Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 71
Illustrative examples

Ok, so in this lecture we will look at a few Illustrative examples. So, we have written down the prescription for solving a second order inhomogeneous differential equation with constant coefficients.

We first wrote down a prescription for solving the homogeneous one and then we also said how if you can find a particular solution for the inhomogeneous differential equation, we can you know couple this with the general solution of the homogeneous equation and write down the general solution even for an inhomogeneous differential equation right.

Specifically we saw how you know there are methods available if the forcing term, you know the term which appears on the right hand side is of a you know special kind right. So, we will look at examples of this kind. So, even when you know the forcing term is of the special kind it is useful to look at a few examples and see how the theory works out in practice. So, that is what this lecture is about.

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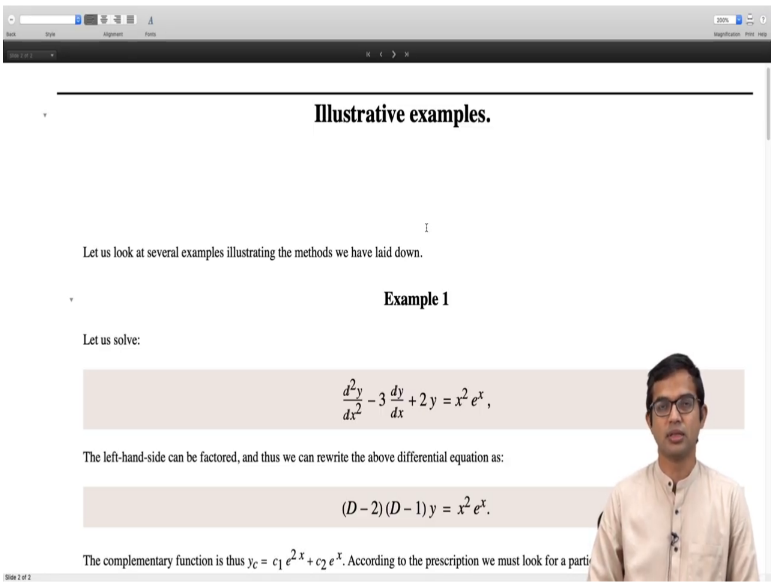

The first example is d squared y by dx square minus 3 dy by dx plus 2y is equal to x square times e to the x right. So, notice that the right hand side is a little more complicated than the kind we have already looked at. So, you have a quadratic expression times an exponential right.

So, the left hand side of course, we should start by factoring it, it is a quadratic form. So it can be factored, and you have D minus 2 times D minus 1 times y and the right hand side is as it is x square times e to the x. So, the complementary function is straightforward to write down.

So it is, you know these roots are available 2 and 1 for this auxiliary quadratic equation. And they are real and distinct. So, it is very straightforward to write down. So, the complementary function is simply c 1 times e to the 2 x plus c 2 times e to the x. So, according to the prescription we must look for a particular solution of the form right.

So, what is the form that we must choose? So, we see that on the right hand side we have an x squared. So, there is a quadratic function. So, we will have to look for a polynomial of the same degree. So, we will have to look for something like a x squared plus b x plus c and then, times you know you have this exponential e to the x.

So, it turns out that you know this coefficient here e to the c x we have so the c matches with one of the roots of the polynomial here. So, we must choose x times e to the x. And so we have this as our ansatz for the particular solution.

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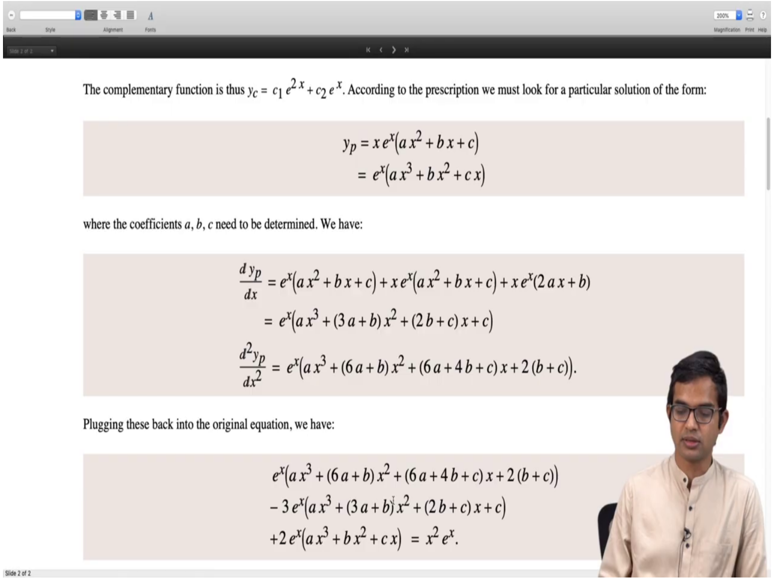

So, we look for a solution of the form x times e to the x times x squared plus b x plus c. So, we need to find these coefficients a b and c such that you know this becomes a particular solution of the differential equation. So, we will have to evaluate what dy by dv dy p by dx is and then collect all these terms carefully so, you have all these terms collected together you have an x cube and x squared x and c there is a common exponential e to the x which comes out.

Then when we do a second derivative once again you know you carefully collect terms. And you know I have worked out the algebra you can cross check this you get e to the x times a x cube plus 6a plus b the whole times x squared plus 6a plus 4b plus c times x plus 2 times b plus c right.

So, then once we have these expressions for the second derivative and the first derivative we have to go back and plug back into the original differential equation. So, you have you know this whole stuff for the second derivative minus 3 times this stuff for the second derivative for the first derivative plus 2 times the function itself must be equal to x square times e to the x right.

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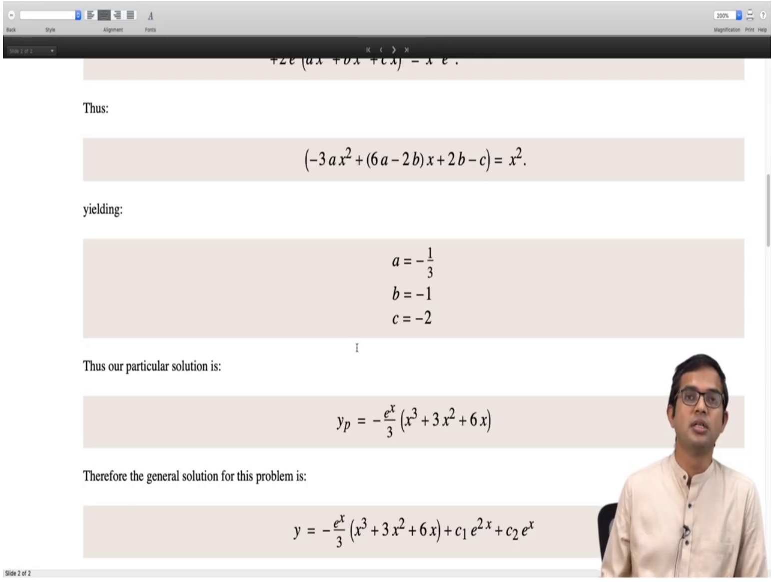

And this must hold for all values of x. So, you can cancel these e to the x is throughout and so, your left with just this equation minus 3a x squared plus 6a minus 2b times x plus 2b minus c must be equal to x squared right. And this again must hold for all values of x. so, it immediately implies that a itself must be equal to one third and 2b equal to 6a which means b equal to 3a.

So, 3 times a is minus 1 and c must be equal to 2b. So, that gives us c equal to minus 2. And so, our particular solution is y p is e to the x divided by 3 into x cube plus 3x squared plus 6x alright. So, you can go back and check where this is indeed you know, if you operate with this stuff D minus 2 times D minus 1 and if you operate with on top of this function.

You must get x squared times e to the x right. So, you can directly work this out and explicitly check that indeed your final answer is a particular solution right. So, this check must be done to be sure that you know, there is all this algebra as you can see even for a fairly simple problem there is a you know a fair amount of algebra is involved and it is possible to make mistakes.

So, you must look at your answer and check whether it is indeed a particular solution and once you have seen that it does not matter, which particular solution you find. And it in fact does not matter how you find it. As long as it is a particular solution, you just string it along with your general solution for the corresponding homogeneous problem which is called the complementary function which we already wrote down and then we have the full general solution for our differential equation which is just given by this you know this expression minus e to the x divided by 3 times x cube plus 3x squared plus 6x plus c 1 times e to the 2x plus c 2 times e to the x ok.

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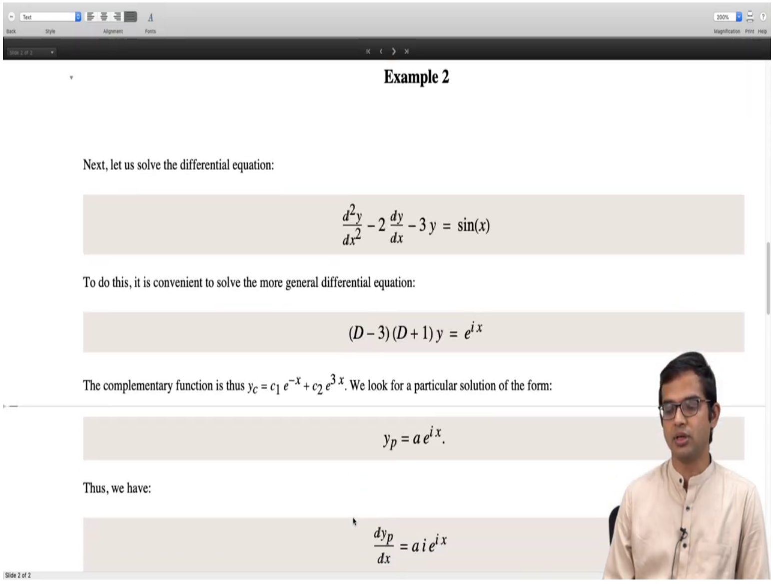

So, let us look at another example. So, here I have a sin of x on the right hand side. I have a fairly you know simple left hand side I am choosing simple left hand sides is it. The same as what we have here. You know, slightly different it does not matter right. So, you can play with this right so, I am just giving you a full few examples.

You know where the factoring is fairly simple. I want to show an example where the right hand side has a sinusoid. So, I told you how if you have a sin of x on the right hand side. It is convenient to first of all you must factor the left hand side so, in this case it is D minus 3 times, D plus 1 times y right as you can check minus 3, 3 y comes in here minus 3 plus one which is minus2. So, it is alright.

So, I have the factorization on the left hand side and right hand side, when you have a sinusoid it is convenient to consider just the exponential right. So, you look at a more general problem. And so you can see that, if you can find a solution for this problem it is going to be a complex function and so, then you argue that you have this linear operator which acts upon this function to give you a you know there is a real part and a and an imaginary part.

So, the real part of this will be a solution to you know this differential equation with cos x on the right hand side. And the imaginary part will be a solution to this differential equation with sin x on the right hand side. So, in our case it is the imaginary part that we care about right.

It is just a you know trick one uses, but if you do not want to use a trick another way to do this is to look for an ansatz of the form a sin x plus b cos x right both sin x and cos x have to be included if you do not want to do it like here. So, let us do it you know with e times e e to the i x here, but if you do not want to you can also try it it is you know it is left as an exercise for you to make the ansatz a sin x plus b cos x and work out the coefficients a and b ok.

So, in this case the complementary function is simple to write down right. We know how to do it, it is just minus e to the minus x and e to the 3x with coefficient's. C 1and 2 which are arbitrary, they are you know this is the solution for the homogeneous differential equation which corresponds to this inhomogeneous differential equation.

So, to find a particular solution we impose the form yp is equal to a times e to the ix right. So, I emphasize once again that this is a particular solution for this differential equation which is a slight way is a variant of the original problem. And now we have the first derivative is just a times i times e to the i x.

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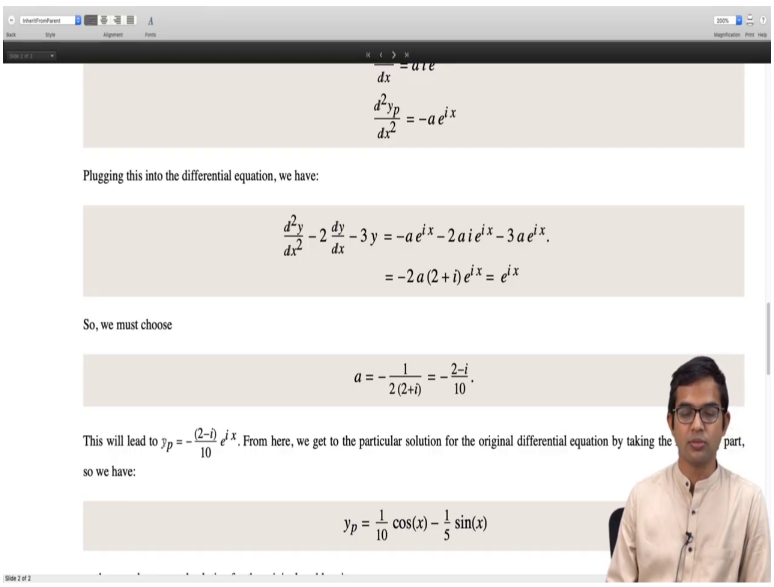

The second derivative is minus a times e to the ix. So, i squared will give you a minus 1 and now, you can plug this in all these expressions back into the original differential equation but with the right hand side equal to e to the i x.

So, you have d squared y up by dx squared minus 2dy by dx minus 3y is equal to you know minus a e to the ix minus 2 times a times i times e to the ix minus 3a e to the ix which is minus 2a times 2 plus i times e to the ix, we want this to be equal to e to the ix when will this happen?

If this coefficient minus 2a times 2 plus i is equal to 1 or equivalently, if you choose your constant a this coefficient a to be minus 1 over 2 times 2 plus i. Which you can simplify right to minus 1over 10 times 2 minus i, but really we are interested in the imaginary part of this right.

So, this will be the particular solution for this modified differential equation where the right hand side is taken to be e to the ix. So, for us it is of interest to find the imaginary part of this because, that is going to be the particular solution of this original problem of interest right. So, the imaginary part of this function is simply just 1 over 10 times cosine of x minus one over 5 times sin of x as you can verify.

You should check that indeed we have got the algebra right you should take this function you know, you have to be careful you know there are these i is hanging around there is a e to the ix so, the simplest way is of course to expand this cosine x plus i times sin x and then collect all these expand out entirely and write it as some real part plus i times imaginary part and so it is the imaginary part that you must work with.

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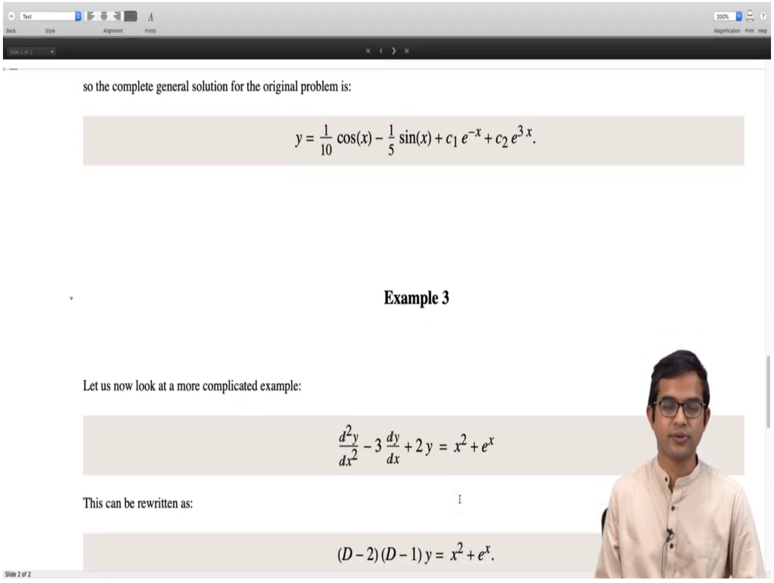

And so in fact, you will get this answer. And then to be absolutely sure you must plug this back into your original differential equation and check that when you operate with d squared by dx squared minus 2d by dx minus 3. When it acts upon this y p it must give you sin of x right. So, as you see this y particular that we have obtained as both cosine and sin right. You could as well have you know started your ansatz with a cosine x plus b sin x and you would still have recovered basically the same particular solution.

So, now you couple this particular solution with the general solution of the corresponding homogeneous equation also known as the complementary function. And you have the full general solution of the original problem, which is just 1 over 10 cosine of x minus one over 5 sin of x plus c 1 e to the minus x plus c 2e to the 3x right. So, we will look at one more example which is a little more complicated than the kinds we have seen right.

You can have a scenario where you have you know x squared plus e to the x appears on the right hand side. You can have even more complicated functions appearing on the right hand side. You can, you know, consider e to the x, e to the 2x, e to 3x.Whatever you can add them all up then you can think of this as you know you must find particular solutions for each type of forcing term on the right hand side and add them up right.

So in this case, let us look at you know the left hand side of course you must factor it out. So, it is a simple factorization. I have chosen the left hand side to be fairly simple : D minus 2 times D minus 1 times y is equal to x squared plus e e to the x. So, to find the particular solution that is where the art form is right.

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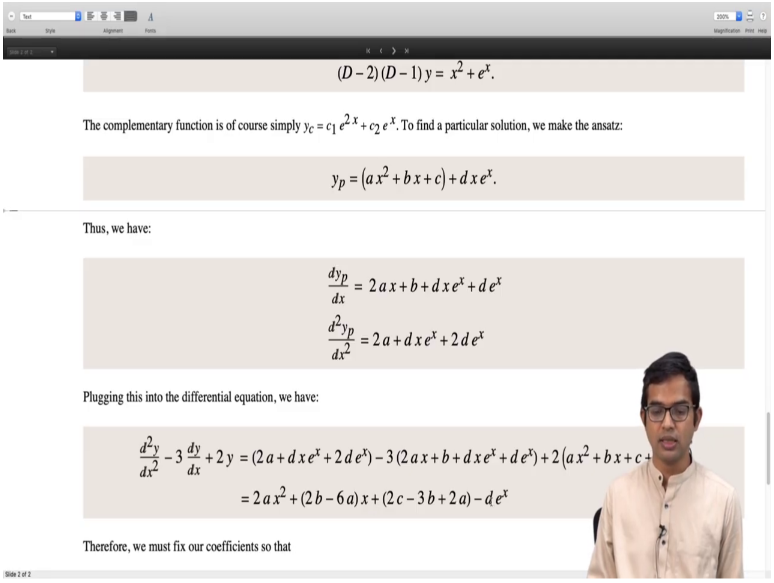

So, the complementary function is of course, simply c 1 times e to the 2x plus c 2 times e to the x. To find a particular solution we make the ansatz. You know, we need a quadratic function because x squared you know appears on the right hand side, but we also have to choose d times x times e to the x right.

So, here you have to be careful. You have e to the x here and you see that there is also an e to the x which appears in the solution of the homogeneous equation. So, then you cannot use e to the x itself as we have seen.

The theory is that if you know the one over roots of your auxiliary equation matches with this coefficient in the exponential on the right hand side you must choose x times e to the x. So, that is what I have done. So, d times x times e to the x so, I see that I have 4 coefficients which I have to determine right.

So, in principle you would have 4 linear equations and 4 variables which can be quite complicated, but you will see that simplifications appear and then it is not as difficult as you might imagine. Even if there are 4 coefficients to be determined you can try even more complicated variants right. So, it is I mean you can cook up problems of your own and to you know really get a solid understanding of this method.

So, the first thing is of course, to write down the answers then you have to find the derivative and the second derivative. So, the derivative you know will have a linear term and then of course, you know this x times e to the x will give you both x times e to the x and also just e to the x. When you take another derivative you will get 2 d times e to the x dx times e to the x remains that as it is and then you have this constant.

So, if you plug all these back into your original differential equation. The left hand side, you will get this complicated looking expression which simplifies. So, you have this stuff which comes from the second derivative minus 3 times the first derivative plus 2 times the function itself. You collect all these terms carefully. So, you have a quadratic term then linear part then there is a constant then you have a minus d to the d times e to the x.

So, you will notice that already we have chosen our particular function in such a manner that you know this term involving x times e to the x already vanishes. So, you do not have to worry about this at all. So, in the end we want only x squared and e to the x squared plus e to the x to be precise on the right hand side. So, immediately we see that d is something we can fix; d should be minus 1 and then you will get an e to the x and others also you can fix.

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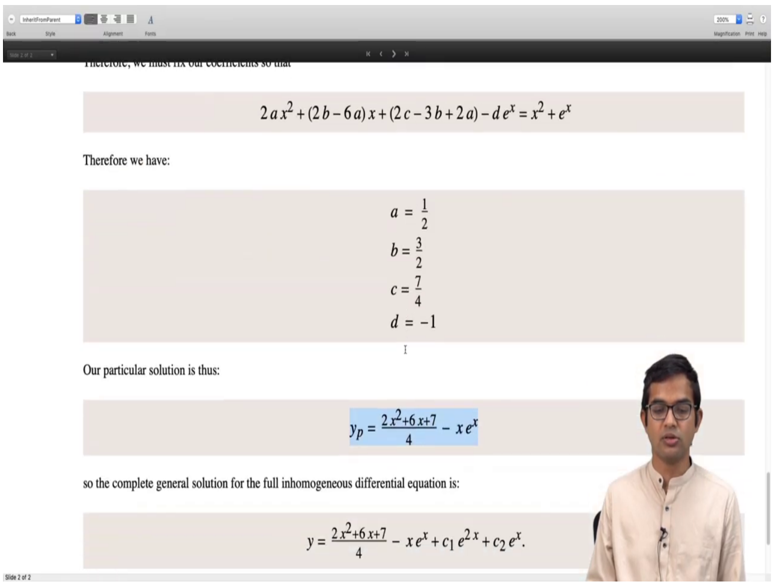

So, a of course, has to be just half because x squared must come on the right hand side. Once you fix a you see that 2b minus 6a must be equal to 0 there is no linear term on the right hand side. So, b is equal to 3a. So, 3 by 2and then if you know b and you know a you can work out c and it works out to 7 by 4 and d of course, we have already seen is minus 1 right..

So, the key argument here is of course, this equality must hold for all values of x. So, term by term this must agree on both sides and therefore, it works out right. So, you can check that the final you know on the final particular solution that we have is indeed a particular solution.

So, to do that you must take this function and directly plug it back into your original differential equation right. So, you must plug it in and operate this function with d squared by dx squared minus 3 times d by dx plus 2. You must operate with this and verify that indeed you get x squared plus e to the x.

And if that holds it does not matter how you have arrived at this particular solution. If it is a particular solution, it is a particular solution and then that is enough for us we can use that along with the complementary function and write down the full answer. The complete general solution for the full inhomogeneous differential equation here, is given to be just 2 x squared plus 6x plus 7 divided by 4 minus x times e to the x plus c 1 times e to the 2x plus c c 2 times e to the x.

So, hopefully these examples will help you get a hang of the theoretical methods we have laid down earlier. I would urge you to tweak these examples, you know, come up with your own versions of this problem and get a really solid understanding of this method. And this will extend also to higher orders, but for our purposes we have found it convenient to just stick to second order. So, that is all for this lecture.

Thank you.