Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 69
Ordinary Differential Equations

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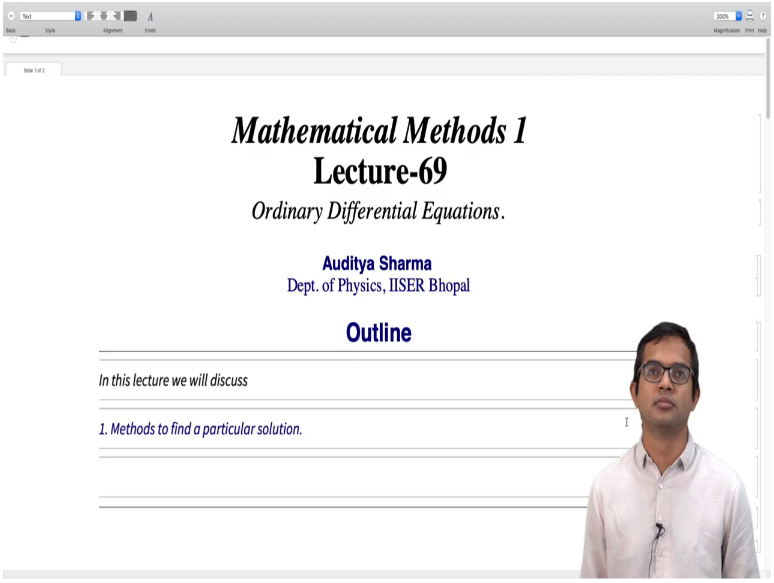

So we have been looking at second order differential equations linear and with constant coefficients. So, we saw how the solution of the inhomogeneous differential equation is connected to the solution of the homogeneous differential equation. So, the key extra ingredient which we need when we have an inhomogeneous differential equation is to be able to find some one particular solution.

If you can find a particular solution, you can just add it to the complementary function and you are done right. So, in this lecture we will look at how to find a particular solution for certain special kinds of inhomogeneous forcing terms right; so that is the subject matter for this lecture ok.

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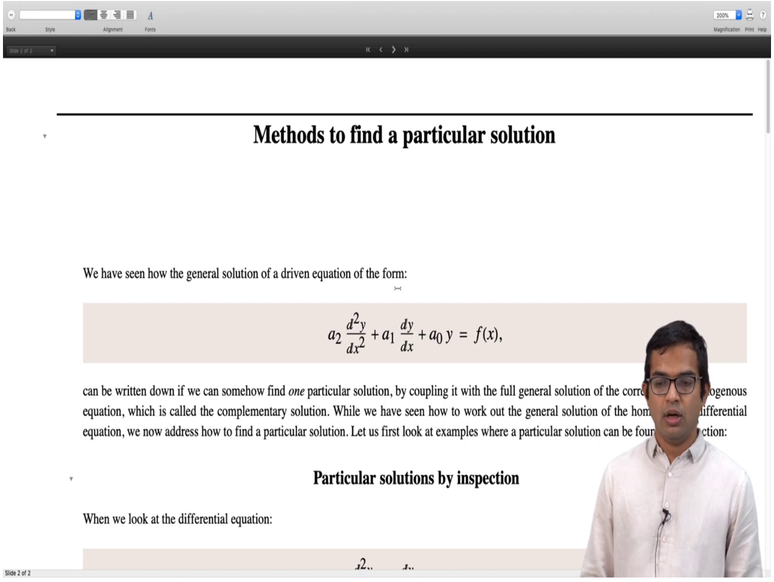

So, we are looking at a differential equation of this kind; a 2 second derivative of y plus a 1; first derivative of y with respect to x plus a naught y is equal to there is a forcing function on the right hand side right. If we can somehow find a particular solution for this differential equation, we already know how to solve the corresponding homogeneous equation. And extract the complementary function; we can just add the two and we are done right.

So, one way to get to a particular solution is to just do an inspection. So, let us look at some examples where you know this inspection will work and that will give us some hints for how to go about it for a fairly big class of forcing functions and a useful class of forcing functions.

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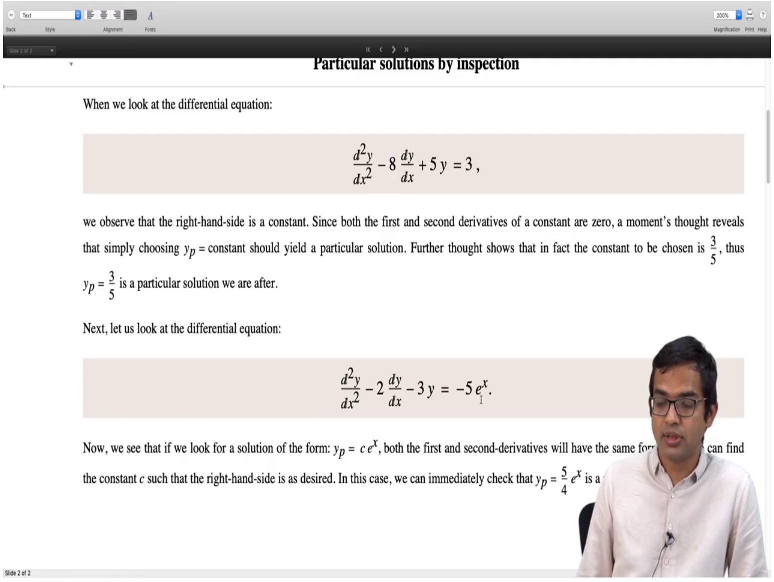

So, suppose we are given a differential equation like this right; so, we see that the right hand side is just a constant right. So, this is you know only marginally more difficult than if you had the right hand side to be 0 right; you it is just it is a constant on the right hand side. So, a little bit of thought reveals that you see you have a first derivative and a second derivative right; both of these will just be 0, if you choose your function itself to be a constant right.

And then it is just a matter of choosing the right constant so that you know; it is just; this must hold right; so indeed that is the right way right. So, we just choose y p is equal to a constant and then a little more thought here; you know tells us that that constant must be actually for this particular example 3 by 5 right.

So, 3 by 5 times 5 is just 3, these guys do not matter because you know the first derivative and second derivative of a constant will be just 0. So, then it is simply a matter of; you know it is like a shift of your homogeneous differential equation solution; you find the solution and then add 3 by 5 and you are done right ok; so this is straightforward.

Now, what about this? So, let us look at this differential equation where we have an exponential of x which appears on the right hand side right. So, this is an important class of problems; whenever you have some exponential of x or you know x in general exponential of some factor times x; you know with an overall factor outside or you can have you know sums of these that is more you know complicated variance; so this can be constructed, we will look at those a little bit later.

But if we look at this differential equation then we have; so we see that ok now if we choose a constant, it is not going to work out right, but some thought reveals that maybe if we choose a function of the form e to the x right. Because we will have e to the x and then we take the first derivative again you will get back e to the x. And then the second derivative also will give you e to the x.

So, if you just choose e to the x and with a factor along with it; it should work out and indeed this is born out to be true. So, let us try y p is equal to c times e to the x; so the first derivative is just c times e to the x, second derivative is also c times e to the x. So, we must; we will get 1 minus 2 minus 3; so it is 1 minus, so it is minus 4 times; c must be equal to minus 5.

So, if you choose c to be 5 by 4 or if you choose y p is equal to 5 by 4 times e to the x; that is going to be a particular solution for this problem right. So, a particular solution here means a full solution for the problem because we know how to solve the homogeneous equation ok. So, this works out fine; so the question is this you know general enough right?

Suppose, we have a function of the form e to some constant times x; will we be able to find a suitable constant times e to the same power; you know times x and will be, will it work out? So, let us look at another example.

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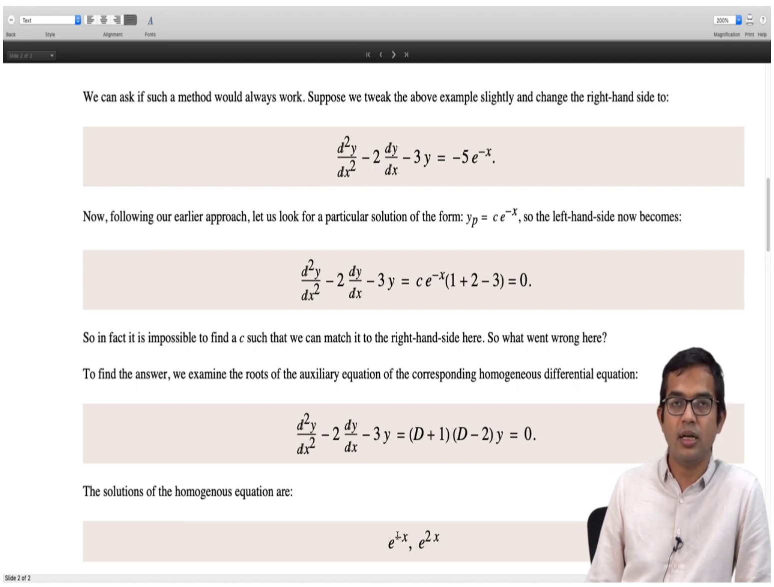

So, how to see if this is; you know; in fact as general as we would like it to be. So, suppose in this example I have the same left hand side; d squared y by d x squared minus 2 d y by d x minus 3 y. But on the right hand side, instead of minus 5 times e to the x; I am going to change this to minus 5 times e to the minus x. So, let us see now whether the same approach will hold.

So, what did we do; so we might guess that now we should try by a particular solution of the form; constant times e to the minus x right, that is a reasonable you know guess. So, if we plug this in then we see that the left hand side becomes the second derivative; you will have a minus sign; times minus x that is going to remain a positive sign.

So, its c times e to the minus x; so times 1, but this first derivative will give you a minus sign; so it becomes 1 plus 2 and then you have minus 3.

You know overall; it is just c times e to the minus x; I have pulled out. So, then I see that this coefficient is 1 plus 2 minus 3 which is 0 right. If it goes to 0, then there is no way that I can choose my c and demand that the right hand side must go to minus 5 times e to the minus x right.

So, how did the previous method work? It worked because you know this coefficient was some other number; other than 0, then I had to just choose my c times this number should be equal to minus 5 and then I was done.

But now if this number is going to 0; then there is no way that I can choose my c such that overall it must give me minus 5 times e to the minus x right; so this is a problem. So, we see that apparently this method is not quite general. So, what went wrong right? So, we will see to examine you know; if we examine the roots of the auxiliary equation, we will find an answer for this question.

So, let us look at the corresponding homogeneous differential equation. What is the homogeneous differential equation? It is this guy; the left hand side is equal to 0, but the left hand side can be factored; right as d plus 1 times, d minus 1; d minus 2, times y equal to 0. So, the roots of this auxiliary equation are minus 1 and plus 2; therefore, the complementary function is you know given by you know; these two are the linearly independent solutions, e to the minus x and e to the 2 x.

So, now we see the crux of the issue is that e to the minus x is what appears on the right hand side. So, in fact one of the solutions of your homogeneous equation is your forcing function.

So, whenever this happens; it is a problem, you cannot blindly find a particular solution of this form. So, you must try an alternate form which in this case is actually just x times e to the minus x right; times some constant to be determined right.

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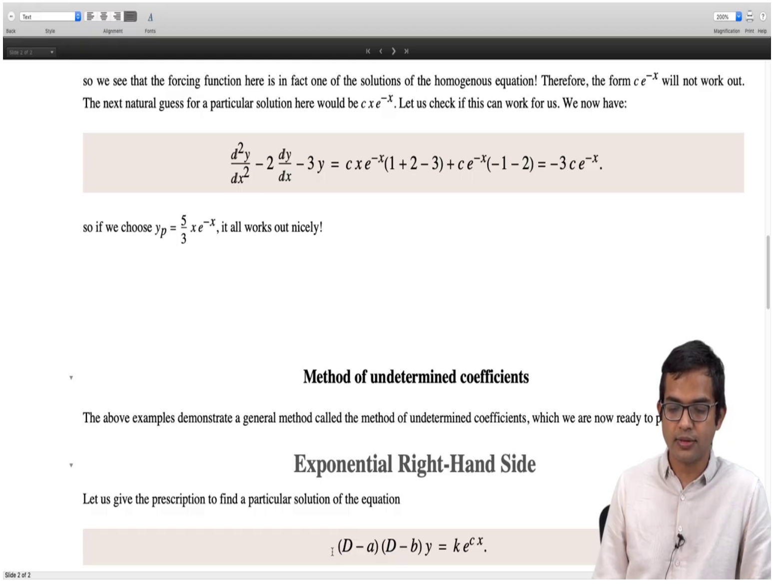

So, let us see if this will work out; if we plug in c times, x times; e to the minus x, now what happens? We have the second derivative minus 2 times, the first derivative minus 3 times y is equal to; now we have two kinds of terms, one term is c x times e to the minus x right, where you know if you take the derivative only with respect to this e to the minus x; in at all levels, you get 1 plus 2 minus 3; so this part will in fact go to 0.

But then, you have another term plus you know the first time you take a derivative; so you will take a derivative with respect to x. So, you have c times e to the minus x; then you take another derivative and you get a minus 1; so you have a minus 1, minus 2 right. So, if you do this; so you get c times; e to the minus x, times minus 1, minus 2; so in fact, you have a minus 3 c times; e to the minus x.

So, it is not going to go to 0 now anymore, but you will get minus 3 times; c times e to the minus x. So, now we can choose minus 3 c to be minus 5 and so if you choose your y p to be 5 by 3 times x times; e to the minus x, we can check; you can check it again that will all it all works out nicely and in indeed we have managed to find a particular solution for this problem right.

So, what we have been doing is an example of a general method called a method of undetermined coefficients right. So, which can solve for you know fairly a wide variety and useful variety of forcing functions right. So, that is what we will describe here. So, whenever you have an exponential right hand side; like we already looked at, the method is you know you first of all write down the left hand side; you factorize it, D minus a times; D minus b times y.

So, these methods you know carry through for even higher order differential equations right. For our purposes, let us just concentrate on second order differential equations, but you know there is a ready generalization available to even higher order problems ok. So, if you have D minus a; times D minus b times y is equal to k times e to the c x. So, the key point now is to find out the relationship between this c and a and b right.

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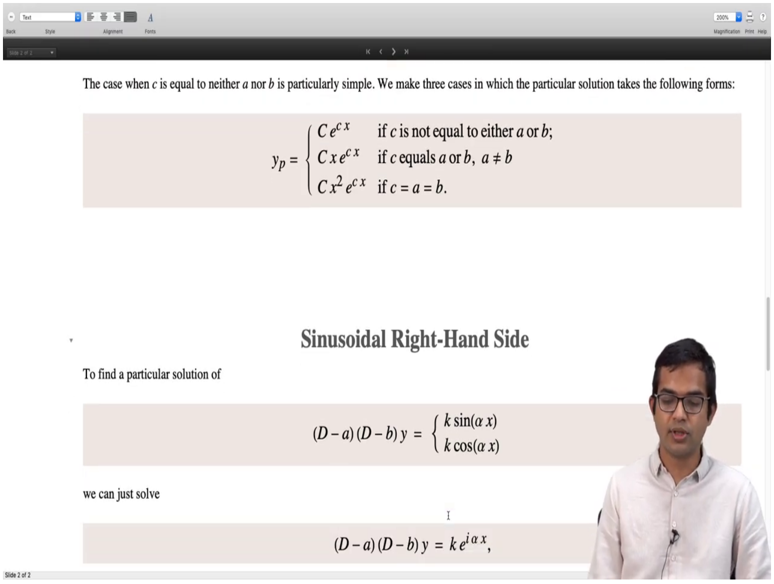

If c is equal to neither a nor b; then it is super simple that is the first type of problem that we have looked at. So, then you are guaranteed that you will have a particular solution of this form; some coefficient times e to the c x right is a solution of this problem. Now, if c is equal to one of these two; c is equal to a or b and a naught equal to b, then that is the second type of problem which we just checked out.

So, then you are guaranteed to be able to find a particular solution of this form c x is equal to e to the c x. And there is also a third case which is if you know you have repeated roots and your c is equal to the repeated root; then you can pick up an example of that kind and verify that. In fact, even x times e to the c x will fail, then you have to go to one more you know higher order in the polynomial.

So, you get c times x squared; e to the c x; for sure you will be able to find a particular solution of this form right. So, this is when you have the exponent; the right hand side being an exponential right. So, another closely related right hand side is if you have a sinusoidal right hand side instead of an exponential.

So, you have a scenario where D minus a; times D minus b acting on y is some k times sin of alpha x or k times cosine of alpha x. So, this is seen to be straightforward; so what you do is, you write you know you look at the solution for this problem D minus a; times D minus v times, y is equal to k times e to the i; alpha x right.

So, here in fact so the c can be complex as well right; it can be you know that is the scenario that I am looking at here.

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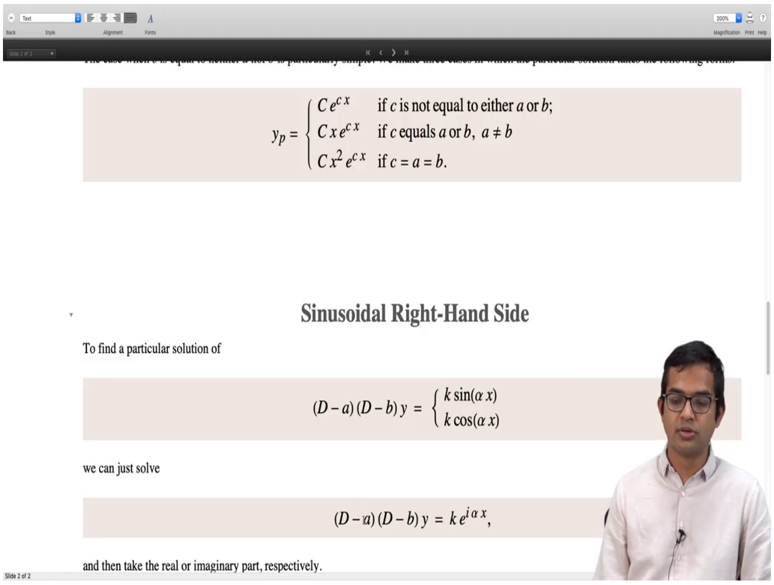

So, i times alpha times x and then you just solve for this problem right. So, a and b in general can also be complex numbers. If they are complex because they are the roots of a quadratic equation, they are going to be conjugate to each other. So, in this case the same kind of conditions will hold and you just solve this problem.

And then what happens is you just take the real part or the imaginary part depending upon which of these problems are of interest. So, you see that D minus a; times D minus b acting on some real part plus i times, imaginary part is equal to k times real part plus you know k times imaginary part.

So, if you take the real part and imaginary part; you know separately you can find the solutions for each of these kinds of right hand sides right. So, this is very very closely related to this type of problem. So, it is best understood with the help of an example. So, which you will find homework on this right; so where you will work this out and convince yourself that this indeed holds.

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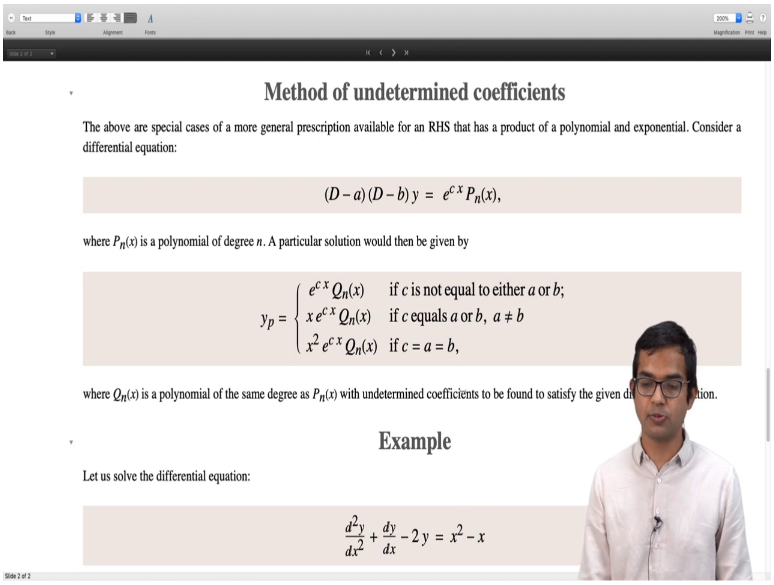

So, let us write down; so in fact, the method of undetermined coefficients is some more, there is some more generality to this. In fact, you can you know look for a solution for this general differential equation; you have D minus a times; D minus b acting on y; is equal to some e to the c times x times some polynomial of x, you can have a polynomial also of x on the right hand side and where P n of x is a polynomial of degree n.

So, in this scenario; it turns out that if you choose a particular solution of this form, again all you have to do is first of all check the relationship between c and a and b right; the polynomial is a separate thing by itself. So, you must make a guess of this form; e to the c x times some polynomial of the same degree right. So, it is an nth degree polynomial; so there will be n unknown coefficients; you have to match those coefficients that is why it is called the method of undetermined coefficients.

So, e, but the key point is that you are guaranteed to be able to find a particular solution of this form. It is just a matter of plugging this form into the differential equation and finding all the coefficients which are unknown right; that is why it is called undetermined coefficients, they can be determined you know. And if c is; if c is not equal to either a or b; it is of this form, but if c is equal to one of the two; a or b and a is not equal to b, then again you will get this x times e to the c x.

So, now instead of a constant; you have a whole polynomial right. So, this constant also gets absorbed inside this polynomial which itself needs to be determined. So, the key is that this polynomial has the same degree as the polynomial which appears on the right hand side in the original inhomogeneous differential equation. And then when c is equal to a equal to b; then you have x squared times e to the c x times q n of x right.

If you make this as your own sort, you will be able to determine the coefficients you know which are unknown and you will have a particular solution for this type of a problem ok. Let us look at one example where, so I have here d squared y by d x square plus d y by d x minus 2 y is equal to x squared minus x. So, I do not have any exponential; here I have purely a quadratic term on the right hand side. So, using this prescription; I will look for a particular solution.

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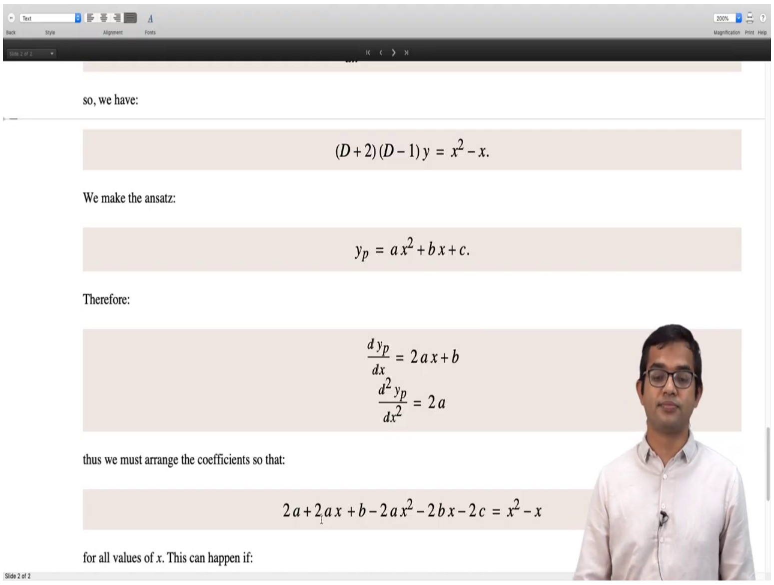

So, I first of all write this down as D plus 2 times; D minus 1 times y is equal to x squared minus x. So, I have just factorized the left hand side and then we make an ansatz which is a quadratic function; a x squared plus b x plus c, where a, b and c are these undetermined coefficients.

So, now the derivative of this is going to be 2 a; x plus b and the second derivative is just 2 a. So, if I must arrange the coefficients such that you know if I subtract; if I add these two and subtract 2 times y p; it must give me x squared minus x. So, I must have 2 a plus 2 a; x plus b minus 2 a x squared minus 2 b x minus 2 c, this entire thing must equal x squared minus x.

So, the key point here is that you know these two expressions are equal for all values of x right; so this is a pretty strong condition. So, what it means is it forces equality term by term. So, the constant on the left hand side is equal to the constant on the right hand side; the first order term on the left hand side is equal to first order term, second order term on the left hand side equal to second order term on the right hand side.

So, that is why you get many equations; although there is just one equation, it is actually it contains many equations; contains three equations in this case.

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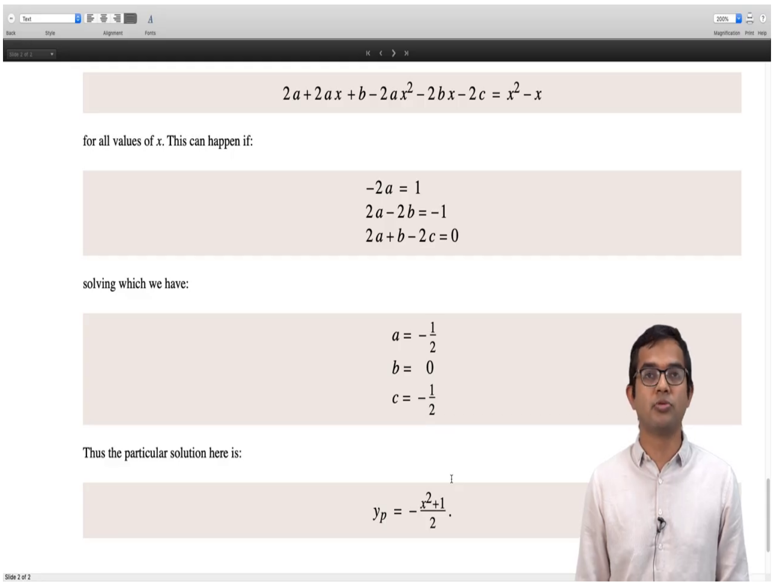

So, you have minus 2 a which is the coefficient of x square on the left hand side must be equal to 1 which is 1 on the right hand side. And then you have 2 a minus 2 b which acts on x; so that must be equal to minus 1 and then 2 a plus b minus 2 c which is a constant term on the left hand side, there is no constant term on the right hand side; so that must be equal to 0 right.

If you solve for this; so you have three unknowns and three equations and you can solve them. In this case, you get a equal to minus a half, b equal to 0, c equal minus half. So, you can use all this information and then write down the particular solution in this problem which is just minus x squared plus 1 by 2.

You can go back and plug this; in fact, you should go back once you have found the solution and plug this back directly into this differential equation and verify that indeed this is a solution of the original differential equation. So, once you have a particular solution; the full general solution can be written down as we have seen ok. So, that is all for this lecture.

Thank you.