Mathematical Methods 1

Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Ordinary Differential Equations
Lecture - 77
Laplace Transforms

So, there is a class of transforms which takes you know functions and gives you other functions, right. We have seen one such transform namely the Fourier transform, it came out naturally and then we were looking at Fourier series, and then we looked for a continuous generalization of the idea of Fourier series and we had a Fourier transform, right.

So, in this lecture, we look at another kind of transform which is called the Laplace Transform. It is very frequently used by electrical engineers and other kinds of engineers. It finds applications in you know many physical problems and applied sciences.

In our discussion, we are interested in making use of Laplace transforms to solve you know ordinary differential equations of a certain kind. And in particular it helps out when you have initial value problems. So, that is coming up a little bit later. But in this lecture, we will discuss what a Laplace transform is and then look at a few examples, ok.

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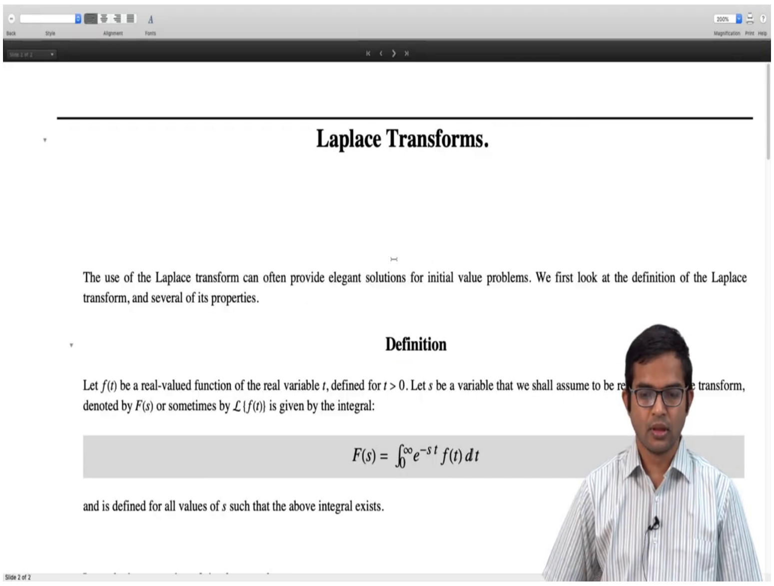

So, let say that you have some real valued function of a real variable t, defined for you know t typically you think of it as time, right. And so, it is a signal of some kind which depends on time. And so, this is something which starts at time t equal to 0, right. So, and so, it is in this context that you know this is useful, right.

So, and so, let us consider s to be some variable which we will take it to be real, right. So, it is possible to consider more general versions where you allow s to be complex and so on. But for our purposes let us say that s is a real variable and so, we define the Laplace transform as this function F of s, right, which is another real function of a real variable which is given by this integral, right.

So, F of s is equal to integral 0 to infinity e to the minus st f of t dt. It looks somewhat like your yeah you know Fourier transform in the sense that it takes. So, there is this exponential factor involved along with f of t. But here now we have restricted s to be real and also the key point is that it goes from 0 to infinity, right. So, this gives you another function F of s, right.

So, you can think of this s as some you know coefficient these e to the minus s, these are weights that you associate with function and you know you look at you know every instant of time from 0 all the way up to infinity. In some sense it captures a global aspect of this function, but for every value of this s, the weights will change and depending upon how you know the weights change and the value of F of s will change.

So, in some sense you are taking all the information contained in f of t and representing in a different way that is represented in terms of this function which is called F of s which goes by the name of a Laplace transform, right. So, it is important to emphasize that you know this is a meaningful definition of a Laplace transform if this integral exists.

So, we must restrict our values of s such that this integral exists, right. So, one can come up with a sort of very sophisticated mathematically intense treatment of you know the converges and so on. But for our purposes let say we deal with you know physically irrelevant types of functions f of t, right, such that; you know this integral will first of all be well defined. So, the only part that we have to take care of is that s lies in an allowed range.

So that this integral converges. So, we will take this function f of t to be you know functions which are; which I will just broadly call reasonable functions, right. So, let us, we will not get into some technical discussion of you know what are the mathematical properties that f of t must satisfy, so that F of s is defined and so on. So, that would be for a mathematically rigorous course, which is not this, ok.

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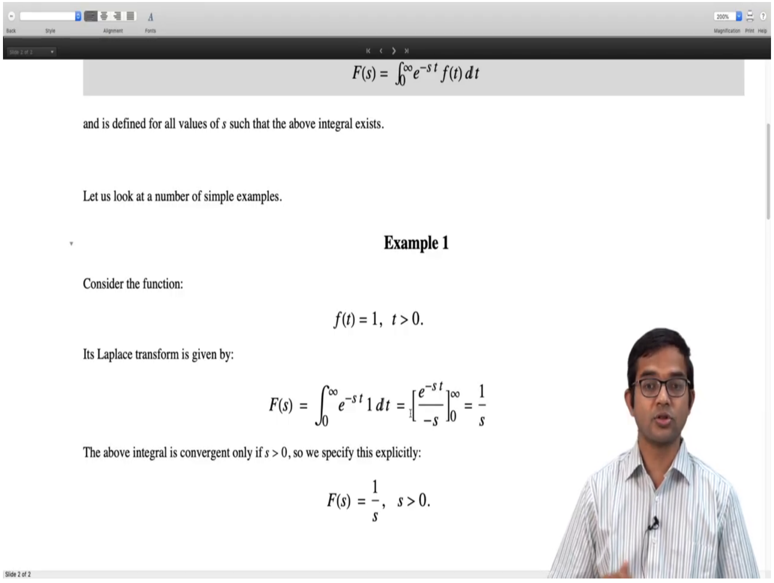

So, let us look at a number of simple examples, right. So, that is what our philosophy is to define a concept, look at a number of examples, and try to apply, ok. So, let us consider this very simple function. So, you may have a function like f of t equal to 1, so it is profitable to think of this f of t as a driving function, right.

For example, so you have some external force that you are providing to your harmonic oscillator let say and so, you can think of a force which is just constant in time f of t equal to 1, for all t greater than 0. And its Laplace transform is given by you know just this integral is very easy to do, so 0 to infinity e to the minus s t times 1 times dt, which will just turn out to be 1 over s, right.

So, it is important to check that this integral is a convergent integral, it is a meaningful integral only if s is greater than 0, right. So, you want this exponential to decay as t goes to infinity and when s is greater than 0, indeed F of s is 1 over s. If s is less than 0, then this is not going to converge, so yeah. So, we restrict s to be greater than 0, for this very simple example.

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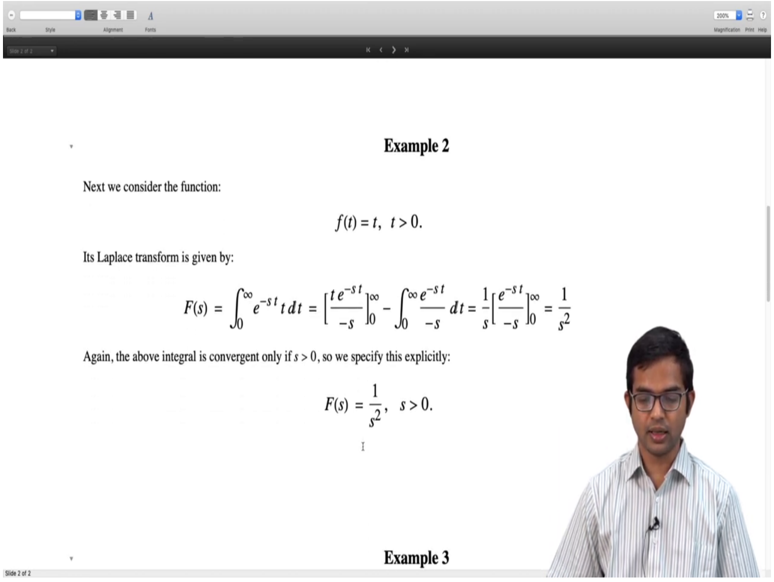

So, let us look at another example. So, suppose we consider the function f of t is equal to t, right, so you may you know put in an external force which goes linearly with time it keeps on increasing. So, now what happens to the Laplace transform of this function? So, once again it is an integral, so it is e to the minus st times t times dt.

So, this is something that can be calculated using integration by parts. So, you have u d v, so you put in u v, so t times e to the minus st divided by minus s from 0 to infinity minus v d u. So, you have e to the minus st by minus s and then when you take the derivative of t you just get a 1. And so, at t equal to infinity this vanishes at t equals 0 also this function because of the presence of this t it goes to 0.

So, the boundary terms go away. And then, we are left with just you know this minus and minus become plus. So, you get a 1 over s. Then, you have to integrate this one, so once again, this is an integral which we have already done, right.

So, you can directly use that in fact, so this is nothing, but the Laplace transform of 1, the function one which we already did. So, it is 1 over s times 1 over s which is 1 over s square. So, once again it is just you know it is convergent if s is greater than 0. So, we must restrict s to be positive, ok.

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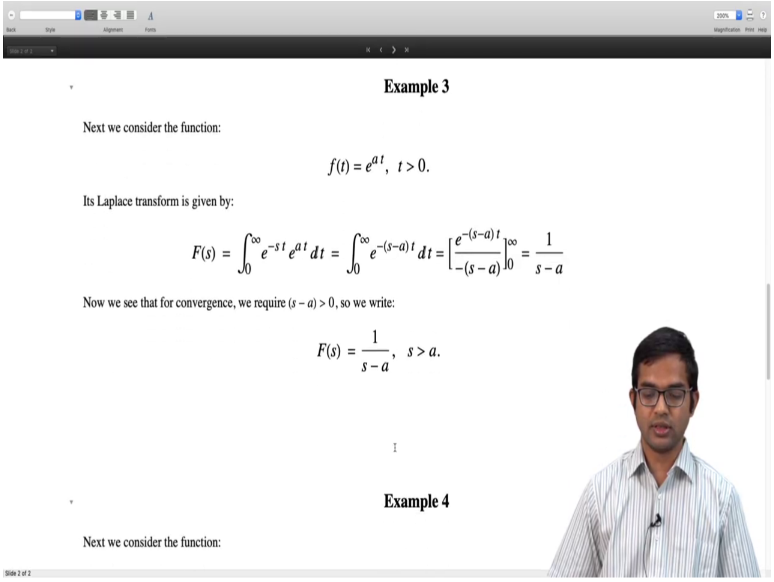

Now, we have the next example which is slightly more complicated. You may imagine turning on a function, which is not just growing with time and or linearly, but in fact, it is going exponentially with time. So, if it is something like e to the at, where for t greater than 0, yeah it is understood that f of t is 0 for t less than 0 or it does not matter anyway.

So, we are interested in the functional information only for times greater than 0. So, F of s here is going to be e to the minus st times e to the a t dt. So, then we write it as e to the minus s minus a times t dt. So, now you can see that basically what has happened is it is like taking a Laplace transform of 1, but now instead of s you have s minus a.

So, the answer is just going to be 1 over s minus a. And then, now the convergence has shifted, so it is not f has greater than 0, but s greater than a, right. So, you cannot have a meaningful value for this function F of s unless s is greater than a. So, that is the restriction on s in this example, ok.

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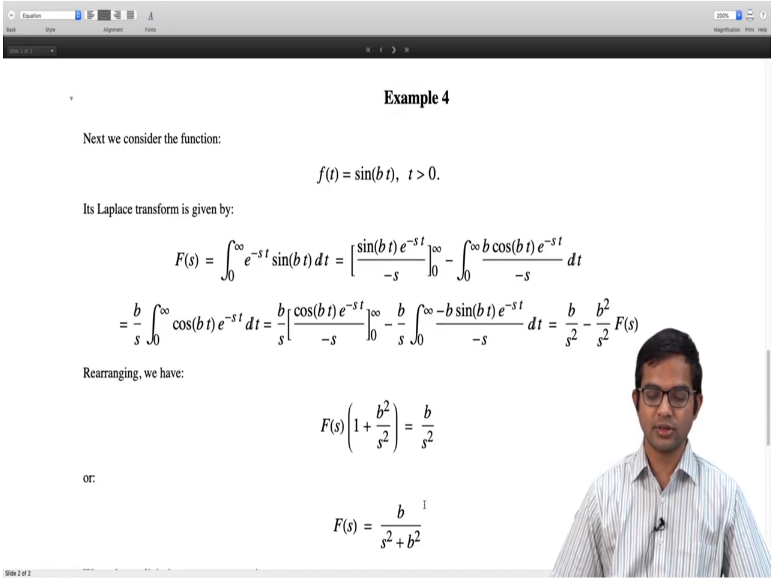

Now, let us look at one more example. So, suppose we consider a sinusoidal function, right. So, it is very common to try to drive your system with a sinusoidal drive, right. We have looked at examples of this kind. And we were looking at inhomogeneous differential equations and looking at the driven harmonic oscillator problem. So, if you have a sinusoidal drive, if you take the Laplace transform of this function you get, yeah; so, a little more work is involved here.

So, e to the minus st sin of bt. So, again we do it by parts. So, it is u d v, so you get sin of bt times e to the minus st divided by minus s, 0 to infinity, minus integral 0 to infinity you have to take the derivative of this, so you get b cosine of bt times e to the minus st divided by minus s. So, once again the boundary term vanishes both at both ends t, i equal to infinity e to the minus still vanish and at t equals 0 sin of bt will go to 0.

So, basically, this can be ignored. So, then you have a minus and minus becoming plus, so you have plus b by s integral 0 to infinity cosine of bt times e to the minus st. So, basically this is a Laplace transform of the cosine function. If we had known this we could directly plug this in, but that is also something which remains to be evaluated. So, therefore, we will integrate by parts once again.

So, you have b by s cosine bt e to the minus st divided by minus s, from 0 to infinity minus you know b by s remains as it is, then you have an integral 0 to infinity minus b sin of bt times e to the minus st by minus s. So, the derivative will give you a minus b and then you have this minus s coming from this other function. Therefore, if you collect all these terms carefully, the boundary term does not vanish at both ends.

So, you have at infinity of course it goes to 0, but at 0 you have to be careful and put in b by s squared. So, b by s and then this gives you 1 over s, so it is b by square. And then this integral is something that is minus b squared over s squared times the Laplace transform of this sin of bt itself.

So, in fact, we have managed to show that the Laplace transform of this function is equal to b by s squared minus b square by s square times the Laplace transform of the same function, right. So, basically we can rearrange and then we have F of s times 1 plus b squared over s squared is equal to b over s square. So, we have F of s itself is just some simple you know algebra after this and you get b over s squared plus b squared, right.

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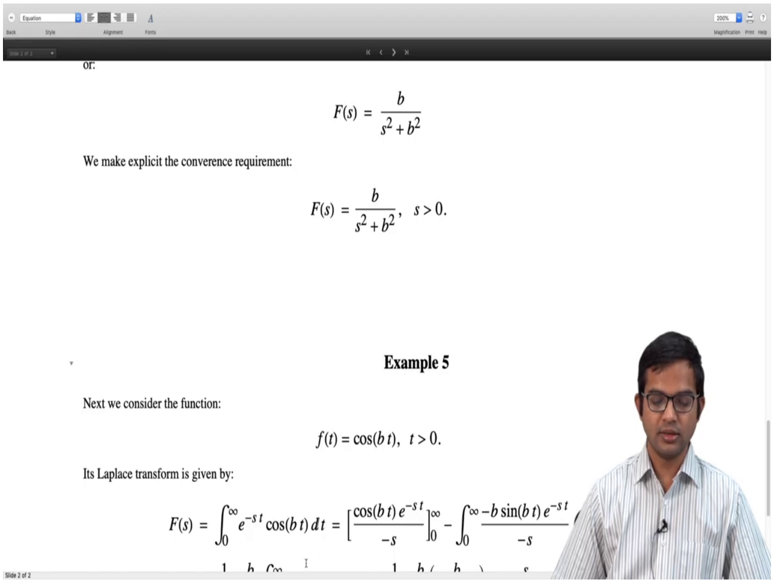

So, this is a convergent integral as you can check, right. So, we have used this during you know when we were doing this integration by parts. So, this is a convergent integral if s is positive. If s for, s greater than 0, F of s is just given by b over s squared plus v squared. So, let us look at one more example which is actually a close cousin of this problem, which is you know if you derive your function, if you have a function f of t which is cosine of bt.

So we have seen that sines and cosines are basically the same, you know some small details are going to be different, but let us work this out. So, its Laplace transform is given by 0 to infinity e to the minus st cosine of bt dt.

Now, again we start by doing this by parts, so we have cosine bt e to the minus st divided by minus s 0 to infinity, minus integral 0 to infinity. If you take the derivative you get minus b sin of bt, then you have to put in minus e to the minus st divided by minus s.

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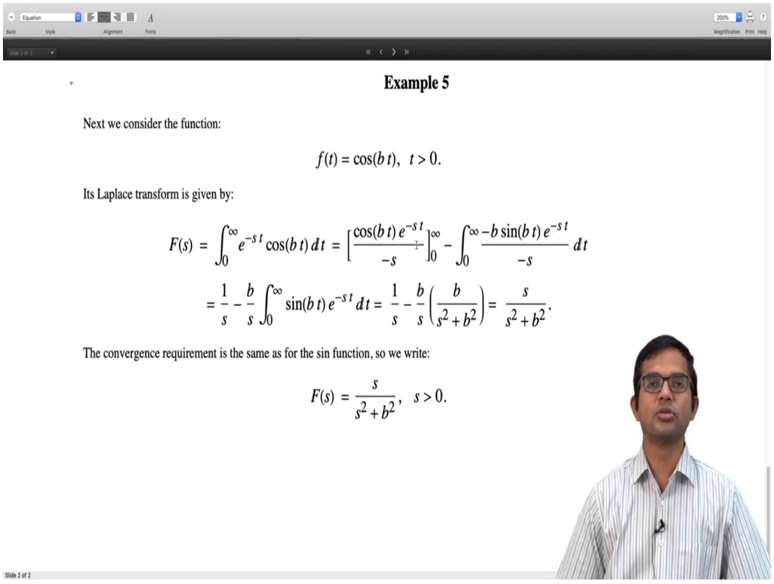

So, now this boundary term does not quite vanish at both ends. It is at t equal to infinity it does, but at 0 you have to write it as 1 over s minus b by s. So, minus n minus cancels, but there is one more minus sign, so you have minus b by s times the Laplace transform of sin of bt which we already have worked out.

So, we can directly get that answer from there instead of doing one more integration by part. So, we will just write it as 1 over s minus b by s times b by s squared plus b squared. And some simplification from here and we get s over s squared plus b squared.

So, we recall that, it was essential that s should be greater than 0 for the Laplace transform of sin of bt to be meaningful, and so the same condition also holds for the Laplace transform of cosine of bt to be meaningful.

So, we see that sine and cosine, you know the Laplace transforms look very similar to each other. You know you have 1 over s squared plus b squared in both cases, but one of them has a factor of b and the other one will have a factor of s. So, that is all for this lecture.

Thank you.