Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 66
The use of a known solution to find another

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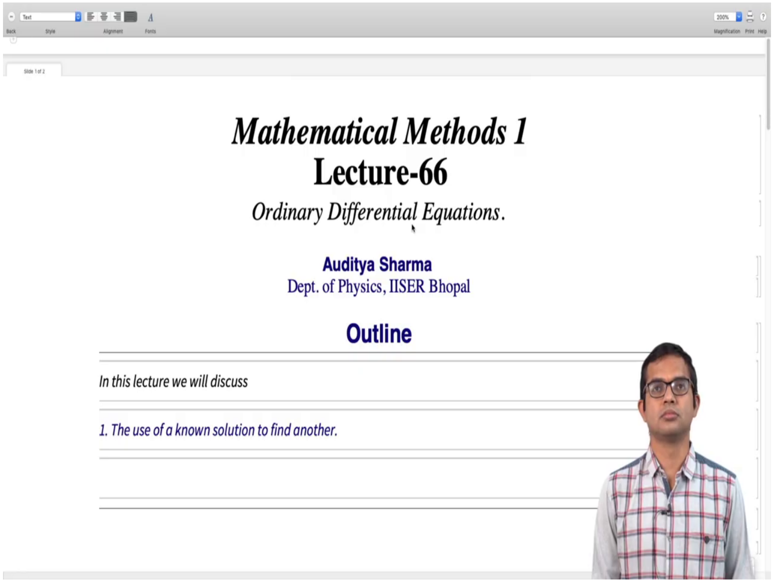

So, we have been looking at second order differential equations. We started with you know the homogeneous version of the second order linear differential equation with constant coefficients right. So, first we said that we are interested in second order differential equations. Then we said that ok, let us look at linear differential equations because they are easier to solve. Then we said ok let us look at just the homogeneous version of it.

And it turns out that even the homogeneous second order differential equation in its full generality you know does not have a completely general solution right. But if you have constant coefficients, we have given the prescription for it, but I mean the more general homogeneous second order differential equation even if it is linear can be quite complicated.

But in this lecture, we will see that if by some means we are able to guess one solution we can also find there is a general prescription to find another linearly independent solution for an arbitrary complicated linear homogeneous second order differential equation right. So, that is what this lecture is about.

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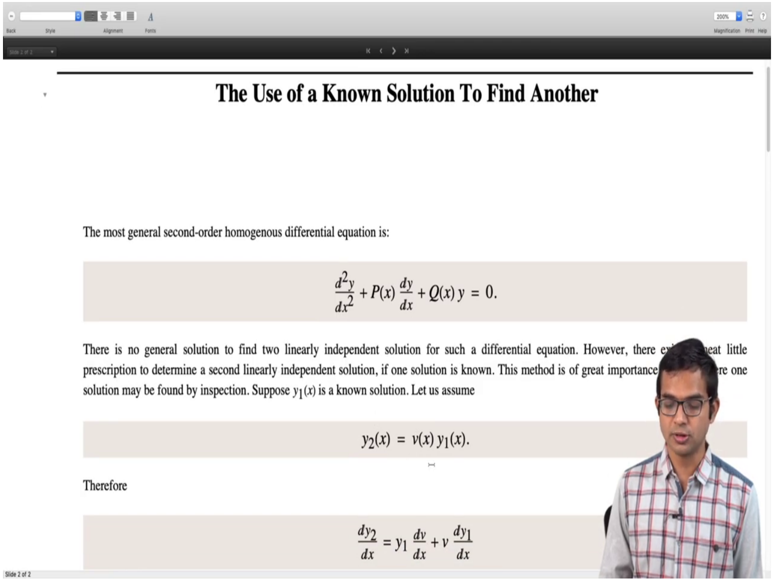

So, the most general second order homogeneous linear differential equation is of this form right. So, you can write it as this: there is a second order term that has to be there d squared y over dx squared, then you can have some function of x sitting here instead of a constant coefficient right.

We looked at how to solve such problems. You have constant coefficients, but in general you can have a P of x here dy by dx and it is linear, and Q of x also here right. So, this can be a hard problem even if it is a homogeneous differential equation.

However, if by some means you are able to guess one solution, we will show how you can also find the other solution right. So, suppose you are able to by some means get one solution let us call it y 1 of x, then you know the method involves writing you know rewriting this differential equation in terms of v.

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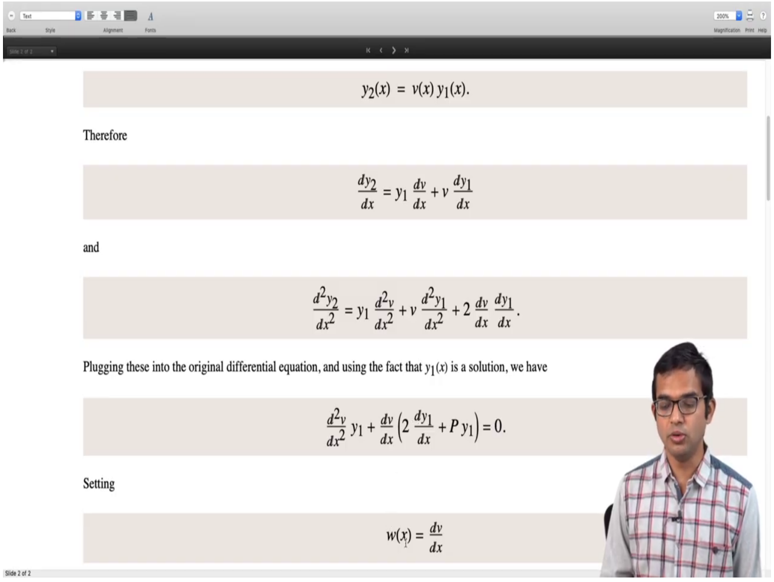

So, we look for a solution v y 2 of x which is v of x times y 1 of x right. So, we plug this y 2 of x suppose y 2 of x is a solution, then now you get a differential equation in v right.

So, you have dy by dy 2 by dx is equal to y y 1 first derivative with respect to v plus v times dy 1 by dx. And then you can also compute d squared y 2 by dx square d which will be y 1 d squared v by dx square plus v times d square y 1 plus dx square, and then you get 2 times you know this dv by x times dy 1 x by dx right as you can explicitly verify.

Now, you should plug these expressions right into your original differential equation, and then use the fact that y 1 is a solution of this right. You are already given that y 1 is a solution. Then you can show that you get a differential equation in terms of v d squared v by dx squared y 1 plus dy by dv by dx times 2 dy 1 by dx plus P times y 1 is equal to 0 right.

So, I will leave it as homework for you to verify this right. So, you know basically all you have to do is plug this expression for, d squared y 2 by dx squared and expression for dy 2 by dx and expression for y 2 into this equation into this differential equation. And use the fact that y 1 itself is a solution of this which means that d squared y 1 by dx squared plus P of x dy 1 by dx plus Q of x y 1 equals to 0.

If you use these two facts it will simplify and your differential equation in terms of v is given by this expression. So, here y 1 is a known function right. We should keep that in mind v is the unknown right. We want to find a v such that ultimately our interest is y 2 right ok.

Let us see what this differential equation is: we have a differential equation in v, which is a homogeneous second order differential equation. So, you might be wondering, so have we accomplished anything? Well, now the answer is yes.

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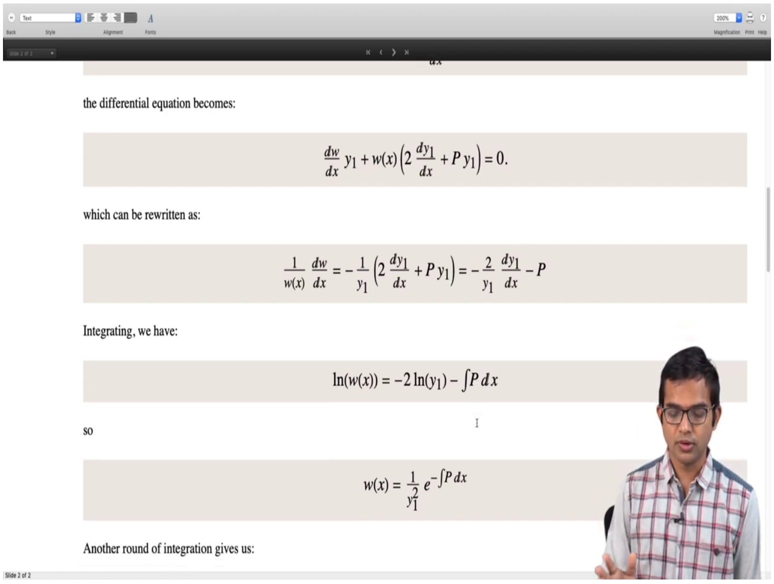

So, if we make the substitution w is equal to dv by dx, we will be able to write this differential equation as dw by dx. So, in place of d squared v by dx squared, we have d w by dx times y 1 plus in place of dv by dx we just putting w of x times 2 dy 1 by dx plus dy 1 is equal to 0. So, the key point is at now it is a separable differential equation in terms of w.

So, you can actually just write this as 1 over w d w by dx is equal to some function of x, which is seen to be minus two by y 1 dy 1 by dx minus p. So, we can just go ahead and integrate. So, I have log of w of x is equal to minus log of y 1 squared minus integral P of x dx or you can you know basically write w of x is equal to 1 over y 1 squared times e to the minus integral P dx right.

So, there is a constant which you know comes out from this integration. But you will see in a moment that, this, this does not matter. So, we will ignore this constant.

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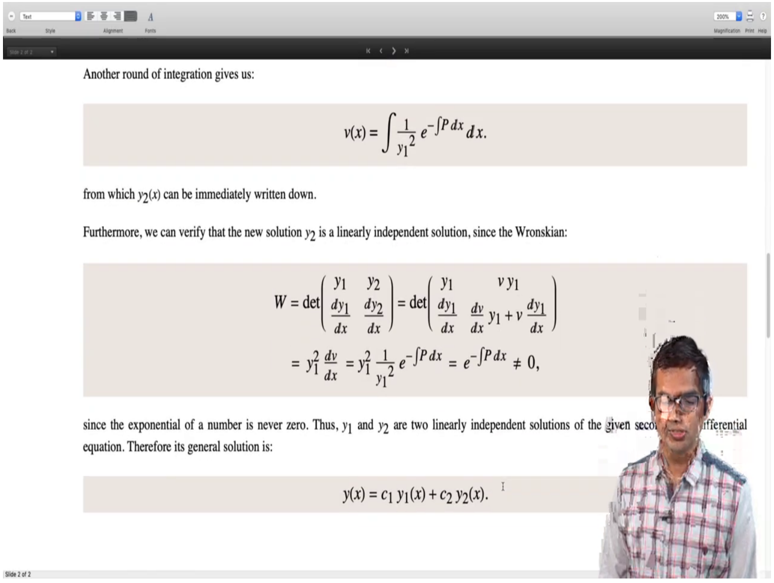

So, another round of integration will give us integral 1 over y 1 squared e to the minus integral P of P d dx the whole thing dx is a formal solution of this differential equation in v. So, from which we can immediately write down y 2, correct. So, the claim is that the solution y 2 that we have found is a linearly independent solution right. We have given a prescription for v. And we can actually verify that y 2 is linearly independent right.

To do this, we have to check the Wronskian. Wronskian as you would remember is given by this determinant of this matrix. So, you must put in y 1 and y y 2 you know you are given two functions y 1 and y 2. And you must take the derivatives dy 1 by dx and then dy 2 by dx if you take the determinant of this function. So, we know that dy 2 by dx is dv by. So, since y 2 is equal to v times y 1, you can write it as dv by dx into y 1 plus v times dy 1 by dx.

And then you expand this, so you see a lot of cancellations, and then you will be just left with this term y 1 squared dv by dx which in turn can be written like here. And the final answer is just e to the minus integral P of x dx. And it is an exponential of you know some function, so this is not identically 0. Therefore, indeed you know you remember the condition that, when a

Wronskian is not identically 0, it means that you are working with linearly independent functions.

So, y 1 and y 2 are two linearly independent functions, and therefore, since they are both solutions of your second order differential equation. So, the general solution for the second order differential equation can be written down as y of x is equal to c 1 times y 1 of x plus c 2 times y 2 of x right. So, all of this is best understood with the help of an example.

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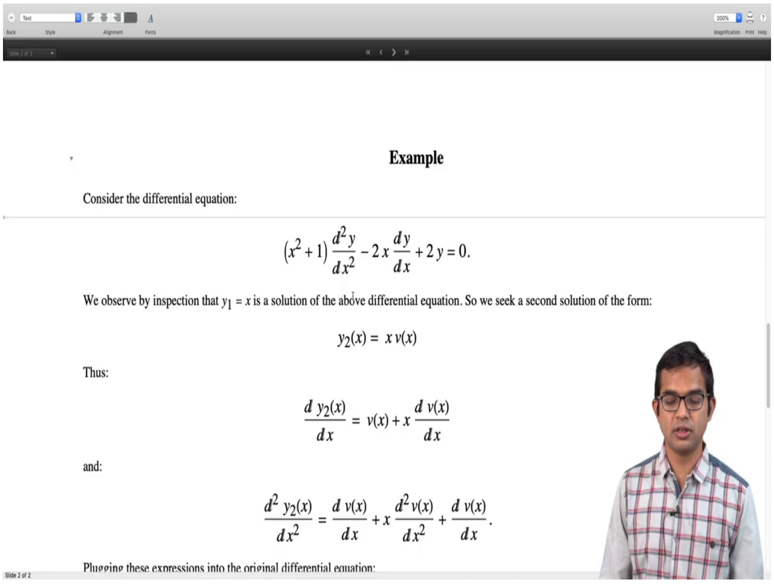

So, let us look at an example. So, I have this equation x squared plus 1 d squared y by dx squared minus 2x dy by dx plus 2y equals to 0. So, as you can see because of the appearance of this x squared plus 1 and minus 2x, you know this is not the same level of an easy problem as you know what we did with constant coefficients. It is linear for sure it is homogeneous, but it does not have constant coefficients.

However, you can see that you know if you if it is a, you know if you are able to find a y if you can guess a linear function which you know just solves this part alone right that is going to be a solution for the full thing. Because if you take the second derivative of a linear function, it is going to be just 0 right; and in this case, it turns out to be completely trivial to find that the linear function is just y 1 is equal to x..

So, you see anyway this part is not going to contribute and y 1 is equal to x is indeed a solution for this differential equation. So, you see that you know this part will give you a minus 2 x plus 2 x which is 0 right. So, we have found one solution just by inspection.

And we want to find another solution. So, let us try out this form y 2 of x is equal to x times v of x. So, if you take the derivative of this, it is going to be v of x plus x times dv by dx. Now, we plugged, so we also find one more derivative d squared y 2 of x divided by dx squared is dv by dx plus x times d squared v by dx squared plus dv by dx.

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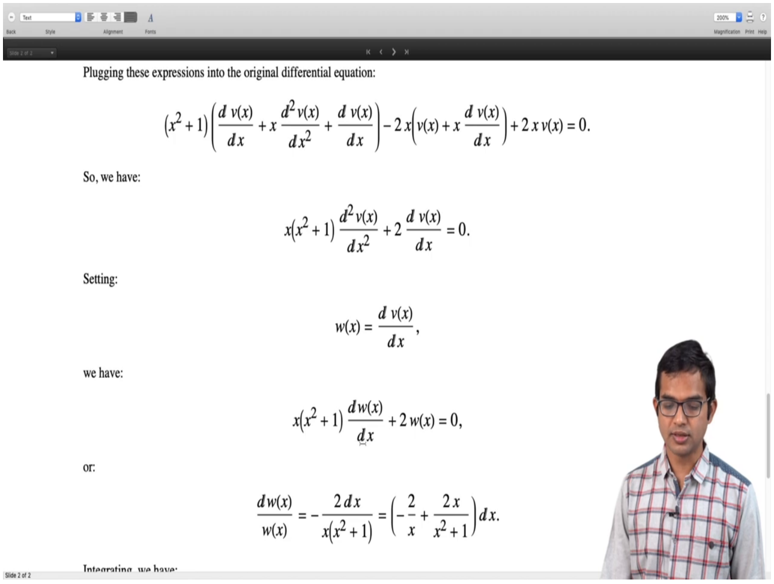

Now, we plug all these three expressions back into the original differential equation. And you know and there are the simplifications which follow and then we get this differential equation which is a second order differential equation in v. So, we have x into x squared plus 1, you should check this to see that all algebra is right. So, you get x times x squared plus 1 times d squared v by dx squared plus 2 dv by dx is equal to 0.

So, now, as we have seen in the general description, the way to make progress is to write this as an differential equation in w which is dv by dx, and then x into x squared plus 1 dw by dx plus 2 w of x is equal to 0 right. So, this is basically you know the type of differential equation where it is second order, but it is effectively first order right as we have seen right.

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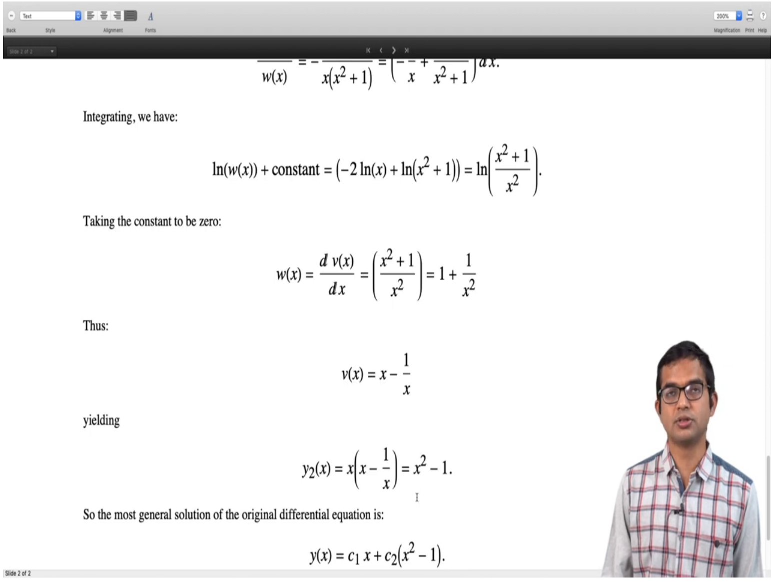

You know work with the derivative and then you get a first order equation in terms of the first derivative, and then that is what is happening. So, you have a first order differential equation. So, you have dw by dx. So, it is also separable. So, it is just a matter of integrating this function right. You will do partial fractions; you separate this out and then you integrate. So, you get log of w of x plus some constant which is we will just put it to be 0.

Since in the end we are going to have only two arbitrary constants. So, if you can find one solution to this differential equation that is good. You do not have to find the completely general solution for v of x. So, it is convenient here to just take this constant to be 0. So, you get w w of x which is at first derivative of v to be just 1 plus 1 over x squared. You should check the algebra, it is fairly straightforward, it is just a matter of integrating and carefully collecting all the factors and everything.

So, since w of x is equal 1 plus 1 over x squared we have to do one more integration and we get v of x is equal to x minus 1 by x where once again we have ignored a constant of which is not very important right. So, because we are not in after finding the most general solution for v, we are only interested in finding one solution of v of x.

So, what we have managed to show is y 2 is equal to x times v of x which is x squared minus 1 is a solution of the original differential equation. And it is a linearly independent solution with respect to y 1. So, we have found two linearly independent solutions of the second order linear differential equation. And therefore, you can just take an arbitrary constant and the first solution plus an arbitrary constant another arbitrary constant times the second solution, and then you have the full solution for the full general solution for the original differential equation ok. That is all for this lecture.

Thank you.