Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 64
ODEs in Disguise

We have seen the notion of an exact differential equation. We have seen how sometimes you would be able to find what is called an integrating factor which can multiply an equation differential equation and make it exact. And we have seen the prescription for finding such an integrating factor if it is a first order differential equation which is also linear.

We have also seen how first order differential equations, even if they are not linear, can sometimes be solved with the help of clever transformations. So, in this lecture, we will look at how you know sometimes differential equations appear in disguise. So, apparently they look to be intractable or you know very complicated, but some thought reveals that they are in fact of a familiar type, right.

So, here also we will use this lecture as sort of a bridge between first order equations and second order equations. We will look at a class of second order differential equations which you know will be seen to be basically effectively first order differential equations right, so which can be solved with the help of first order differential equation techniques effectively right, so that is the content for this lecture.

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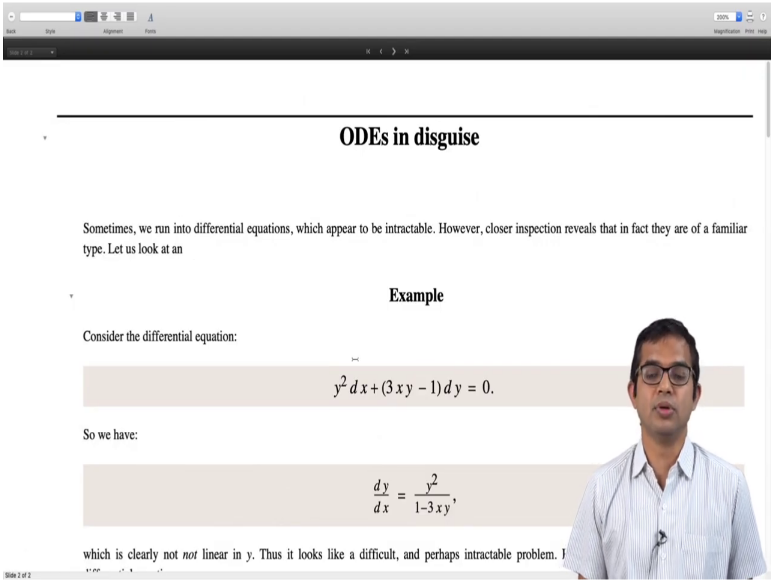

Let us start with an example of a differential equation which is written in terms of differentials of x and y. So, you have a differential equation y squared dx plus 3 x minus y dy is equal to 0 right. So, naturally, we would like to write it down as you know this derivative dy by dx is equal to y squared divided by 1 minus 3 x y in a bit to bring it to one of the standard forms that we are already familiar with right. Clearly, this is a first order differential equation.

The highest order of the derivative is just 1 dy by dx the first the first order differential equation, but we see that it is a nonlinear differential equation. So, because of this appearance of y squared and you know the complicated function of y appears on the right hand side. So, it looks like this is a hard problem, perhaps an intractable problem.

However, some thought reveals that, in fact, this problem can be turned on its head literally, so that in fact, you should think of this as a differential equation in x. So, if you write down dx by dy, it is 1 minus 3 x y divided by y squared. And now immediately you see that this equation is first order for sure in x, but also it is linear in x.

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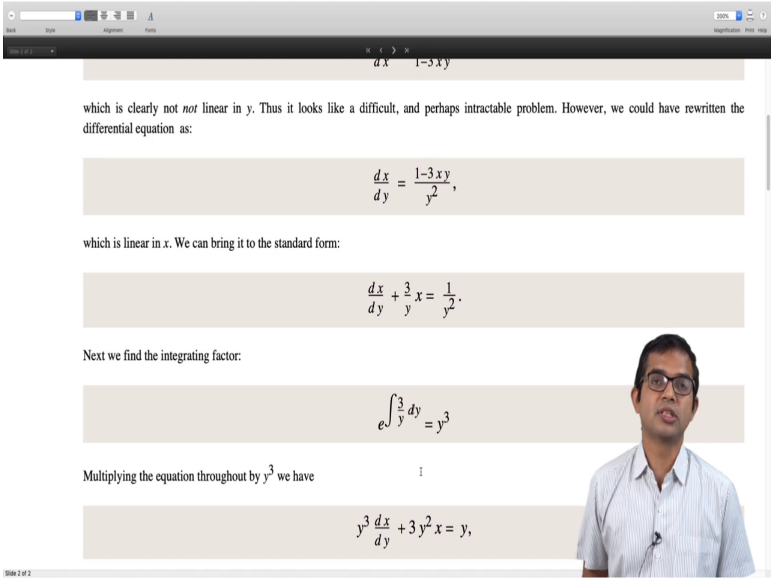

So, now, in fact, you can go ahead and bring it into the standard form dx by dy plus 3 by y times x is equal to 1 over y squared. So, this is in the form you know the derivative plus some function of y which is the independent variable times x equal to some other function of the independent variable on the right hand side. So, it is an in homogeneous first order linear differential equation. We know how to solve such differential equations.

So, the first step is to find the integrating factor. So, the integrating factor in this case turns out to be rather straightforward to compute; it is just e to the power integral 3 by y d y. So, integral 3 by y dy is 3 times log y which is the same as log of y cube, so e to the power log of y cube is just y cube, so very straightforward. So, you know the next step is to go ahead and multiply throughout this differential equation with y cube. So, then you have y cube dx by dy plus 3 times y squared x is equal to y.

So, although we have written down you know the prescription to find out you know the answer, the final solution for a first order differential equation which is linear. It is still useful to work it out step-by-step. So, work out the integrating factor multiply throughout, and then explicitly you will see that the left hand side becomes a perfect derivative, right.

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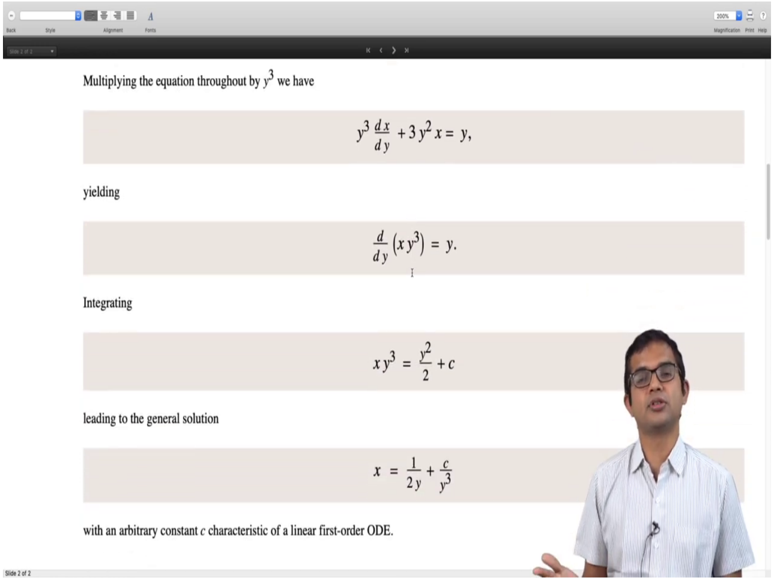

So, in this case, we see that it is a, the derivative d by dy of the function x y cube. So, the right hand side is a very simple function that is just equal to y. And then from here it is just a matter of integrating the right hand side function, which in this case is completely trivial. So, xy cube is equal to y squared over 2 plus c. So, therefore, we can go ahead and write down the solution as x is equal to 1 over 2 y plus c over y cube right.

So, the solution is obtained in as you know with x as a function of y in this case, and so the moral of this exercise is that we should you know when we are confronted with a differential equation we should try to you know bring it in you know tweak it around and see if there is some you know form in which it is more suitable for investigation right.

So, the simplest thing is to see which of the two variables should be treated as the independent variable and which of them is to be treated as a dependent variable ok. So, an apparently hard problem is with the help of a, you know a closer look is revealed to be just a first order linear differential equation, ok.

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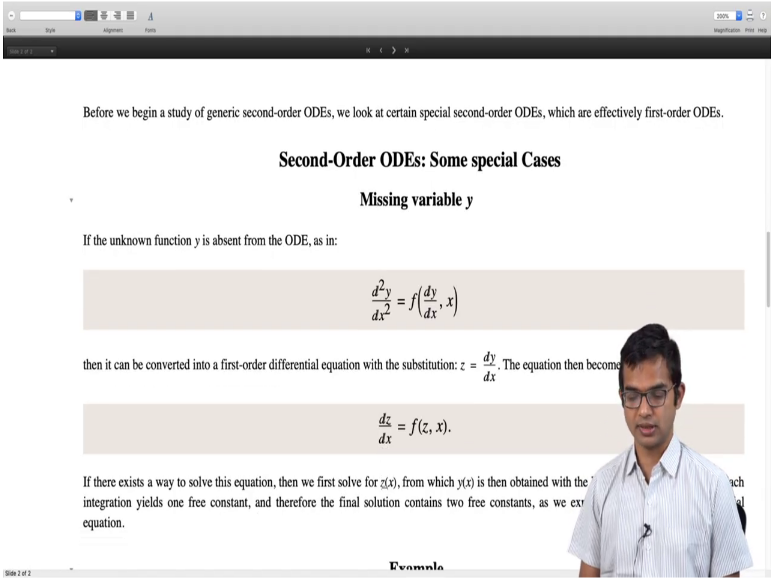

So, now we will enter our study of second order ordinary differential equations, you know in a gentle manner right. So, we will look at a bunch of cases, which are special cases of second order differential equations, but which will be seen to be effectively first order ok. Let us look at the first type, which is when you know the variable y is missing.

So, we are looking at a differential equation of the form d squared y by dx squared is equal to a function of only x and dy by dx. So, there is no y which appears on the right hand side. And when you have such a form, write a simplification and choose if you can make the substitution dy by dx is equal to z, then this equation becomes dz by dx.

So, the left hand side is just dz by dx which is equal to f of z comma x. So, dy by dx in place of dy by dx you have z comma x. And so immediately you see that now this is a first order differential equation in z, right. So, assuming that f is not some extremely violent function perhaps there is a way to solve for this right.

So, we will be able to write down z in terms of x, z as a function of x. If we can do it, then we are still not done right, presumably this will give you one free constant right. So, and then one more integration is involved. And therefore, you will end up with two free constants, so that is how you know the characteristics of the second order differential equation comes in that is going to be two free constants in the end. So, because the ultimate answer is to be written as y as a function of x.

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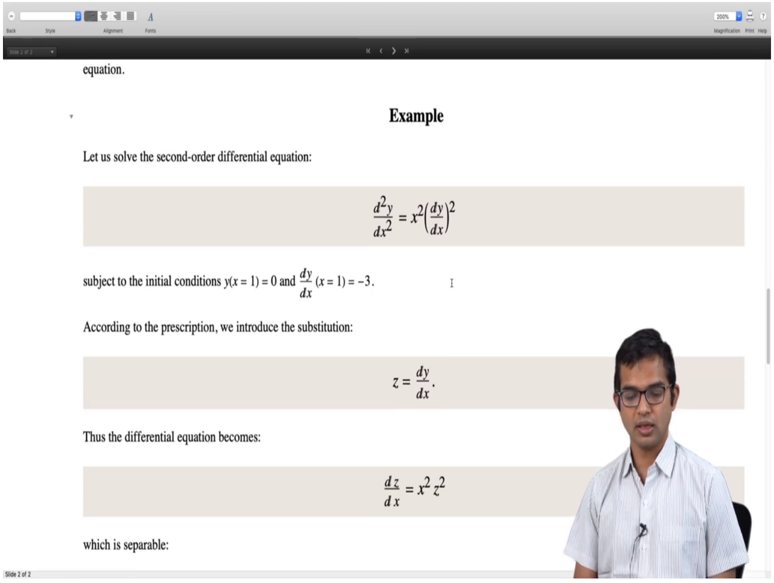

So, let us look at an example. So, suppose, we are interested in solving for this differential equation d squared y by dx squared is equal x squared times dy by dx the whole squared. And I am giving two conditions y of x equal to 1 is equal to 0 and dy by dx of x equal to 1 equal to minus 3 right.

So, if I put in these you know two conditions, then this will eliminate the two free constants which will come out. So, according to the prescription given, we must introduce a substitution z equal to dv by dx. So, the differential equation becomes dz by dx is equal x squared times z squared.

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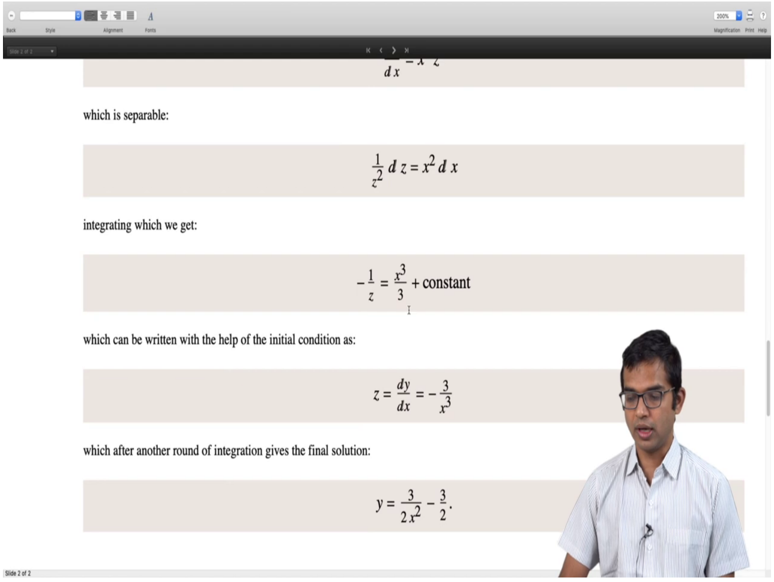

Now, which is a separable differential equation, the first order separable differential equation. So, you have 1 over z squared dz is equal to x squared dx which can be integrated, and we have minus 1 over z is equal to x cube divided by 3 plus constant. So, but we have this initial condition dy by dx of x equal to 1 is equal to minus 3.

So, when x equals 1, z is going to be minus 3. So, this means that this constant is, so in other words, so you can just remove this, this constant does not count. So, you have z is equal to minus 3 over x cube right. So, if you put this constant to be 0, then you will see that when x is equal to 1, z is equal to minus 3.

So, it is just the derivative dy by dx is seen to be minus 3 over x cube. One more round of integration and then we get y is equal to 3 by 2x squared, and then there is another constant c 2 which I have already put in this condition x equal to 1, when y of x equal to 1 must be equal to 0. And you can check that if you put x equal 1, 3 by 2 minus 3 by 2 is 0. And, and so you can implant this directly into the original differential equation, and check that indeed you have got the correct solution ok.

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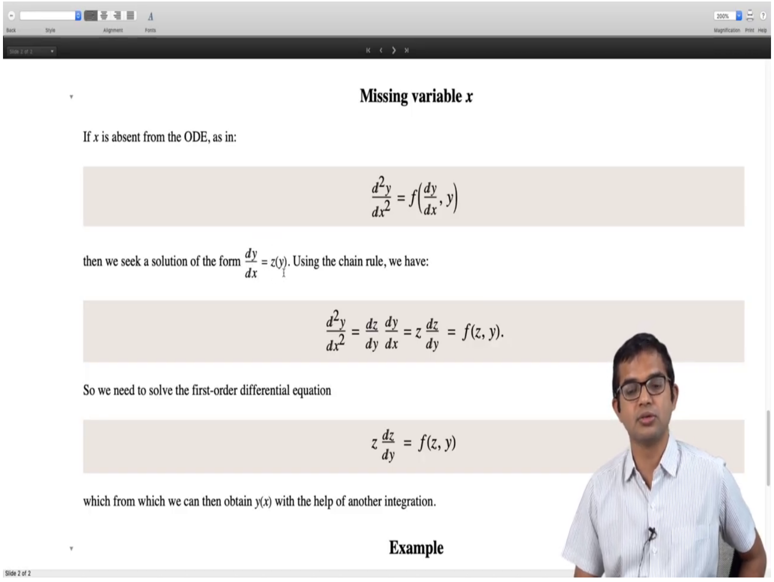

So, is one type of you know second order differential equation which is actually a first order differential equation. There is another type which I will describe now which is when you have a missing variable x right. So, here d squared y by dx squared is going to be the form f of dy by dx comma y. So, earlier we had you know y was missing you had dy by dx and x on the right hand side, but now you have x is missing.

So, you are allowed to have dy by dx, and y can be a function of these two. And these two can be you know considered or treated as a first order differential equation. So, give with the help of the same substitution, but you know with a slight modification right. So, we seek a solution of the form dy by dx. This is to be treated now as z as a function of y not as a function of x like we were doing earlier right.

So, what we do is, we use the chain rule. And write this as d squared y by dx squared is actually it can be thought of as dz by dy times dy by dx. So, which is equal to z times dz by dy. So, you want to completely eliminate x from this because on the right hand side there is no explicit dependence on x.

So, now, you see that you have a differential equation z times dz by dy is equal f of z comma y. So, this is a first order differential equation in z you know with respect to y which can be solved. Once you have found z as a function of y, you have to do one more integration dy by dx is equal to z of y, and then you can at least formally write down y as a function of x, ok.

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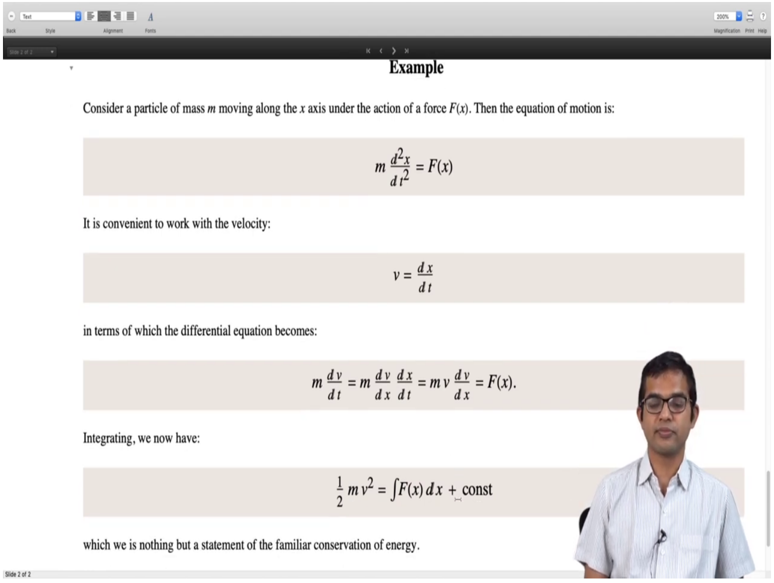

Let us look at an example. Consider a part which is actually a familiar example we have all seen and you know this trick this very trick has perhaps been used by us. So, consider a particle of mass m moving along the x-axis under the force F of x. So, then the equation of motion is m times d squared x by dt squared is equal to F of x. So, here we should not confuse the notation x here and the x here.

So, here I am using x and t like a standard in mechanics; x refers to distance, and t refers to time. So, t here is the independent variable, and x is the dependent variable. So, you see on the right hand side what I have called F of dy by dx comma y here is you know F of just x. So, this y play role is taken over by x here. So, in fact, the equation in the example that I am considering is even simpler than the general case that I have given here.

So, this is a special case of the special case where you know the right hand side is purely a function of the dependent variable alone. There is no dependence even on the derivative, right. So, I have m times d squared x by dt squared is equal to F of x. So, it is convenient to work with velocity right. So, the point is that you know this substitution is really like thinking about it in terms of velocity, v is equal to dx by dt.

And so now, comes the clever part of this trick which is in order to write this differential equation as m dv by dt is equal f of x will not do, but you have to you know realize that this m dv by dt is the same as m dv by dx times dx by dt. So, in fact, you can think of this as m dv by mv times dv by dx, so velocity as a function of position. So, time is eliminated entirely, and so you see you have mv dv by dx is equal F of x.

So, if you integrate both sides, you get half mv squared is equal to the integral of F of x dx plus constant right, which seems to be in fact nothing but the statement of the conservation of energy that the kinetic energy plus potential energy is equal to constant. So, there is this negative sign you know which comes in because you know work done by the force, and you know work done on the body. You know there is this sign convention involved.

But basically the point is that you have kinetic energy plus potential energy is equal to a constant. There are many problems where we have directly use this principle. But if you analyze this from the point of view of a differential equation, you see that here there is this trick used where we convert a second order differential equation into an effectively first order differential equation, and solve for v in terms of x right.

You need explicitly what F of x is to be able to go one more step and to write down x as a function of t right which is possible only if the function F of x itself is known, and if the integration is possible. And you know sometimes it can be extremely messy, but the general prescription is available right. That is all for this lecture.

Thank you.