Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 62
Solutions of Linear First Order ODEs

So we have seen how we can solve for an arbitrary linear first order differential equation. So, we have seen that it's useful to first start with the homogeneous counterpart of the same differential equation and then go to the more general inhomogeneous differential equation, right. So, in this lecture we will take a look at you know the solutions and you know discuss a couple of properties of these solutions, ok.

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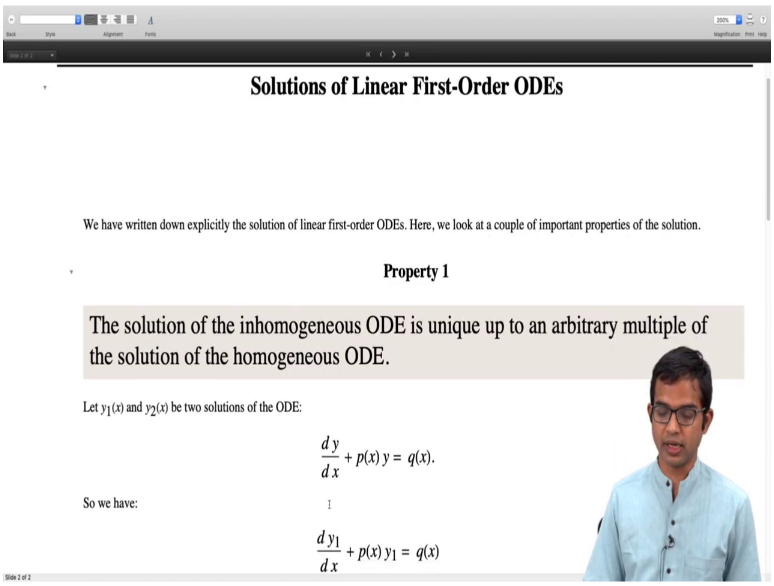

So, we have written down the solutions and we know everything about the solution right. So, we know explicitly how to find a solution; at least formally. Now, there are a couple of properties which relate to solutions of you know first-order linear ODE’s; the first property is that the solution of the inhomogeneous ODE is unique up to an arbitrary multiple of the solution of the homogeneous ODE, right.

So, there is this intimate connection between the solution of the homogeneous equation and the inhomogeneous equation and this will turn out to be true even for higher order equations. And so we will return to similar ideas later on right when we look at more you know higher order differential equations. So, let us go over the argument; suppose you have two solutions, you have found y 1; y of y 2 of x to be two solutions of your full inhomogeneous differential equation right.

So, we have given the differential equation dy by dx; plus p of x times y is equal to q of x. So, y 1 and y 2 are both solutions; so you can write d y 1 by dx plus p of x times y 1 is equal to q of x and likewise d y 2 by dx plus p of x times y 2 is equal to q of x right; since both of them are solutions of the ODE.

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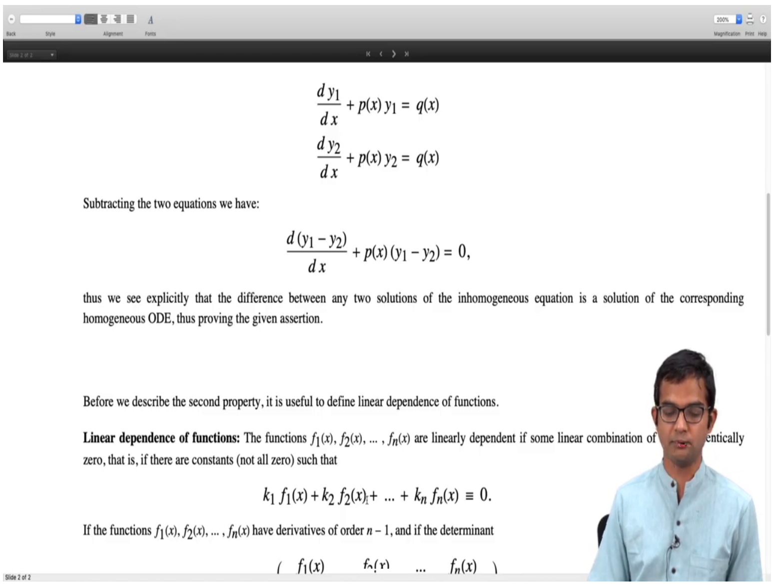

Now, if you subtract these two equations right; so right hand side will go to 0. So, this equation dy by dx of y 1 minus y 2 plus p of x times y 1 minus y 2 is equal to 0. So, immediately we see that in fact the difference of any two solutions or the full inhomogeneous differential equation is in fact, a solution of the homogeneous differential equation right.

So, which basically proves the assertion right; so the assertion is that the solution of the inhomogeneous ODE is in fact, unique after an arbitrary multiple of the solution of the homogeneous ODE, which is another way of saying you know what we have just shown, which is that if you manage to find any two solutions, their difference necessarily must be a solution of the homogeneous differential equation, right.

So, it is here that you know the power of you know first of all finding the solution of the homogeneous equation and then using this as you know as a you know something free that can be added to your solution of the full inhomogeneous equation. So, in general finding the solution of the full inhomogeneous equation is a harder problem, right.

So, if you can somehow find you know one solution of that right; if you want to find the general solution of this which is a harder problem, what this tells you is that you know you can find a solution of the homogeneous equation and then you know you can just tag that along for free.

So, you see that if you are able to find one solution for the full inhomogeneous equation; if you acted if you added a solution of the homogeneous equation to it or any arbitrary multiple of the, of a solution of the homogeneous equation, that is still going to be a solution of your full inhomogeneous equation. Because you are just adding 0 to the right hand side and it remains unchanged right.

So, the left if you think of this d by dx plus p as an operator, it acts upon y 1 to give you q of x. And then if you have managed to find some other you know the solution of the homogeneous equation that can be added for free to a solution of the inhomogeneous equation; that will allow us to write down the full general solution for inhomogeneous ODE.

So, it is a trick that we will use later on alright. So, in its and it's going to become much more important when we do second order and higher order differential equations ok. So, there is another property which we want to discuss. So, but before we discuss the second property; let us you know look at the concept of linear dependence of functions right.

So, we have said that these functions can be thought of as vectors in a vector space. We have also discussed the notion of linear dependence of vectors and so on right. So, let us just discuss linear dependence of functions. So, functions f 1, f 2 so on all the f n of x are said to be linearly dependent, if you can find some set of coefficients; set of constants, not all of them 0 right. If all of them are 0, then that does not suffice.

If you can find some non trivial set of coefficients; k 1, k 2 all the way up to k n such that k 1 times f 1 plus k 2 times f 2 plus; all the way up to k n times f n is equal to 0, then you say that these functions f 1, f 2 all the way up to f n are linearly dependent right. So, and they are linearly independent if you know this equation such an equation implies that all these coefficients k 1, k 2; so are all 0 right. So, there is only a trivial set of coefficients which will make this possible; if that is the case then you say that these functions are linearly independent.

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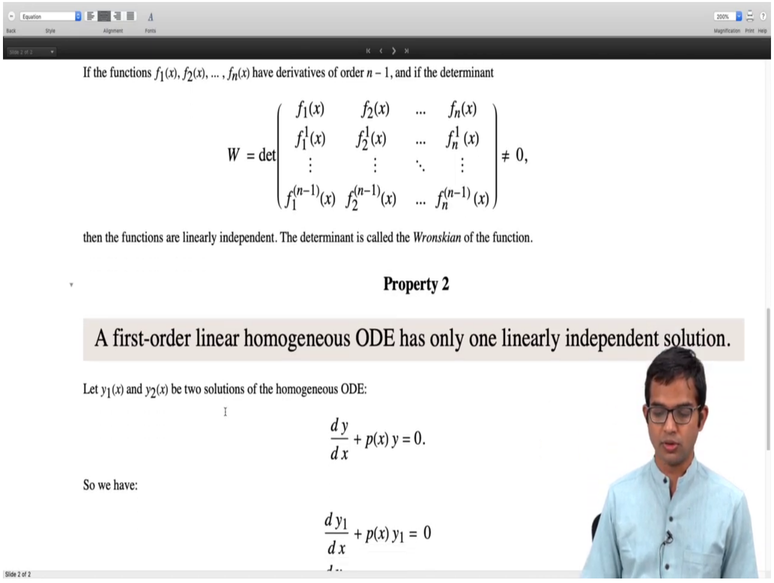

Now, so there is; so if this holds right; so there is a way to recast this condition for linear independence of such functions. If k 1 this holds, then we can just take the first derivative; then you get k 1, times f 1 prime of x plus k 2 times f 2 prime of x so on all; all the way up to k n times f n prime of x is equal to 0, that is second equation.

You can write a third equation or fourth equation; so in fact, you have n different equations, you can write down k 1; f you know f; f 1, the n; n minus 1 th derivative of x so on; all the way up to f n of n minus 1 of x right. So, there are going to be n equations; it is a system of n linear equations in n variables; so k, k 1, k 2 all the way up to k n being the variables right.

So, we know that if the determinant of this quantity is nonzero; then there is a unique solution for these k 1's. So, you have a; you have you know you for example, you know you can think of it from the Cramer's rule point of view, but even other way is; we know that if this determinant is nonzero right.

So, then it immediately follows that you know these functions are linearly independent because if they are linearly dependent, then you will be able to find some non-trivial set of coefficients k 1; all the way up to k n such that this condition holds, right. So, this determinant is; there is a name for this determinant and this it is called Wronskian right.

So, it turns out that this is a necessary condition right; it is not a, I mean you; it does not always work in the opposite direction. If the determinant of; if the Wronskian is 0, you cannot say that the functions are linearly dependent right. So, well if the functions are linearly dependent; then the Wronskian is going to be 0 right.

But on the other hand the opposite way does not quite work. And so there is property 2 that I want to discuss, but so we will see that we will talk about the Wronskian in a moment for this property 2. So, the property 2 is that if you have a first order linear homogeneous ODE; it has only one linearly independent solution right.

So, again let us start with two solutions; suppose there are two solutions y 1 and y 2; let us say there are two solutions of this differential equation. This is a homogeneous differential equation, so the right hand side is 0. So, we have d y 1 by dx plus p times y 1 is equal 0; d y 2 by dx plus p; y 2 is equal to 0.

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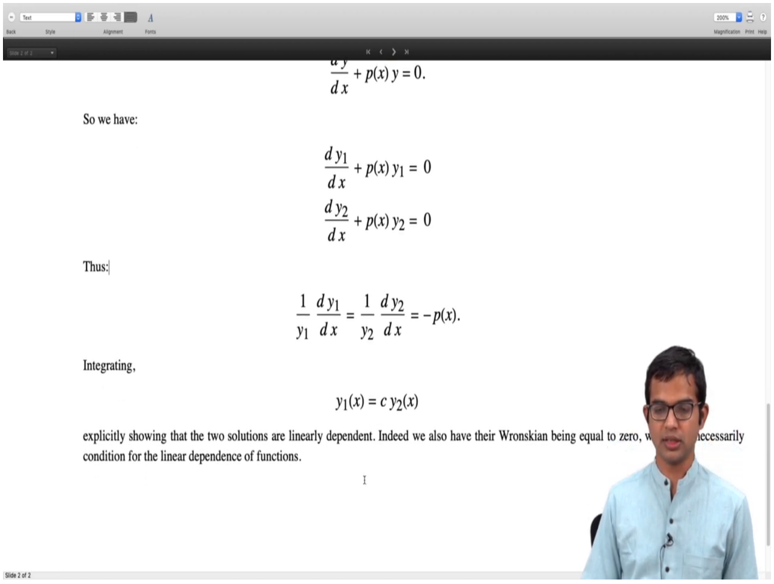

So, we can rewrite this as 1 over y 1 times d y 1 by dx is equal to minus p of x is equal to 1 over y 2; d y 2 by dx right. So, if you look at just the first you know this part of the equation; so we see that immediately we can just integrate both sides. So, you get log y 1 is equal to log y 2 right, and so plus a constant and basically which in turn means that y 1 must be equal to some constant times y 2 right so, which is the condition that these two are linearly dependent right.

So, if these functions are linearly dependent; so I said that the Wronskian is going to be 0 right because if the Wronskian is not 0, then the functions are linearly independent; that is what this condition is right. It is right, but there probably will be a homework problem where you will be given a set of functions, which are for which the Wronskian is 0, right and yet they are linearly independent.

So, if the Wronskian is 0 by itself; you cannot it is you cannot say whether you know the functions are linearly dependent or independent. But on the other hand if the Wronskian is nonzero, then for sure these functions are linearly independent. So, in this case you see that you managed to show that in fact, these functions are linearly dependent and then you can verify that the Wronskian is 0 in this case right.

So, you I mean it is straightforward to see this because you know basically the idea is the Wronskian is given by in this case f 1 times; so y 1 times y 2 prime minus y 2 times y 1 prime right which is basically this condition. So, you have y 2 times y 1 prime minus y 1 times; y 2 prime is equal to 0 right. So, indeed the Wronskian of these two functions is 0 ok. So, that is all for this lecture.

Thank you.