Mathematical Methods 1

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Ordinary Differential Equations
Lecture - 68
Inhomogeneous second-order equation

Ok. So, in this lecture, we will look at second-order equations which have a term on the right hand side. We will see how the solution for the inhomogeneous second order equation or in general for any order equation in fact is connected closely to the general solution of the homogeneous equation, the corresponding homogeneous equation right.

So, we spent a fair amount of time writing down the prescription for you know the solution of the homogeneous, linear homogeneous second order differential equation. And here we will see how we can use this to write down the solution for the inhomogeneous second order equation, if you are able to find something called a particular solution. So, that is what this lecture is about.

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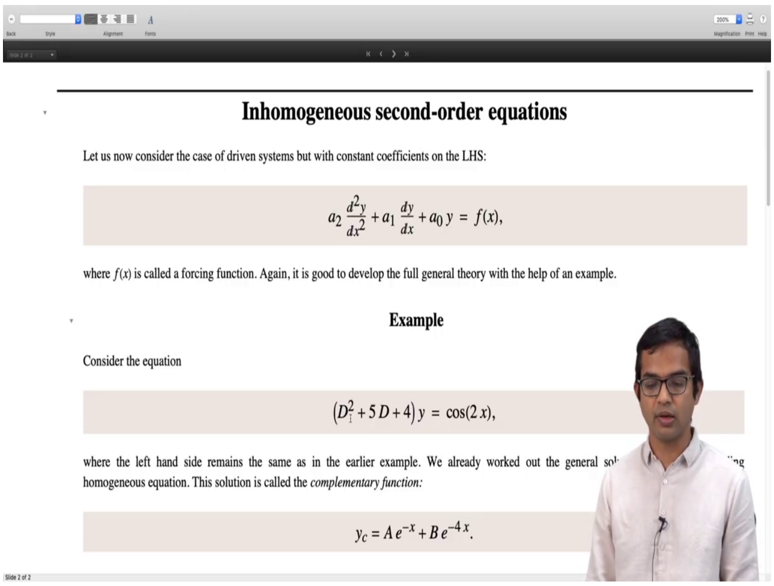

So, we are interested in a differential equation of this type: a 2 d squared y by dx squared plus a 1 dy by dx plus a naught y is equal to a function of x on the right hand side right. So, a 2, a 1, and a naught are constants, and f of x is called a forcing function right. So, sometimes this is also called a driven system.

So, you can think of the right hand side or as some you know a physics motivated example as an externally applied field of some kind right. So, the differential equation you set up is to see what is the response of your system to an external driving force right.

So, oftentimes you know special kinds of external force are of interest like you know periodic functions you know sometimes functions, which have some small tweaks around the periodic function, or you know some kind of randomness and so on right.

So, there are a lot of interesting applications, which come off from differential equations of this kind and where one can play with the form of f of x on the right hand side. But as far as the theory of differential equations is concerned, you know whatever function f of x appears on the right hand side as long as it is not 0, then it becomes an inhomogeneous second order equation.

And so the point of this lecture is to show how the solution of this inhomogeneous equation is related to the solution of the general solution of the corresponding homogeneous equation. So, let us look at an example. So, we have given some particular differential equation like this D squared plus 5 D plus 4 acting on y is equal to cosine of 2 x right.

So, like I said, forcing functions which are sinusoidal in nature are common. So, let us look at cosine of 2x. Now, this is an example we have considered before where we looked at just the homogeneous equation. But anyway we will return to we will look at the homogeneous differential equation right solution down. So, we already worked this example out.

So, if you have not seen this ok it is a matter of taking this factor you know write this as D plus 4 times D plus 1 acting on y is equal to cosine of 2x. But the homogeneous differential equation is when there is no cosine of 2x, you just put equals 0, and then you write down the answer as just e to the minus x and e to the minus 4 x both of them are roots are solutions to this homogeneous equation. And this is called a complementary solution right.

So, the general solution of the corresponding homogeneous equation is called the complementary function y c subscript c is equal to A times e to the minus x. So, this minus 1 and minus 4 are roots of this auxiliary quadratic equation. And we have seen how you just put these exponential e to the minus x and e to the minus 4 x both of them are valid solutions.

So, an arbitrary coefficient A times e to the minus x plus an arbitrary coefficient B times e to the minus 4x will be the general solution of the homogeneous equation. So, it is called the complementary function.

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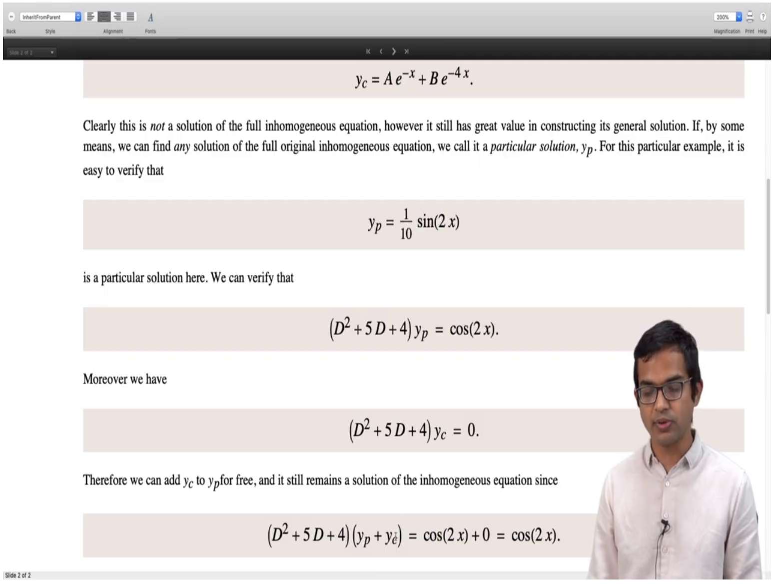

So, now comes clearly this is not the full solution of the homogeneous equation right. So, however, it has great value because if you can somehow find what is called a particular solution. If you can find one solution of this full differential equation, you can add the solution y c for free. And the resulting linear combination is still a solution of this full differential equation right.

Why does this happen? Because you can think of this as some operators there is a linear operator which acts upon y p right, you have managed to find a y p such that when this linear act operator acts upon y p it gives you cosine of 2x. You have already found a particular solution.

Now, if you add this y c to it right, so which has these two arbitrary coefficients associated you know we have just seen that it is a linear operator. So, this operator when it acts upon y p plus y c, it will act upon y p and y c separately. When it acts upon y p, it gives you cosine of 2x. But when it acts on y c where you are going to get 0, because y c is obtained as a solution of this linear operator acting on y equal to 0 right.

So, it is for this reason that if you can find one particular solution of the full inhomogeneous differential equation, you can just simply add this particular solution to the complementary function and you are done right. So, that is going to be the full general solution of the full problem and it has two free coefficients, so that is as general as it comes for a second order differential equation, ok.

So, we can verify right. So, in this case y p is equal 1 over 10 times sin of 2x turns out to be a particular solution right. Let us say you have by some means you managed to find this right. So, later on, we will discuss a method for finding these particular solutions right. So, if so in this case you can verify that y p equals 1 over 10 sin of 2x is a particular solution right.

So, you can verify that if you act upon this function, you know take the first derivative, take the second derivative, you know use all these coefficients appropriately add them up, you are going to get cosine of 2x right check this right. So, like I said we also have D square plus 5D plus 4 acting on y c is equal to 0.

Therefore, we can just add these two for free. And if you take this linear operator and act it upon y p plus y c, you know it gives you the linear acting operator acting on y p which is cosine of 2 x plus this linear operator acting on y c, which is equal to 0. Because that is how we have found y c. And therefore, this is equal to cosine of 2x.

So, therefore, if y p is a particular solution, then y p plus y c is also in fact the general solution of this second order inhomogeneous differential equation right. So, this is the answer. So, you go ahead and write down y is equal to 1 over 10 times sin of 2 x plus A times e to the minus x plus B times e to the minus 4x right. It is the full solution for this problem ok.

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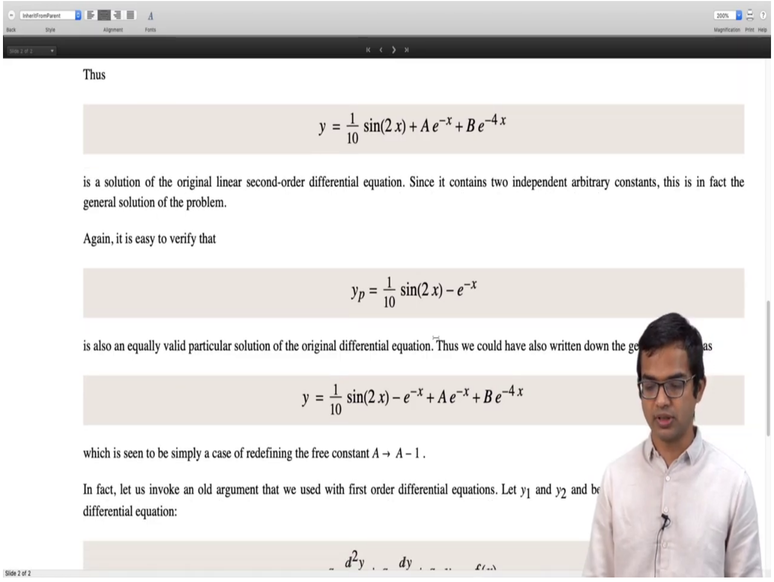

So, I mean we use something similar even for first order differential equations where we use the solution of the homogeneous equation to get to the solution of the inhomogeneous equation right. So, this is a general trick it goes to even higher order problems as well, this will hold. And there will be some other details which will come in.

But in any case so the main message here is you know now we are beginning to see why it is important to have a good solid understanding of homogeneous differential equation, because even inhomogeneous solution inhomogeneous differential equations are connected to the underlying homogeneous equation, ok. So, it is easy to verify that this y p you should check this right.

So, this y p is not unique. If you had chosen y p to be 1 over 10 sin of 2 x minus e to the minus x this two could be a completely valid particular solution right. If we had used this instead of the other y p, the answer we would get is 1 over 10 sin of 2x minus e to the minus x plus you know the same prescription A times e to the minus x plus B times e to the minus 4x, which is also a completely valid general solution right.

So, you observe that all that has happened is in place of A, we seem to have shifted this to A, A minus 1 right. And it is not a surprise. So, in fact, we can invoke this old argument, which we used with first order differential equations.

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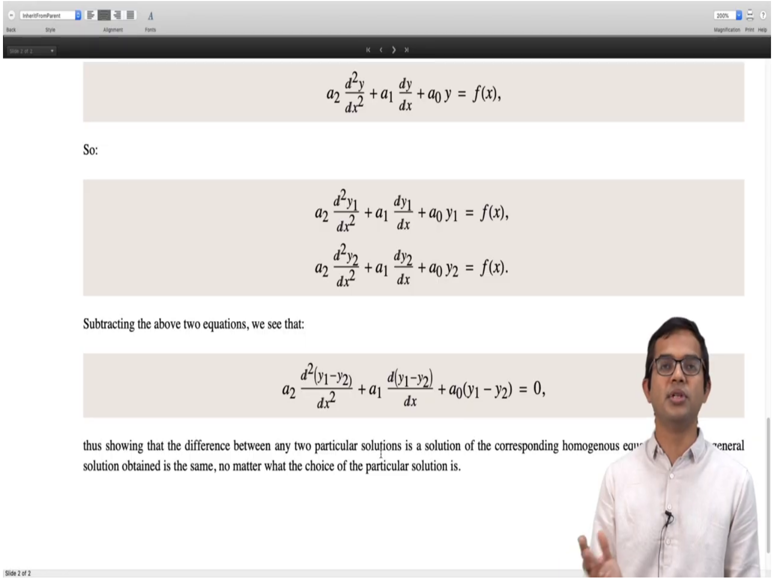

If y 1 and y 2 let us say there are two particular solutions of the full inhomogeneous differential equation, then we have these two conditions a 2 acting on d squared y 1 by dx squared plus a 1 dy 1 by dx plus a naught y 1 is equal to f 1 f of x. And likewise similarly with the function y 2 right, so, you have one more condition.

So, you can go ahead and subtract these two equations. So, then what do you get? You get a 2 times the second derivative of y 1 minus y 2 plus a 1 times the first derivative of y 1 minus y 2 plus a naught times y minus y 2 is equal to 0. So, if you think of y minus y 1 minus y 2 as a function right, you see that it satisfies the homogeneous differential equation right.

But, we have seen that you know the general solution is already available for us, so y 1 minus y 2 is going to be the general solution. So, therefore, you know any two particular solutions of your inhomogeneous differential equation you know are different from each other by some you know by an amount, which is a solution of the homogeneous differential equation right.

So, it is in this sense that it does not matter which particular solution you find. You find any particular solution and you just teach it with the homogeneous equations general solution, and then you get the full general solution right. Sometimes you know this particular solution that you obtain looks much more deceptive than here.

So, in this case, you see that you know we have just subtracted some part of the solution from taken from the complementary function right e to the minus x appears here. We have just pulled out this e to the minus x. And of course, it is going to be this - it is not a surprise that this object minus this is also a particular solution right.

But sometimes it is a little more deceptive than this. So, it does not matter. If you just need to find one particular solution and the complementary function, and then you can just add them and you get the full solution of the normal linear differential equation ok. That is all for this lecture.

Thank you.