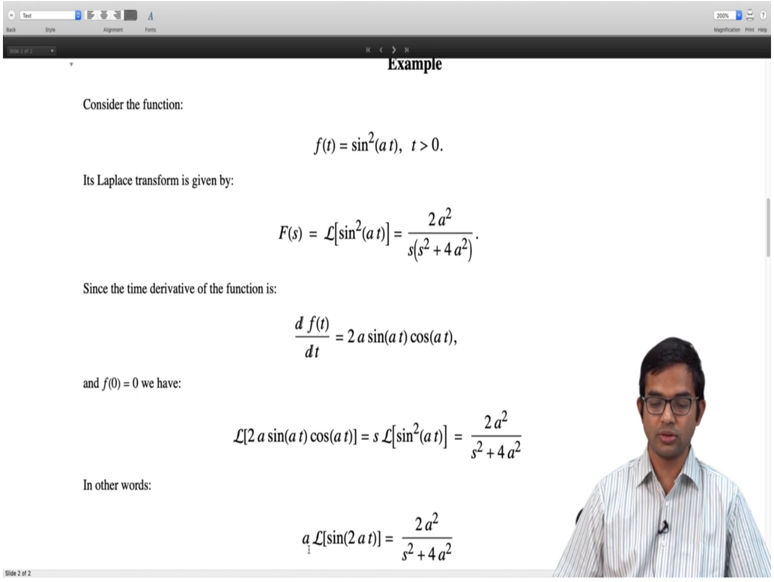


So, let us take another example and see how this plays out. So, example is again we can take the same function f of t is sine squared of at for t greater than 0. Now if we take a derivative of this function you see that I mean we already know that the Laplace transform of this function we worked it out and we saw that it is 2 a square divided by s times s squared plus 4 a squared.

So, if you take a derivative of this function it is 2 a sine of at times cosine of at and also we observe that the value of this function at t equal to 0 is just 0. So, if we want to work out the Laplace transform of the derivative in other words Laplace transform of 2 a sine of into at times cosine of 2 at, it is just going to be s times the Laplace transform of the function itself. Which is very simply written as 2 a squared by s square plus 4 a squared right.

So, but now we observe that in fact this 2 a sine of at times cosine of at can be written as you know there is a factor of a which I can pull out which is linear operation. So, you know constant factors will come out without any issues, so but 2 sine 2 at times 2 sine at times cosine of at is the same as sine of 2 at. So, what we have managed to show is you know a times the Laplace transform of sine of 2 at is equal to a squared by s squared plus 4 a squared.

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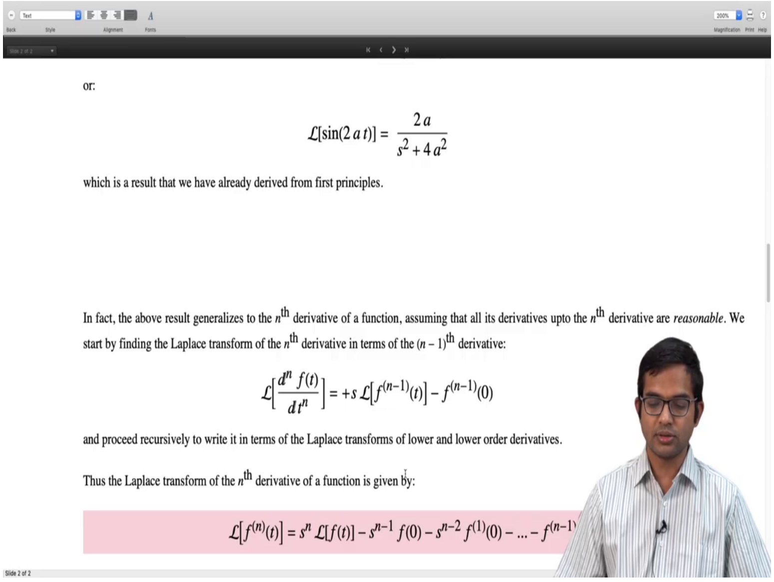


So, if I cancel a on both sides, so I am left with you know the Laplace transform of sine of 2 at is equal to 2 a by s squared plus 4 a squared, so which is not a surprise right. So, this is something that we already know right, because we know that the Laplace transform of sine of b t we worked out is just b divided by sine squared s squared plus b squared. So, in place of b we have 2 a here right. So, this is just an alternate way of seeing something which we already know ok.

So, let us look at how you know this idea can be generalized. So, you do not have to work out just the Laplace transform of the first derivative; you can look at the second derivative, the third derivative and in fact the nth derivative.

So, if you take the Laplace transform of the nth derivative you see that you we use the same logic as just above and say that it is equal to s times the Laplace transform of the n minus 1 th derivative, because this is like taking this first derivative of this function f n minus one of t is d n by dt n of f of t. So, minus you know the value of this the n minus 1 th derivative of this function at t equal to 0.

So now, but we can go ahead and you know apply the same ah same formula if you wish or you know the same logic in a recursive way. So, again you apply it to n minus 1 th level then apply to n minus 2th level n minus 3 th level and so on.

So, in the end you can convince yourself that you are going to get this result, which is that the Laplace transform of the nth derivative of a function f of t is you know s time s to the power n times Laplace transform of f t minus s to the n minus 1 times f of 0 minus s minus s to the n minus 2 times the value of the first derivative of the function at 0 and minus so on; all the way up to minus you know the value of the function the nth n minus 1th derivative of the function at 0 right.

So, this is something that you can quickly cross check and convince yourself of right.

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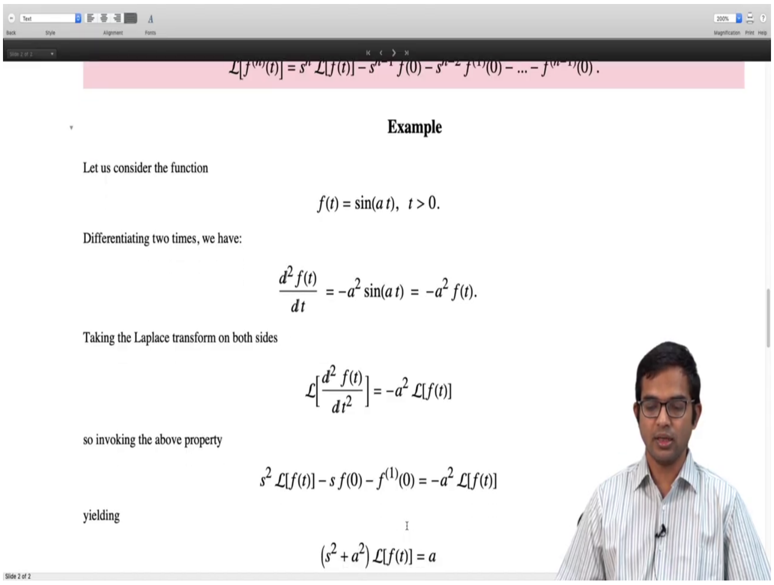


Once again I have assumed that not only is the function reasonable, but all its derivatives up to nth derivative or all reasonable right. So, that we do not run into any you know technical issues with regard to whether some particular order derivatives Laplace transform whether it is defined or not and so on right.

So, let's agree to not worry about these technical issues at this point. So now we look at an example. So, let say you have this function f of t is equal to sine of at greater than 0. So, the second derivative so we have already worked out the Laplace transform of this function, but let us look at it from a different perspective.

So, if I take a derivative once I get a times cosine of at, then if I take another derivative then I get minus a squared sine of at which is the same as saying minus a squared f of t. So, if I take the Laplace transform of this equation on both sides I have the Laplace transform of the second derivative is equal to minus a squared I have invoked linearity minus a squared times Laplace transform of f of t.

But the Laplace transform of the second derivative according to this property that we just discovered it I can rewrite the left hand side as s squared times Laplace transform of f of t minus s times f of 0 minus the first derivative of this function evaluated at 0 which is equal to minus a squared l of f of t.

Now, if I so f of 0 itself is 0, but the first derivative if I take it then I get cosine of cosine of at times a and if I put t equal to 0 I will just get a, so I bring this a to the right hand side. So, I have s squared plus a squared times Laplace transform of f of t is equal to a.

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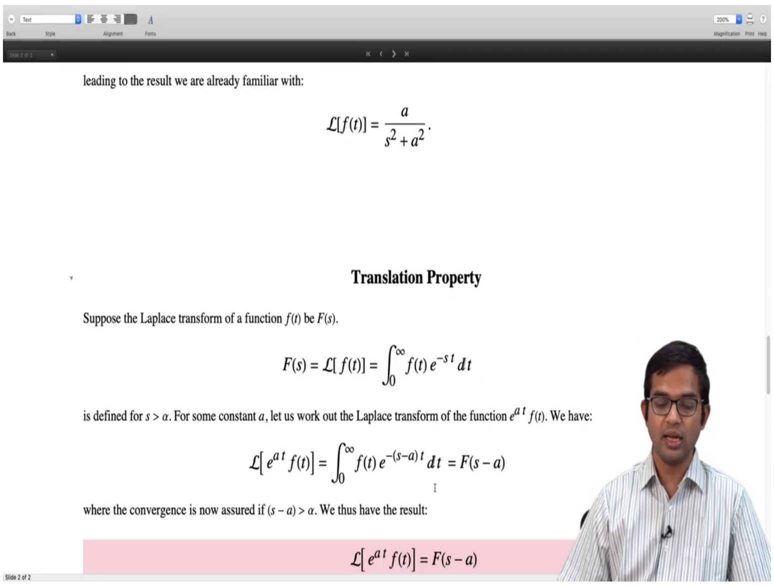


And immediately we have this result: Laplace transform of sine of at is a by s squared plus a squared which is something we have worked out already right. So, this is just an illustration of how we can use these properties and many times the evaluation of Laplace transforms can be made very efficient. If you if you use the right property for the right situation ok

So, let us look at another property. So, this is the Translation property. So, suppose we have the Laplace transform of function f of t and its defined by this equation and its defined for some s greater than alpha right. For some constant a if we work out the Laplace transform of this function e to the at times f of t. So, we see that we have a to e to the at. So, in place of e to the minus s t it becomes e to the minus s minus at. So in fact, immediately we see that this is going to be a shifted version of the Laplace transform of the first function.

So, you get there is a translation of the Laplace transform which happens. So, which is f of s minus a and now convergence is assured if s minus a is greater than alpha.

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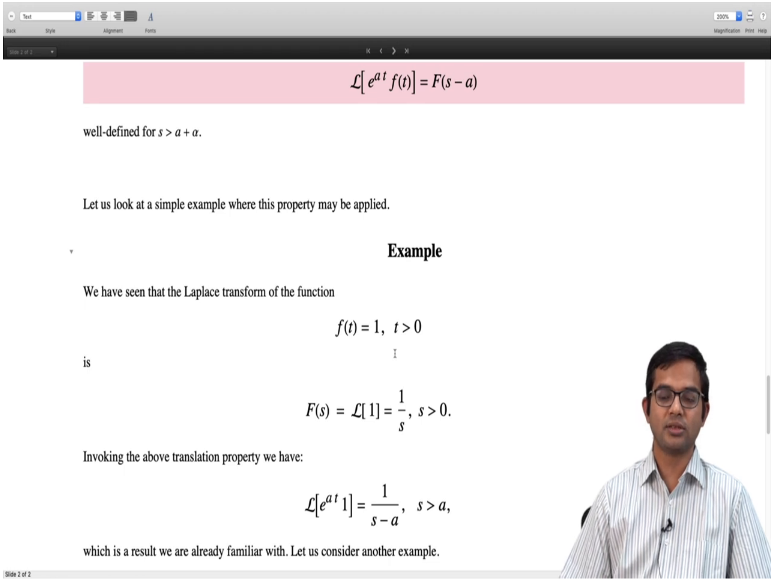


So, we have this result: the Laplace transform of this you know e to the at times f of t is equal to f of s minus it is just a translated you know version of the Laplace transform of the original function itself. And now convergence being guaranteed is if s is greater than a plus alpha ok. Let's look at an example where this is applied. A very simple example if you just take f of t equal to 1 for t greater than 0 this function is just constant.

Now, if we take the Laplace transform of this function we have seen that this is one over s for s greater than 0. So now, if I multiply the original function with e to the at then invoking this property I have 1 over s minus a and this is defined for s greater than a. But this is a result which we have directly worked out. So, the Laplace transform of e to the at is indeed 1 over s minus a s greater than a right. So, this is something which you are familiar with.

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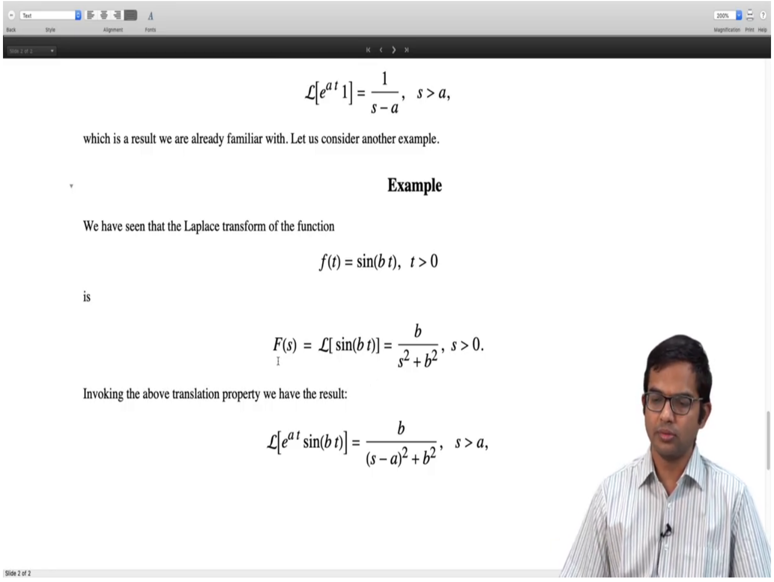


Now let us look at another example: if I have sine of b t, then we know that its Laplace transform is b over sine squared plus b squared. Now if I multiply this function with e to the at times sine of b t, then I get b over instead of s I have to write s minus a. So, s minus a the whole squared plus b squared and now s is restricted to be greater than a right. So, this is another example where this is applied ok.

So, you can play with this. You can create your own interesting functions and find the Laplace transforms using these properties.

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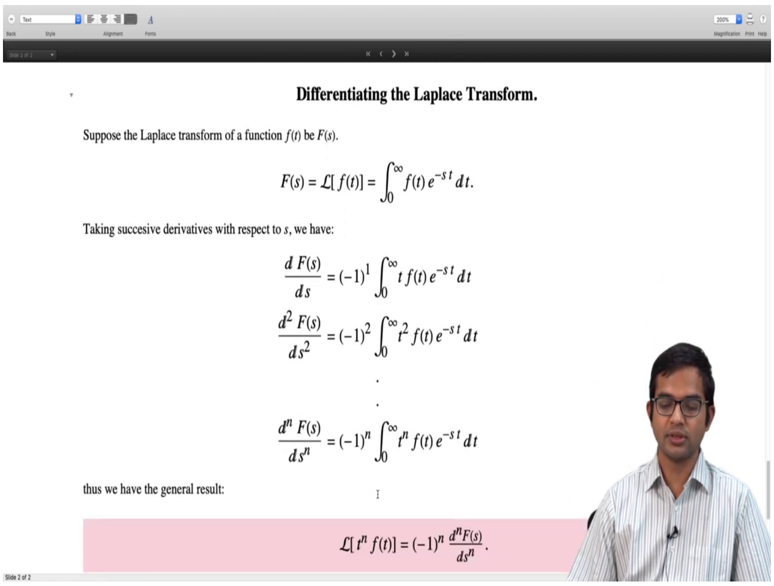

So, there is one more property which I want to describe which is the differentiating the Laplace transform itself. So, we saw f of s is given by this integral right and if I take successive derivatives of this function the Laplace transform with respect to s. So, we have you know the first derivative will give me a minus 1 times e to the minus s t times t. So, every time I take a derivative I will get a minus t out.

So, if I collect all the minus signs outside here, I write it as minus 1 to the 1 integral 0 to infinity t times f of t e to the minus s t d t. So, you see that now this is looking like the Laplace transform of t times f of t, if I take a second derivative I get minus 1 squared of the whole squared times the Laplace transform of t squared f of t.

So, in general you can continue like this and we have this result: the Laplace transform of t to the n times f of t is minus 1 to the n the nth derivative of f of s right. So, this is also a very useful property.

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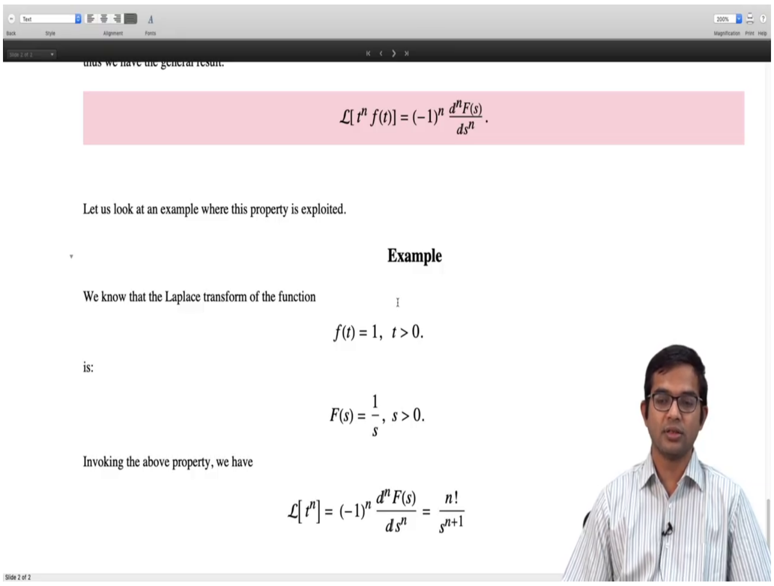


And you can exploit it to work out interesting Laplace transforms of functions of interest.

So, let us look at one example that is very simple, so we know that f of t is equal to 1 has a Laplace transform 1 over s. So, if I take a derivative of this function then that must correspond to t right. So, indeed that should measure and also I have to put in this minus sign.

So, 1 over s squared right we have already seen this happen. So, in general in fact we can work out the Laplace transform of t to the n right. So, it is going to be just minus 1 to the n times the nth derivative of f of s which in this case we just turn out to be n factorial divided by s to the n plus 1 right.

We have already seen that if you have t you will get 1 over 1 over s squared. So, if I put n equal to 1. So, I will get 1 over s squared right. You can look at you know higher powers, but basically we have a general expression directly as a consequence of this result involving the derivatives of the Laplace transform ok. So, that is all for this lecture.

Thank you.