Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 73
Vibrations in mechanical systems

So, we have already look at a bunch of techniques for solving differential equations. We saw how it is useful to work out the solution for the homogeneous equation. And then if you can find a particular solution for the full inhomogeneous equation, we can you know put these together to get the full general solution for the inhomogeneous differential equation. And then we looked at special techniques for special kinds of problems and so on.

So, in this lecture we will look at some applications right. So, the core applications in you know of this kind of differential equation is in the context of mechanical systems, but you know completely analogous equations also hold in electrical circuit style systems right. So, we will just look at the mechanical problem and you know the corresponding circuit problem is something that you can work out as either homework or I mean it is really the same differential equation you can look it up ok.

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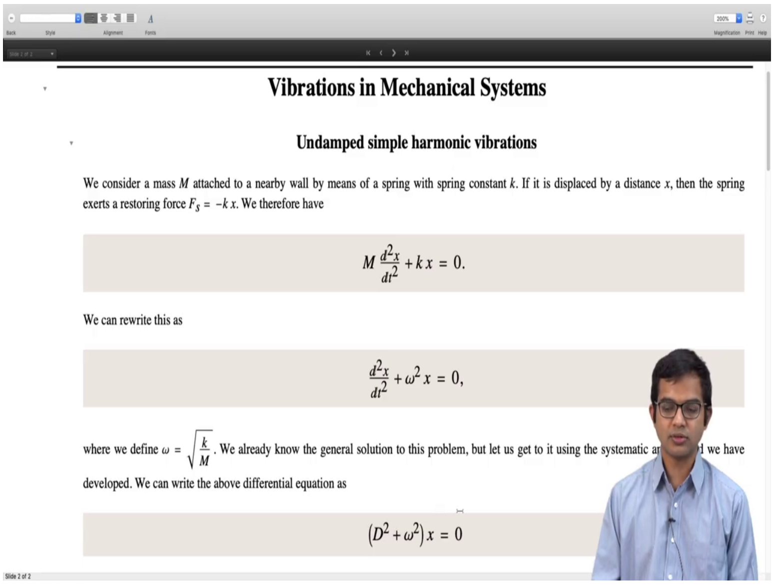

So, we start with something extremely familiar and very simple. So, it is a mass connected to spring. So, you have a mass M which is connected to spring with spring constant k. And so, the differential equation which comes out is simply M times d squared x by dt square plus k times x is equal to 0 right.

So, if it is at equilibrium then there is no force and then the particle would be at rest, but if it is moved by an amount x then there is restoring force. And so, it is so we have this differential equation which we have seen since high school days. So, then it is useful to non dimensionalize this to introduce this quantity omega which is equal to the square root of k by M.

And so, then our differential equation takes this form and I mean although we know the solution it is useful to work this out from the principles that we have laid down. So, we have a differential equation which is second order. It is a linear differential linear homogeneous differential equation, a very simple differential equation you know exactly how to solve. So, we start by writing it down as d squared plus omega squared acting on x is equal to 0 and now the left hand side is a quadratic form right.

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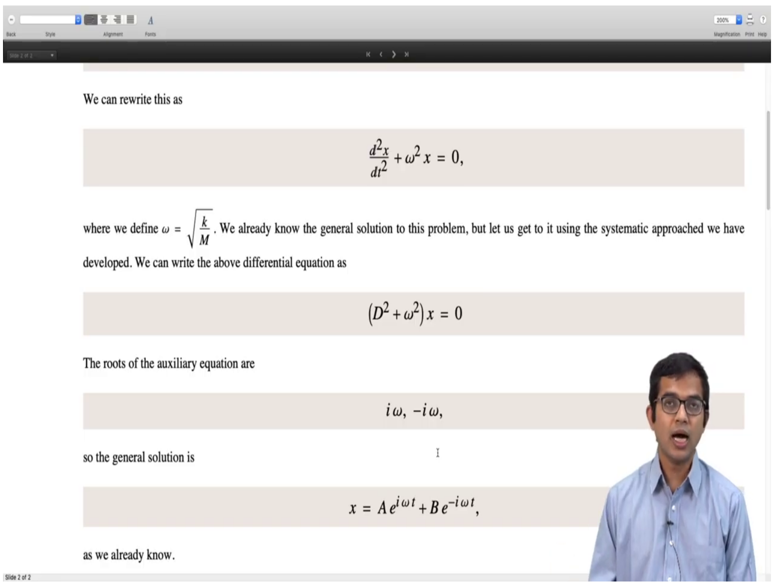

So, whose roots the auxiliary equation has roots i omega and minus i omega. So, we can factor the left hand side and rewrite it as d plus i omega times d minus i omega acting on x is equal to 0 from which immediately we are able to write down the general solution. Which is just x times x is equal to A times e to the i omega t plus B times e to the minus i omega t.

So, equivalently we could have written this as you know cosine omega t and sine omega t is a linear combination of cosine omega t and sine omega t, whose coefficients you can work out in terms of a and b right. So, all very familiar or equivalently you could have also thought of this as cosine of omega t plus some phase times an appropriate amplitude right.

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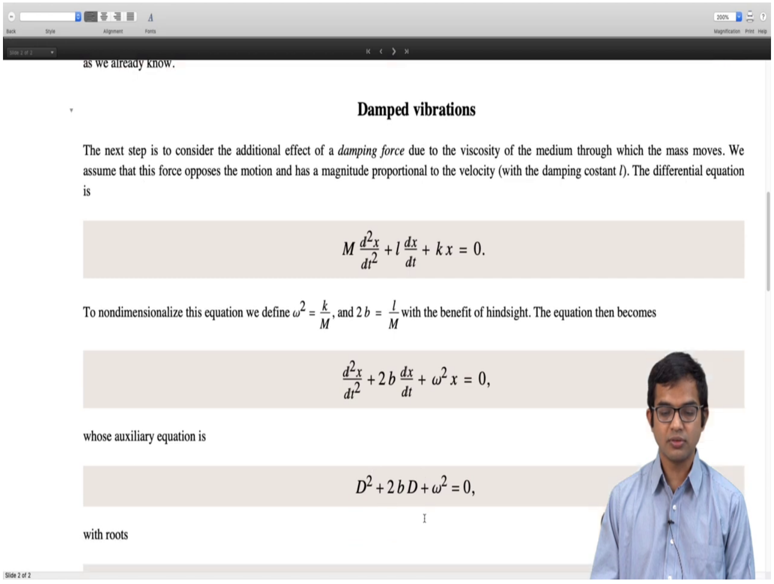

So, this is something we are familiar with. Now, if we put on we introduce some damping into our problem. So, you have a damping force which is a frictional force you know which comes in if your medium has some viscosity associated with it. So, the faster your particle is moving you know the greater is this frictional force.

And so, the simplest assumption is that your viscous force is proportional to the speed right, and in a direction opposite to the direction of motion and it is if you assume that it is proportional to the velocity or the speed. So, the differential equation becomes M times d squared x by dt square plus l times d x by dt plus k x is equal to 0 right.

So, if you do not have l then it is just the earlier differential equation, which we have already seen. Now, it is useful to introduce the quantities omega square and 2b right. So, it is convenient to write to think of this as 2b you know for convenience as well as convention.

So, let us just use 2b is equal to l over M right. So, we can pull out this M throughout and then we can rewrite this differential equation as, d square x by dt squared plus 2b d x by dt plus omega squared x equal to 0. So now, once again it is a differential equation of a familiar type right. So, we have worked out the solution for more difficult problems than this. So, here it is a homogeneous differential equation of the right hand side is 0. So, all we need to do is factor the left hand side.

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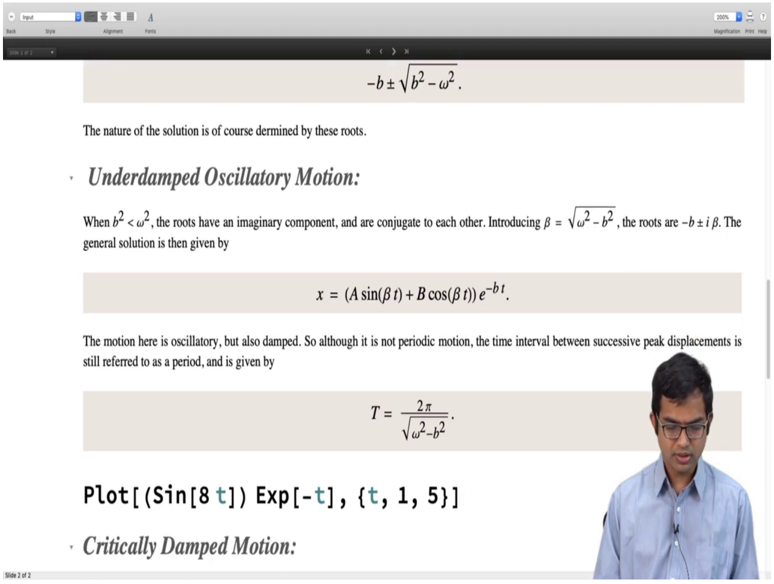

So, we have D squared plus 2b d plus omega squared equal to 0 that is the auxiliary equation with roots minus b plus or minus square root of b squared minus omega squared right. So, the nature of this solution will be determined by what kind of roots you have right.

When you have seen that there are three different kinds of roots possible. One of them will give you what is called under damped motion: and it is oscillatory in nature right. So, it comes about when b squared is less than omega square. So, if b squared is less than omega squared. So, then you have an imaginary component.

So, the roots are complex and there are 2 roots that have to be conjugate to each other. So, we have a pair of complex conjugates as the roots of this quadratic equation, and so the roots of the differential equation are simply given by you know this solution. So, you have A sine beta t plus B cos beta t the whole thing multiplied by e to the minus b t alright.

So, you know we could have written it in a different manner, but here it is transparent to pull out this factor e to the minus bt right. So, b is real. Now, since omega squared minus beta squared is greater than 0. So, beta is also a completely well defined square root of a positive number. And so, you have you know A is unknown B is unknown it is a free constant.

And you know there are 2 of these because it is a second order differential equation and then you have this b e to the minus b is this where the damping is coming in. So, the more there is the decay in your time evolution of your position this way, damping leads to decay and but there is also this oscillatory aspect right.

So, there are these two forces in nature. One is if there were no damping of course, you know you have this is a simple harmonic oscillator which will perform oscillations about the mean and so that part is still in here because; damping is not so large. You see that this b squared less than omega squared simply means the damping is present, but it is not it is not the dominant force so.

In fact, the oscillatory aspect of the motion is still retained, but it is also damped to the right. So, it doesn't matter how small these are as long as it is not 0. There is going to be a decay in the amplitude. So, there is going to be oscillating motion, but it is damp.

And so it is so, there is a periodicity to this kind of motion because; of the oscillatory nature although it is not really periodic in the sense that the opposition does not come back to exactly where it started every you know time period it is still useful to think of you know to define the notion of a time period as the interval between two successive peak displacements right.

So, and that is given by simply 2 pi divided by square root of omega squared minus d square. You know you have one should be careful about you know how one interprets this here it is you know, it is completely well defined as the time interval between two successive peaks of your motion right although there is it is not periodic motion strictly speaking ok.

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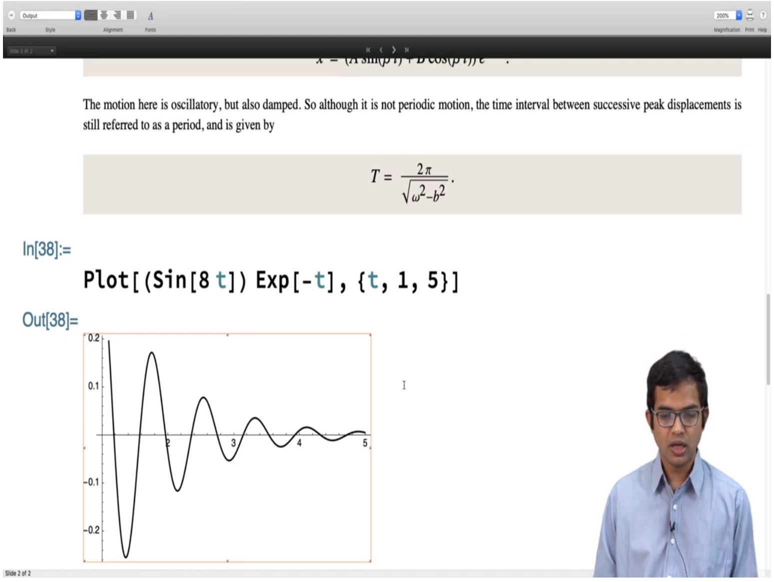

So, let us plot it and see what it looks like. So, if I plot this so this is what it looks like right. So, this is when I so now, you see that the time interval you know that elapses between a particle going from one peak to another is the same as a time interval that elapses for it to go from this peak to the next peak although it never really comes back to where it started right.

So, it is in that sense that it is not periodic, but there is a you know oscillatory aspect to it there is something which repeats and that has a very well defined period associated with it ok.

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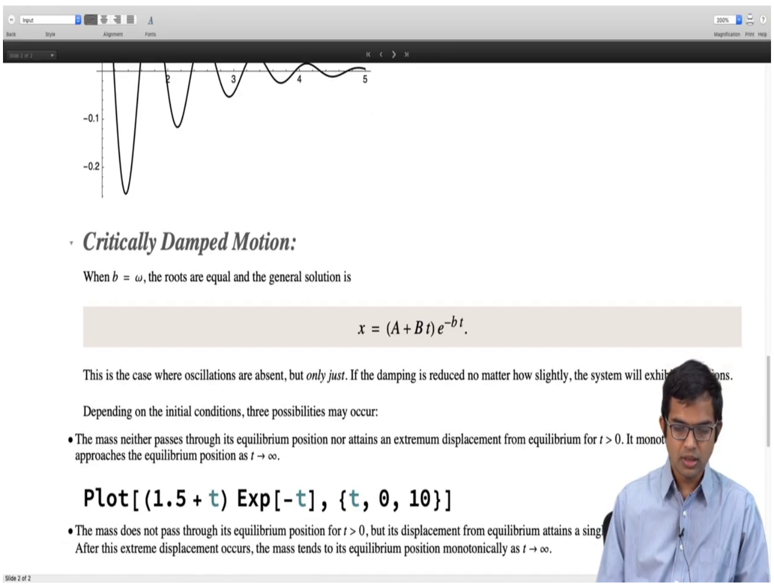

So, this is something that we are sort of familiar with, but it is useful to review it with all the machinery that we have developed. So, then there is this second case which is called critically damped motion. If you think about increasing the damping you know if you keep on cracking up the damping there comes a critical value of the damping at which the roots become equal and so, the general solution now is A plus B t times e to the minus b t.

So, the damping is so large now. That oscillations are not possible right so, but this is a critical damping which means that oscillations are absent, but only just right. If the damping, where reduced even by the smallest amount then the system will again show oscillations right. So, depending on the initial conditions, there are three possibilities. So, one is if you so I will show you know qualitative aspects.

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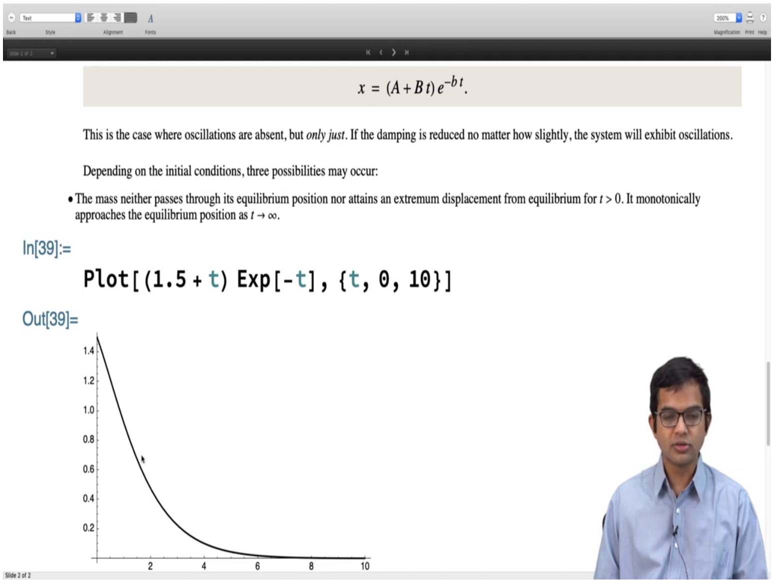

So, one is that your system may just keep on decaying. So, the positive equilibrium position is never attained in the middle of its journey; it will eventually just die down to 0. But it is monotonically decreasing - there is no peak or anything of that kind and it will just monotonically approach the equilibrium position as at time going to infinity.

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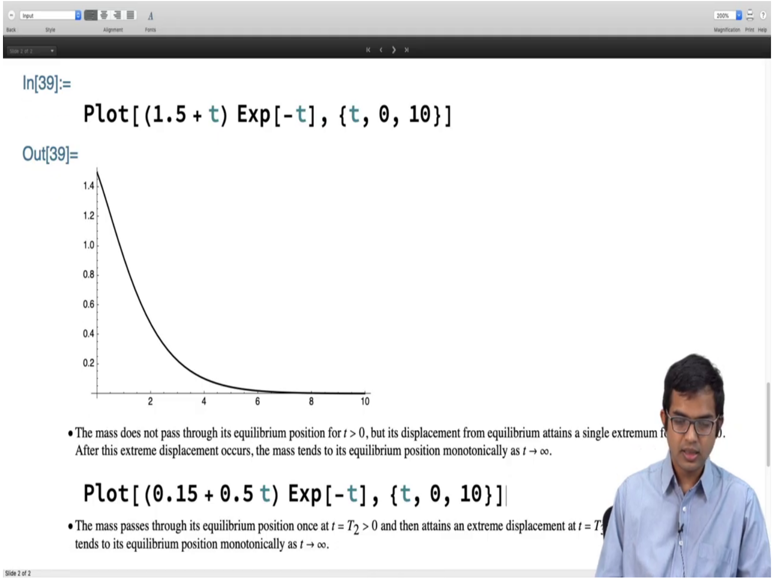

So, then the second possibility is that it tends to go up before falling down right.

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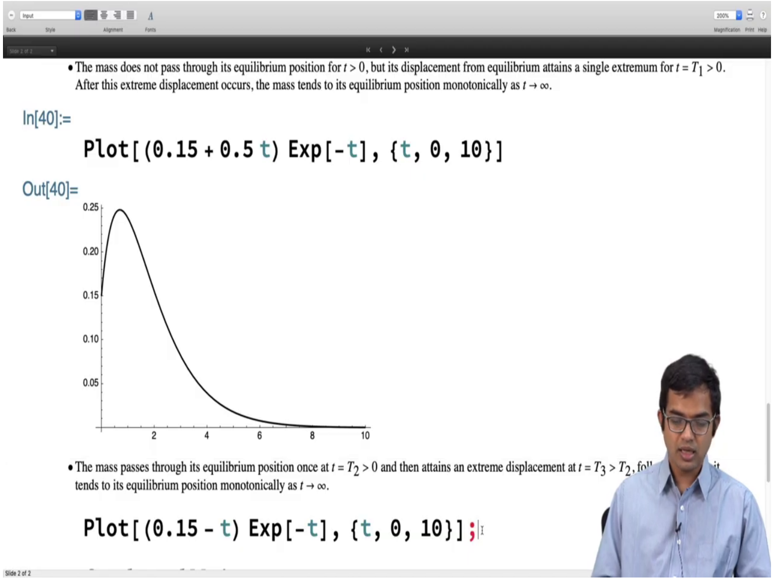

So, this is so there is a characteristic time scale t 1 associated with it at this point it hits an extremum. So, it is increasing in value, but then the damping begins to overtake it and after that it turns around and then it is a steep fall after that and right. And then after that it is a monotonically decreasing function.

Which he just goes to eventually the particles, that is down to it is equilibrium position at t equal to infinity. Then there is a third possibility which is you know if it goes in the other direction right.

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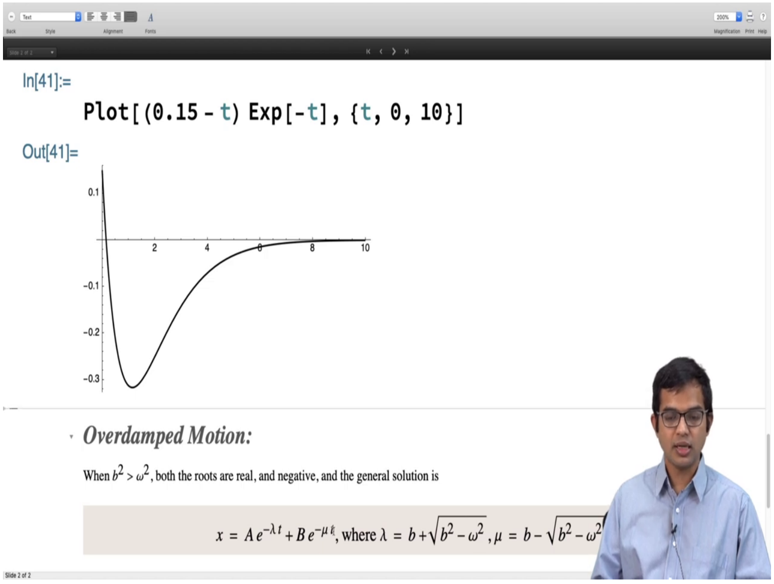

So, if you the mass may actually go below the equilibrium position or go in the other direction right. So, of the equilibrium position. So, this will happen when typically you know the direction of motion, at the initial point is in the opposite direction of you know of the previous case yeah right.

So, it will there is enough velocity in it that it can actually shoot pass the equilibrium point, but then damping takes over and then it has to make a turn and then there is a minimum here in this case and after that it will keep on heading towards the equilibrium position. Now, it comes in from the opposite direction and then eventually goes to zero. So, but it is only one time right it can never there is not enough energy in it for it to perform even one oscillation right.

So, that is the critically damped case and over damped case is basically the same, but it is a more exaggerated version of it in the sense that damping is really dominant here and. So, there is no question of you know even if you slightly decrease the damping your motion is not going to be periodic in nature right. So, that is what is different here. And so, the nature of the solution is also slightly different here: you have A times e to the minus lambda t plus B times e to the minus mu t right.

So, it is not A plus B t like here, but you will have A times e to the e to the minus lambda t plus B times e to the minus mu t right. So, right we recall how if you have repeated roots we have to put in a on sorts of you know this kind t times e to the minus B t comes in here because of the repeated root right.

So, you have to find these two independent solutions. And so, whereas here both of these are you know you just work with e to the minus lambda t and e to the minus mu t. So, that is your particular solution and then you get your general solution by attaching these two different kinds of independent solutions.

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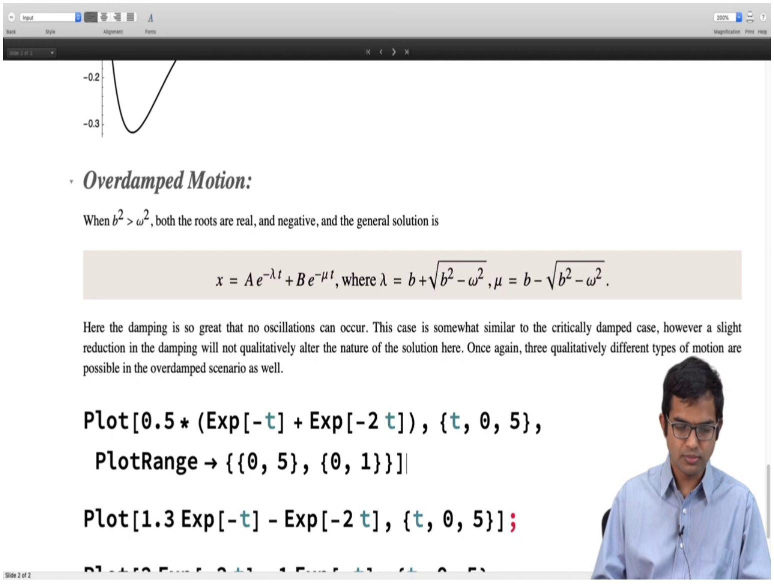

Now, once again the qualitative nature of the solution there are three different possibilities. So, I will just show you some plots.

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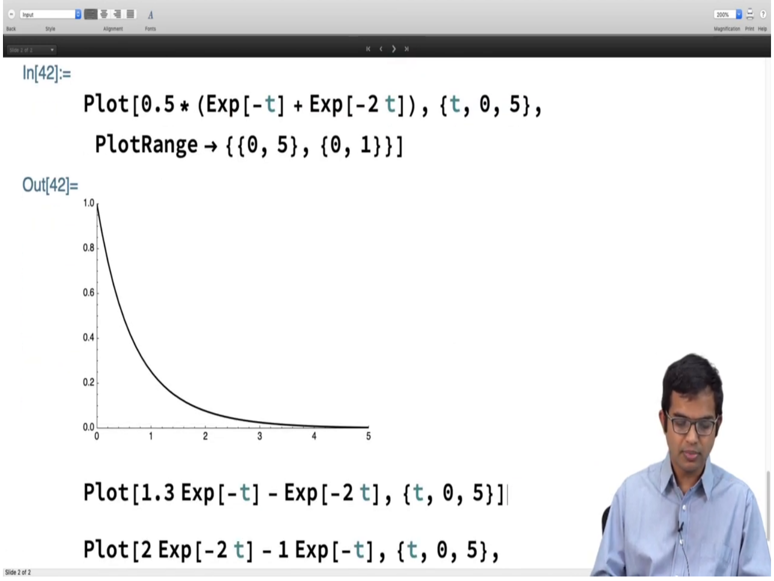

Where I just played with the parameters and.

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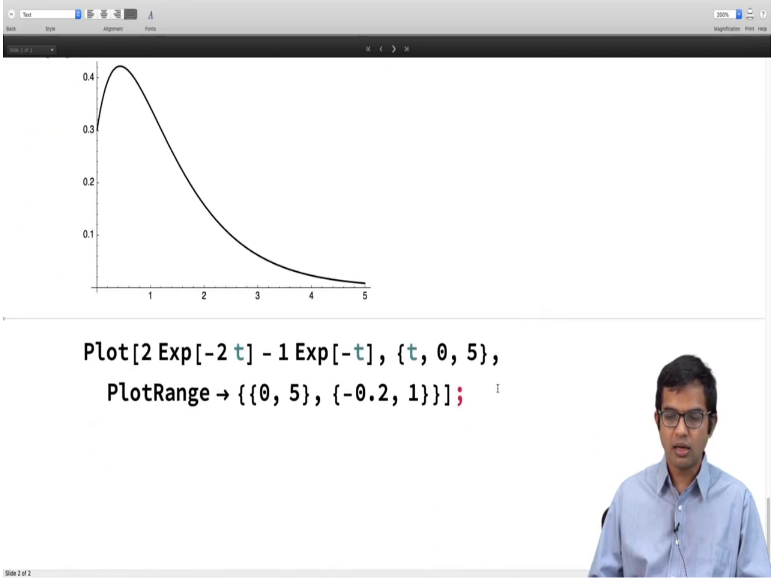

I am able to generate you know these three different kinds of motion ok.

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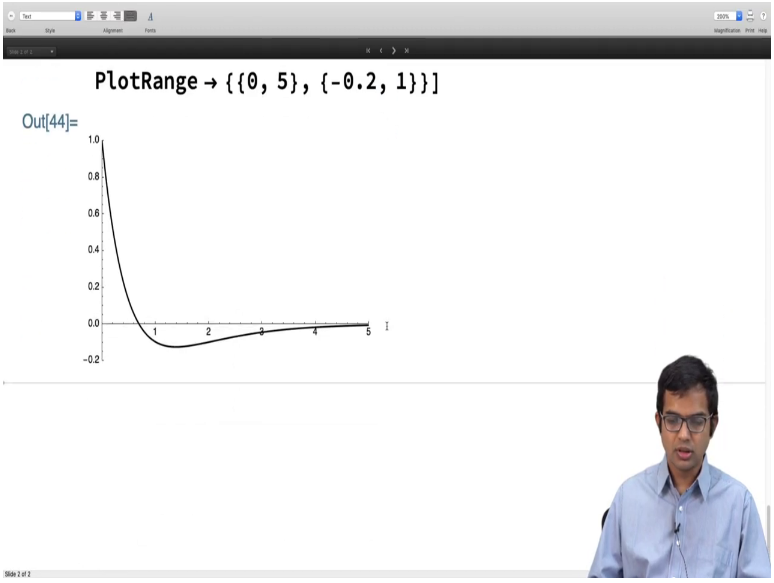

So, that is all for this lecture, we discussed what happens to a you know a vibrating mechanical system in the absence of damping in the presence of damping. And when there is damping depending upon the value of the damping you make it three different kinds of you know possibilities, or you know the critically damped lower damped case each of them are you know of a very similar nature. When all oscillations in the system are completely washed out.

So, that is all for this lecture.

Thank you.