Mathematical Methods 1
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Linear Algebra
Lecture - 07
Cauchy-Schwartz inequality for vectors from LVS

So, in the previous lecture, we saw how we could derive Cauchy-Schwartz inequality when we are looking at vectors in three-dimension right. So, there it turned out to be a result which was not a surprise right. So, but we came up with a method right, we outlined a geometric approach which something similar can be used to generalize you know the Cauchy-Schwartz inequality, we can get a generalized version of the Cauchy-Schwartz inequality for you know vectors drawn from a linear vector space.

At the moment, a scalar product is defined, the Cauchy-Schwartz inequality will come into play and you know the way to obtain it is analogous to the method that we use to get to the Cauchy-Schwartz inequality with three-dimensional vectors. So, that is the subject of this lecture ok.

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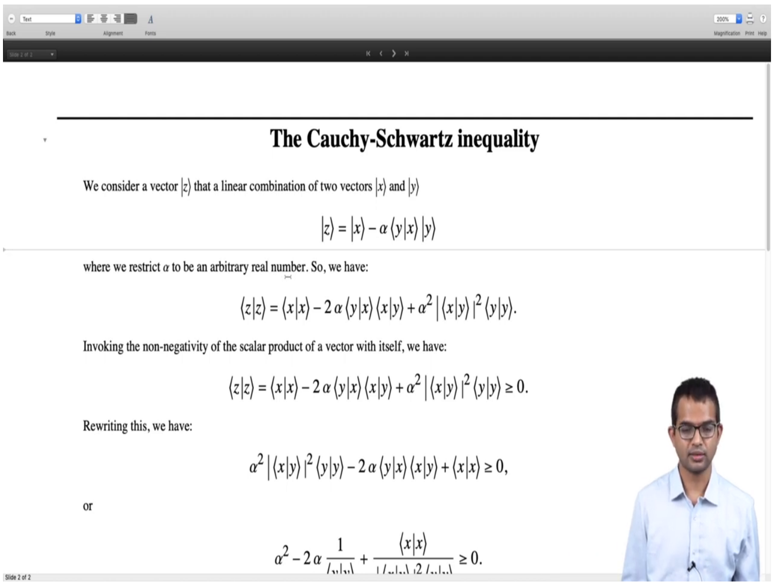

So, we consider some vector z which is a linear combination of two vectors x, y right. So, as always like in the previous you know lecture, we saw we took two vectors a and b. So, likewise we are taking you know two vectors x and y so, generic vectors right. So, there will be some special cases you know where x is related to y in a certain way or if it is another vector, these are all special cases which you can think about you know separately.

But let us say that these two are you know generic vectors x and y and we take the special linear combination x and y right I mean in general, we could have put some arbitrary coefficient here, but we already have this you know wisdom from hindsight or we have used it. And then, we saw there that there was some nice result that could be extracted by considering you know this a dot b.

So, we will consider the inner product of y, x will already put in an inner product y, x I mean there was a geometrical interpretation which we gave for why we went about considering this particular vector. So, you know to get to an inequality, we are going to use you know the one inequality which is sort of encoded in the definition of inner product right.

So, that is where any other you know interesting inequality is going to come from the fact that you know every inner product of every vector with itself must be greater than or equal to 0 that is what we are going to use. But we are going to first of all try to get a you know some nice linear combination of two vectors which is like here and then invoke the property that we care about ok.

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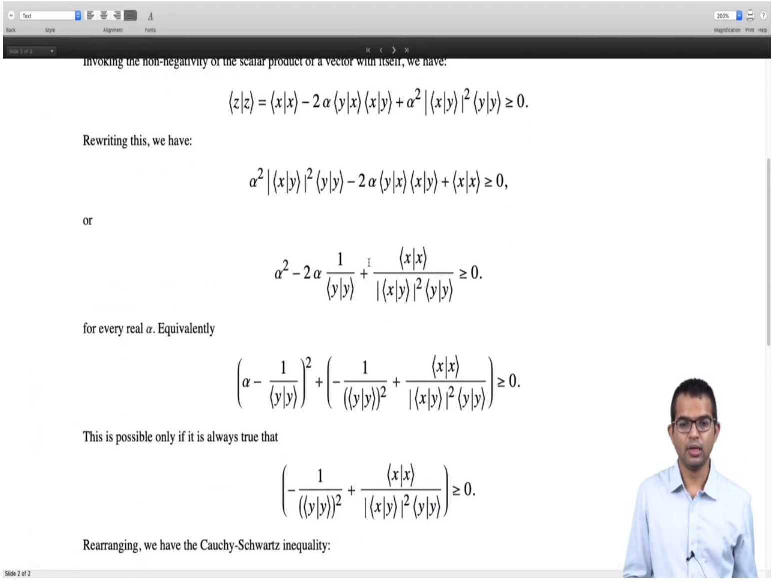

So, let us restrict alpha to be some real number. So, if we take the inner product of z with itself, what do we get? Ok. So, if you take the inner product of z with itself, you have this inner product of x with itself minus 2 times alpha times y, x times x, y right why does this happen? Right, so, you have to work through this carefully right, this comes from the fact that alpha is real right.

So, if I do inner product of z with itself so, when I take the bra vector z, alpha is going to remain alpha there is no alpha star because it is a real number and then, but y, x will become x, y right you have to carefully work out this algebra and then, you will see that you have x, x minus 2 alpha y, x, x, y.

So, you are going to get x, y here and the other one will come from you know you get y, x times x, y and x, y times y, x does not matter. So, you can combine couple both of them, you will get minus 2 times alpha times y, x times x, y and then, you have the final term which is plus alpha squared mod of x, y the whole square times y, y right all of this you will have to convince yourself by working through this right, you have to do the algebra.

Not very difficult, but already there is an application of the properties of the inner product that are already used here. Now, comes the you know the property that of non-negativity of a scalar product of a vector with itself. So, if you just apply this so, we have this you know this entire quantity must be greater than or equal to 0 and then, we write it in terms of as a quadratic expression in alpha square right.

So, now we recall that alpha is some arbitrary number for any value of alpha, any real alpha this condition must hold right. So, we will write this as a quadratic expression in alpha. So, you have alpha squared minus 2 alpha times some coefficient plus some other coefficient must be greater than or equal to 0 right. So now, we will complete the squares.

So, you have alpha minus 1 over y, y the whole squared and then since we have got an extra 1 over y, y squared, we are going to subtract that and then, you have minus 1 over y, y squared here plus this other constant must be greater than or equal to 0.

Now, this must hold for any real value of alpha, for any real value of alpha, this quantity here right y, y is a real number, alpha is a real number the difference of two real numbers is the real number, the square of a real number is going to be real and positive alright. The sum of a positive number with another quantity which must be greater than or equal to 0, for any value of alpha. So, this can happen only if this quantity itself is greater than or equal to 0 right. So therefore, it forces this quantity must be greater than or equal to 0.

Now, if we rearrange these terms, we have the Cauchy-Schwartz inequality which is the statement that the inner product of any two vectors x, y, the modulus squared of the inner product of any two vectors must be less than or equal to the product of the inner products of each of these vectors with themselves right, inner product of x with x and inner product of y with y the product.

This product is going to be greater than or equal to the modulus squared of the inner product of these two vectors right. This is a Cauchy-Schwartz inequality which has very beautiful applications in many fields of mathematics right and it appears in you know when you are studying complex numbers, it appears in all kinds of contexts.

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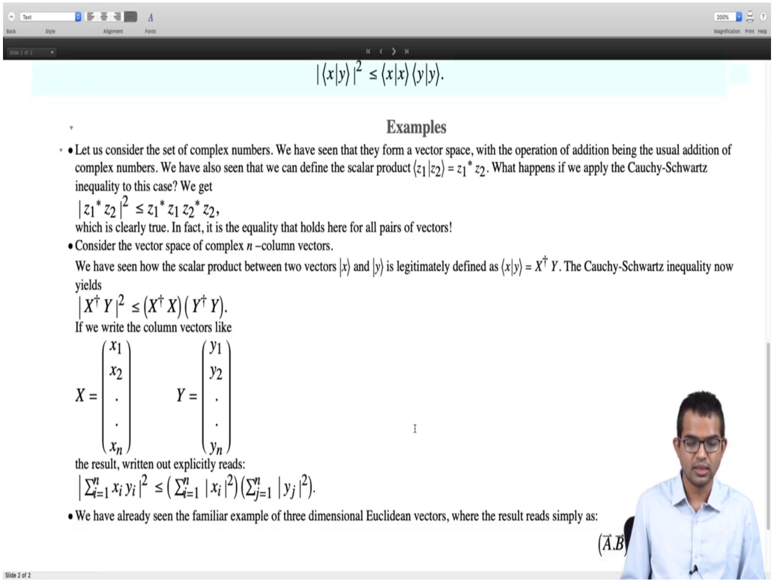

So, let us look at a few examples where you know drawn from within you know the theory of vector spaces. So, we have seen that we can think of the set of complex numbers as a vector space right. So, with the operation of addition being the usual addition of complex numbers.

So, we also defined a scalar product z 1 and z 2 which is just going to be z 1 star z 2 it is a legitimate scalar product. So, if we simply impose this Cauchy-Schwartz inequality to this scalar product right, so, it is important to realize that the Cauchy-Schwartz inequalities is very general and it will hold for any definition of a scalar product as long as it is a legitimate definition of a scalar product you must have all these properties corresponding to the scalar product satisfied and then, it should work out.

So, look at what happens here. So, for this case it turns out that this inequality becomes like a trivial equality right. So, why does this happen? So, you have z 1 star z 2 the whole squared mod of this squared is must be less than or equal to z 1 star times z 1 times z 2 star times z 2, but the right-hand side and the left-hand side are the same for any two vectors z 1 and z 2.

So, it is never going to be a hard inequality, it is always going to be an equality right. So, it reduces to an equality in this particular case. Now, but there are other cases where it is actually quite a you know it is a strong result. For example, if you have n complex numbers which form you know we saw that n-column vectors form a linear vector space right, consider any two such vectors.

We also said that you know there is a legitimate inner product which one can define as X dagger Y, you take the rho vector corresponding to x multiply with the column vector corresponding to y, you know you are multiplying all these complex coefficients x. And then, the Cauchy-Schwartz inequality now says mod of X dagger Y the whole squared is less than or equal to X dagger X times Y dagger Y right.

So, explicitly if you want to see this, let us say you have a, you know you have two column vectors like x and y you know x 1, x 2, x and all the way up to x n, they are all complex coefficients right. So, and y 1, y 2, y n all of them are also complex numbers, arbitrary complex numbers. So, what this result says is, it does not matter - you just have some n arbitrary complex numbers here and another n arbitrary complex numbers you tie them up in this particular manner, this inequality necessarily holds right.

So, x, y x i y i you add them up, take the square of them and so, this must be necessarily less than or equal to summation of i over i mod of x i squared the whole thing times summation over j mod of y j squared right, so this product. This inequality can be derived directly by other methods, but the beauty is that we have this very general result and it finds application in all these particular cases.

And then of course, we have seen one example which is the familiar result of Euclidean vectors, where the result simply reads as a dot b the whole squared is less than or equal to a dot a times b dot b right. So, there I mean we saw that this was simply a consequence of the fact that cos squared of theta is always less than or equal to 1 right. So, it does not seem like a big deal, but the point is that this has the power of application in a much more general context.

And in fact, one can combine these ideas right cos square of theta is less than or equal to 1 implies the or is seen to be equivalent to the Cauchy-Schwartz inequality and so, you can use this as a way to define a cos theta between any two vectors in a you know in an abstract vector space right. So, maybe we will look at that in another example, but the point is that so, there is this analogy which one can make right. I will just make that as a statement, but maybe we will go into it elsewhere and so, that is all for this lecture.

Thank you.