Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

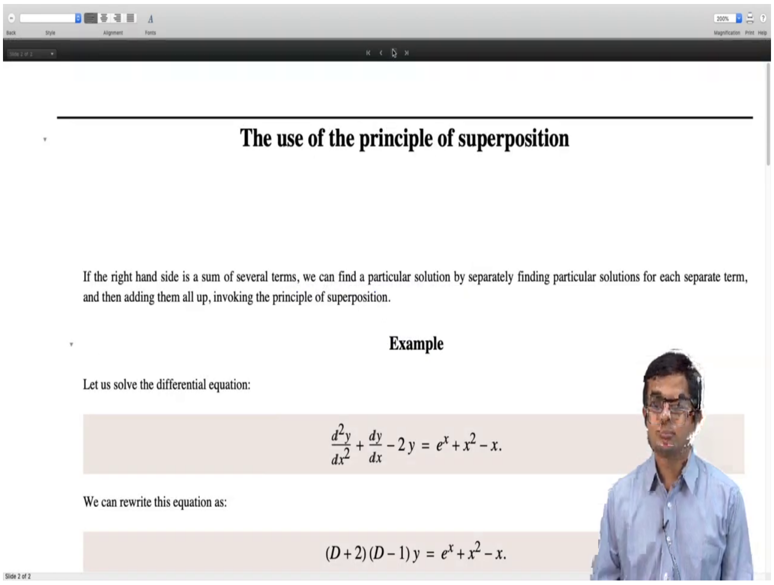
Ordinary Differential Equations
Lecture - 76
Linear superposition

So, in this lecture, we will see how the principle of Linear Superposition can be used to string together rather complicated stuff which can appear on the right hand side of our differential equations, right.

So, we have the you know stuff which we call the homogeneous the stuff which goes to form the homogeneous equation which appears on the left hand side and then if it is equal to some function of the independent variable on the right hand side, then it is a it is an inhomogeneous differential equation, right.

So, in this lecture we will see how the linear superposition principle can be invoked to work out the particular solution for rather complicated kinds of drives which can be applied on the right hand side, ok.

(Refer Slide Time: 01:08)



So, the principle of superposition is just that you know you have linear systems here. So, therefore, you can take the solution of one external drive and the solution of another external drive and just add the two that are going to be a particular solution for the sum of these two drives right.

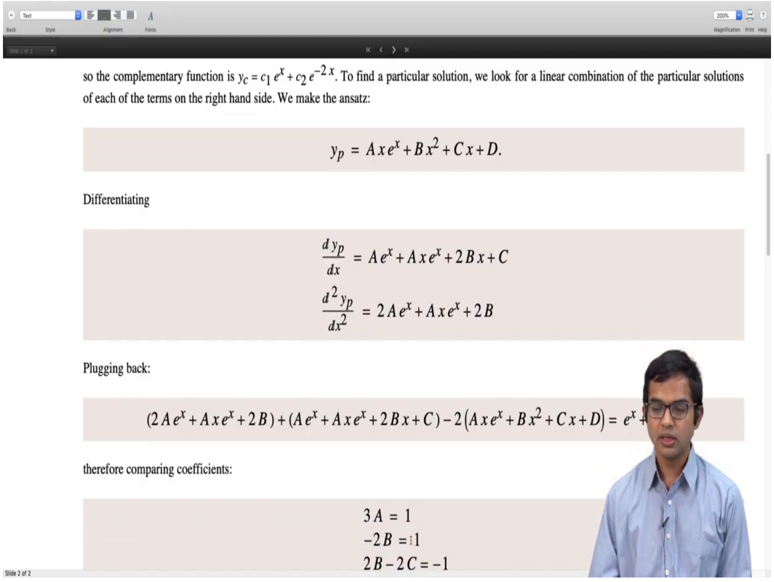
So, in fact, you can have as many different terms as you want on the right hand side. So, it is a linear problem therefore, you can just add all of them and you will get a particular solution for the full problem. So, since we have you know we have laid down a prescription for obtaining particular solutions for you know a whole class of you know inhomogeneous terms, we can add in all of these kinds of different kinds of inhomogeneous terms on the right hand side and still work out a particular solution right.

So, working out of the particular solution itself is usually the hard problem. So, the homogeneous part we know how to do, right. So, ok so, well, I mean we have seen other ways of getting to it, suppose you have you know the variation of parameters and so on, but this is another approach, right.

So, you will see that depending upon the situation you know one technique may be better or another one ok. So, here let us look at an example right we have probably tried out similar problems in the past, but let us look at one example here which is you know d squared y by dx square plus dy by dx minus 2 2y on the left hand side and this is equal to e to the x plus x squared minus x on the right hand side.

So, the right hand side is a little more complicated. So, the first step of course, is to rewrite the left hand side you know factorize it and rewrite it as D plus 2 times D minus 1 acting on y is equal to e to the x plus x squared minus x the right hand side appears as it is. So, there is this you know the linear operator acts on y to give you this complicated you know external drive on the right hand side.

(Refer Slide Time: 03:23)



So, the complementary function is of course, very straightforward it is just simply c 1 times e to the x plus c 2 times e to the minus 2x. Now, to find a particular solution we must look for a linear combination of the particular solutions of each of the terms on the right hand side, right. So, we can directly make a notes either you can start and say ok let me first work out the solution for just e to the x then work out for just x squared, then work out for minus x and then add them all.

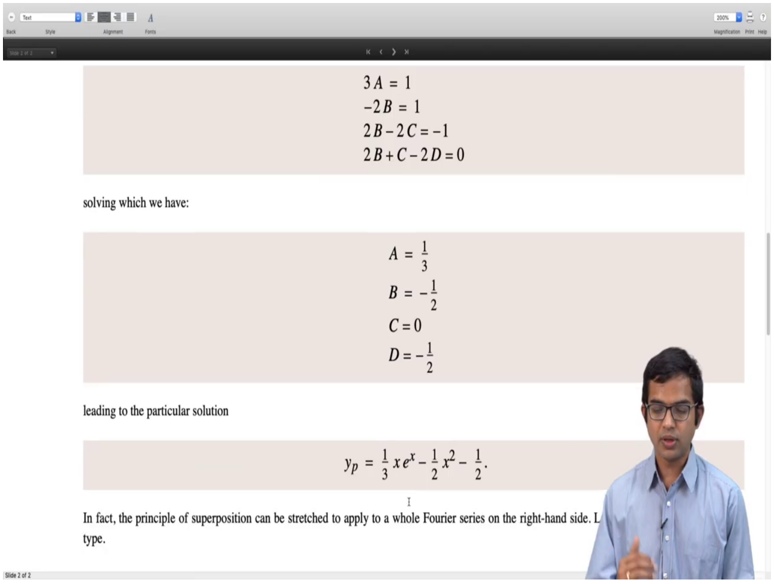
Or you can use this knowledge to directly make an ansatz of this kind y p is equal to A times x times e to the x. So, why do we need A times x to the times e x because you see that e to the x is a, is part of the complementary function. So, you cannot just use e to the x here on the right hand side. So, you must try x times e to the x.

So, let us look at A times x times e to the x plus a quadratic function right that is the rule. So, we must put in B x squared plus C x plus D. We do not know B, C, D, we do not know A, we need to find it. This is the method of undetermined coefficients. So, we differentiate. So, you get Ax Ae to the x plus Axe to the x plus 2Bx plus C differentiate again. You get 2Ae to the x plus Axe to the x plus 2B.

Now, we plug back into the original differential equation and then you know collect all these terms carefully, the left hand side. So, you collect terms corresponding to e to the x. So, of course, you can check that immediately x times e to the x will vanish. So, you have Axe to the x plus Axe to the x minus 2 Axe to the x. So, that is anyway gone e to the x the coefficient is just 3A. So, that must be equal to 1.

And, the coefficient corresponding to x squared is 1 on the right hand side, but it is minus 2B on the left hand side. So, minus 2B should be equal to 1, the coefficient corresponding to x is minus 1 on the right hand side. And on the left hand side is 2B minus 2C that should be equated.

(Refer Slide Time: 05:11)

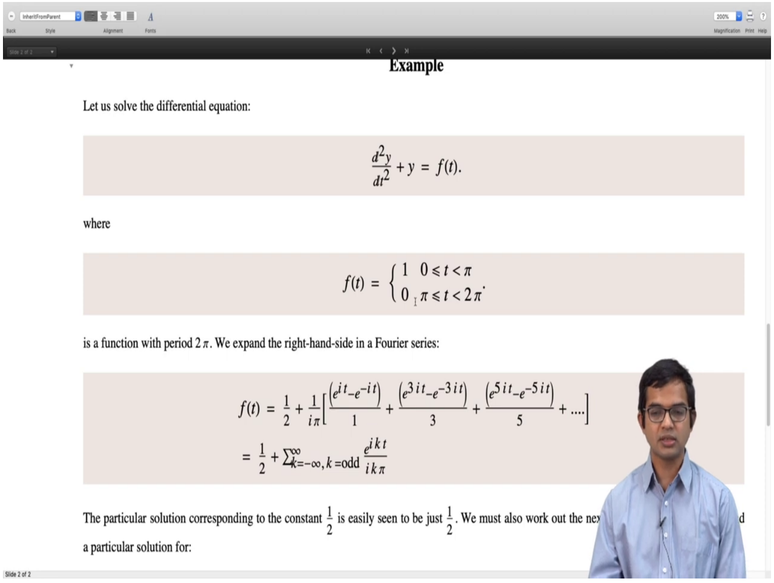


Coefficient corresponding to the constant part is just 0 on the right hand side. So, 2B plus C minus 2D must be equal to 0, right. So, we just solve these four different four equations in four unknowns, it is very straightforward because two of them are already solved.

And, then you just substitute, and then you get A equal to one third, B equal to minus half, C equal to 0, D equal to minus half and then you get this particular solution which you must take this final particular solution and directly apply the linear operator on it and check that indeed you have got the right particular solution right. Because there are possibilities of making mistakes, but it is always best to cross check that indeed you have got a particular solution.

So, it does not matter how you get a particular solution, any particular solution is good enough you just tag it along with the complementary function and you are done. So, that is we have implied we have applied this linear superposition principle. So, in fact, this linear superposition principle can be stretched. So, instead of just one term or two terms or 10 terms on the right hand side, you can actually have an infinite series sitting on the right hand side.

(Refer Slide Time: 06:37)



So, the kind of infinite series that we are familiar with is the Fourier series right so, special kind of infinite series and so, let us look at an example where we have a Fourier series on the right hand side. So, suppose you know this is the standard harmonic oscillator style problem d squared y by dt square plus y, but you are driving the system with a square wave of this kind.

So, f of t is 1 for an interval from 0 to pi and then it goes to 0, then again it becomes 1 and then again it goes to 0 and so on, right and with a time period of 2 pi. So, now if we expand this right hand side so, the way to solve this type of a problem is to expand the right hand side in a Fourier series and then use linear superposition principle.

So, with this is a Fourier series that we have already done, we have expanded. So, you are going to get a bunch of sines, right. So, you can check this. You could have worked this out in a Fourier sine series, but it is convenient to work out the exponential series. So, let us write down the exponential series.

So, you can verify that indeed you get f of t is equal to half plus 1 over i pi, I mean it is basically the sign series and then there is this constant right it is it is not exactly just the sign series. So, there is a, it is like a shift which has happened. So, it is half plus 1 over i pi times you know this is a factor that has been pulled out then you have e to the i t minus e to the minus i t by 1.

So, that is like sin t by 1 then you have sin t 3 t by 3 then sin phi t by 5 so on. So, you get e to the 3 it minus e to the minus 3 it by 3 i mean I have to make use of this i and also some factor of 2 must be adjusted if you want to write it in terms of signs right which is an equivalent version.

So, it is convenient to write this as half plus summation over k going from minus infinity to plus infinity, where k is odd and it is e to the i k t divided by k pi, right. So, k can take negative values and positive values, but not any even ones ok. So, the particular solution corresponding to this constant, so, how do we solve this problem?

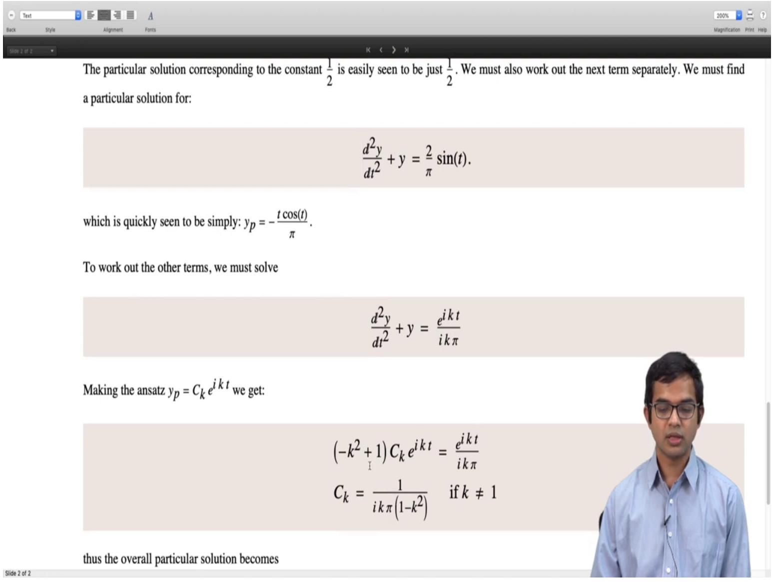
So, you say ok, d squared y by d square d t square plus y is equal to half solve it you get one solution then you have d squared y by dt squared plus you know the first term on this side only on the right hand side then solve for it and so on, right. So, the particular solution corresponding to half is very easy. It is just half if you choose y p equal to half then d squared y by dt squared will just kill it and so, y is equal to half. So, there is no problem, just half. Now, there is this one term which we have to work out separately.

So, this is in fact, what is happening is we have chosen to drive our system here at resonance in some sense because the frequency of you know the drive is the same as the natural frequency of this problem. So, I constructed it in such a way that you know I can show you that one has to be careful.

And to see that you know instead of driving such a system with a cosine drive which we have seen in the past it is also possible to drive it with a square wave. And, in fact, here the square wave turns out to be a rather messy object to work with. It is much more convenient to get the physics out if we applied a sine wave or cosine wave.

But, here the point is to illustrate how we can work with a sine wave with a you know an arbitrary periodic function and then expand it in a Fourier series and then work out the particular solution term by term. So, since that is our goal a square wave is a particularly instructive type of drive to consider, and also to consider it such that its frequency matches with the natural frequency.

(Refer Slide Time: 10:26)



So, now, because the frequency matches I cannot blindly you know just use e to the, i t minus e to the minus i t. So, I should solve this first term alone separately as you will see in a moment the solution for the other parts you know can be written down in one shot. So, let me solve this.

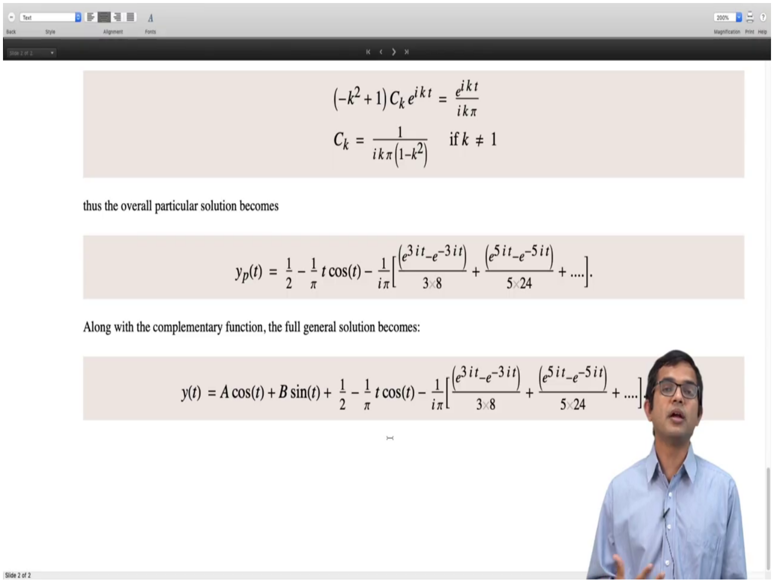
So, I have d squared y by dt squared plus y is equal to 2 by pi sin of t. So, I cannot blindly put you know a particular solution of the kind sin of t because that is going to be part of the complementary solution. So, but on the other hand I should try something like t times sin t or you know t times cosine t, a linear combination of both of these that is what I should try.

And, so, in fact, I did this and I checked that in fact, the particular solution for this is this minus t cosine of t by pi as you can directly verify by you know just plug this in here into this differential equation and verify that indeed you know if you do d squared by dt square plus 1, if you act upon this then it should just give you 2 by pi sin of t. So, this is a particular solution for this problem.

Now, to work out the particular solution for the other terms we should just solve for this kind of a problem d squared y by dt square plus y is equal to e to the ikt divided by ik pi. So, this is the generic term. Now, if I do this, the ansatz that I will have to make is C k times e to the ikt.

So, now, I get minus k squared plus 1 times C k e to the ikt d is equal to e to the ikt divided by k pi. So, C k is equal to 1 over ik pi times 1 minus k square. So, now, you see the difficulty we would have run into if we had put k equal to 1 or k equal to minus 1 as well, right. So, that is why the first time I have treated it separately and otherwise it is going to give me terms of this kind.

(Refer Slide Time: 12:21)



So, the overall particular solution becomes you know the sum of the particular solutions of all of these. So, I have y p of t is equal to half minus 1 over pi t times cosine of t minus 1 over i pi e to the 3 i t minus e to the minus 3 i t divided by you know I have to take into account this 1 minus k squared. So, which is you know 3 comes because of a k and 1 minus k squared will give me 8 and then I have a 5 which comes in here phi squared minus 1 to the 24 and so on, right. So, there is an overall minus sign.

So, of course, the book keeping is somewhat messy, but the point is the principle involved here, right. So, here we see that using the superposition principle we can work out a term by term particular solution for each of the terms which appears on the right hand side. Just add them all up and we get the particular solution for the full problem.

So, once again we note that this solution is also going to get dominated by this term t cosine t because you are driving your system at resonance. So, everything else has an oscillatory behavior, but this has this linear behavior as a function of time. So, everything else will get suppressed or will be less important compared to this and then you get this t cosine of t.

You can plot this function and verify for yourself that indeed resonance leads to like very very large amplitudes being achieved for you know even relatively small times and definitely for large times. And, so, in general when you drive your system at resonance you know something very dramatic is going to happen.

So, very very likely you are going to just completely destroy the system if it is going to break down at some point right. But, that breakdown is sometimes of a desirable kind of breakdown which you want and so, in such applications you know systems need to be run resonantly then there are other scenarios where the goal is to avoid resonance.

But, the main message from this lecture is that using this the principle of superposition we can work out the particular solution and therefore, the full general solution for differential equations involving you know forcing terms which are rather complicated which involve you know some of many objects, sometimes the sum can be an infinite series as well.

Thank you.