Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 60
Exact differential equations

So, we have seen how first order linear differential equations can be solved. And so we did some kind of a magic trick there. We managed to find this function i and then we took the exponential of this and multiplied throughout and then suddenly your equation you know is transformed into a very nice perfect derivative.

So, we want to look a little deeper into what was going on here. And in this lecture, we are going to introduce what are called exact differential equations. So, the concept of an integrating factor is described in this lecture.

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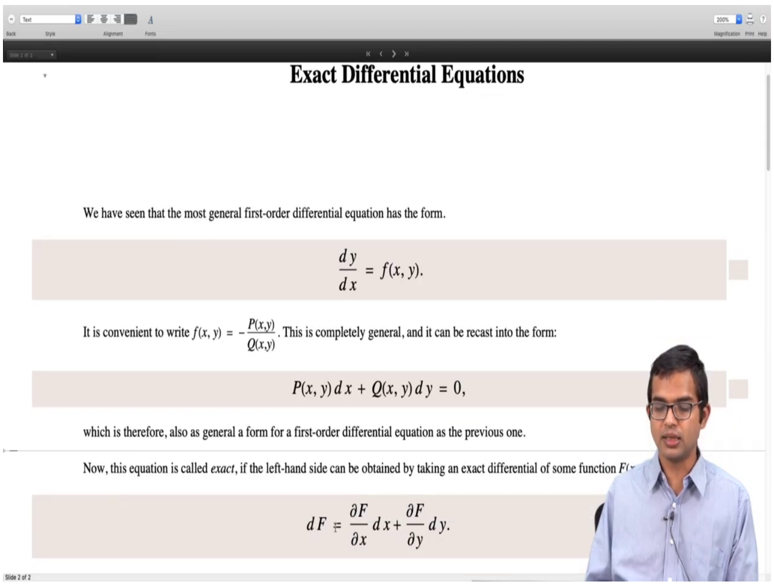

So we have seen that the most general first order differential equation has this form right, dy by dx is equal to f of x comma y and so, we have the freedom to write this as minus P of x comma y divided by Q of x comma y there is no loss of generality here right. So, only if you make these functions you know functions purely of x and purely of y, which we did when we were looking at you know separable forms.

Then you are making a concession, but at this point it is completely general right. So, this allows us to recast this first-order differential equation in this form P of x comma y dx plus Q of x comma y dy equal to 0 right. So, this equation is as general as the first equation itself. So, this is a you know general first-order differential equation.

Now, we will introduce the notion of an exact differential equation. So, we would call this differential equation as exact if it is something that can be obtained from some function of x comma y. So, if you are able to find some function f of x, y; such that the derivative of this function right the differential of this function right, which we know involves these partial derivatives.

So, since F is a function of both x and y. So, if you take the differential of this F it is given by dou F by dou x dx plus dou F by dou y dy right. So, you see that the left hand side has some function times dx plus some other function times dy right. If it so happens that this left hand side can be obtained as the exact differential of some function F then, we say that the differential equation is exact, right.

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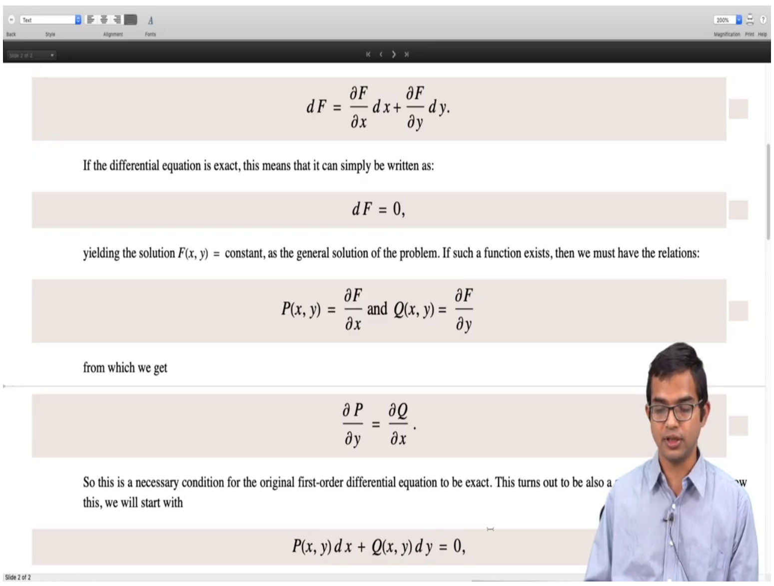

If a differential equation is exact this means that it can simply be written as dF equal to 0. Now, the left hand side is an exact differential. So, immediately what it implies is f of x comma y equal to constant right. So, which is in nothing but the solution for the problem itself, right.

So, if such a function exists then if you can somehow find it then that is also the problem immediately right. If f of x comma y equal constant is the solution. Now, of course, it would be nice to tell immediately as soon as you are given a differential equation if it is already exact right. Is there a prescription to find out, whether it is exact and so the answer is yes right.

So, the condition is the following. So, we are saying that you know there is an F such that dF is equal to this so, P of x comma y is equal to dou F by dou x and Q of x comma y is equal to dou F by dou y. So, if a differential equation is exact, then if you take the y derivative of P then you get dou P by dou y.

And if you take the x derivative of Q, then you get dou x dou Q by dou x and immediately we see that both of these must be the same because, they are both the second derivative of F with respect to x and y the partial derivative dou square second derivative dou square F by dou x dou y. And therefore, dou P by dou y is equal to dou Q by dou x, which is necessarily true if your differential equation is exact right. So, this is a necessary condition for your original differential equation right.

So, when you are given a differential equation you do not know what F is whether there is such an F or not, but if you can immediately test you know with this if you are able to test this then your differential equation is exact - this must hold. But it turns out. In fact, that this is even stronger there is also a sufficient condition right. So, what we can show is that if you are given a differential equation if you start with a differential equation, like this some arbitrary differential equation.

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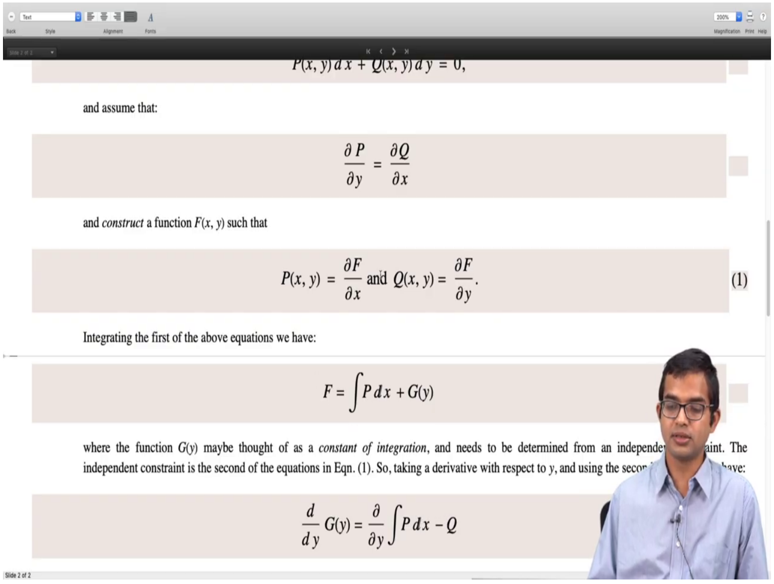

And if you are also able to show that if you can explicitly see that dou P by dou y is equal to dou Q by dou x. Then for sure there exists a function F of x we can construct a function F of x y; such that P of x comma y is equal to it is a the differential of this f will give the left hand side basically.

So, you will be able to construct an F of x comma y such that P of x comma y is equal to dou F by dou x and Q of x comma y is equal to dou F by dou y. So, integrating so, how do we see this? So integrating this first-equation; so, let us say yeah we are looking for a function F of this kind.

So, let us write down this equation and then integrate both sides. So, we have F is equal to integral of P dx plus you know some you can put instead of a constant you can actually put an arbitrary function of y right because, all you need is the partial derivative of this function with respect to x must give you P and that it will do. No matter what function of y that you choose here to add to this.

Now, but this function itself must be something that you should be able to determine from some independent constraint and it should be compatible with this other constraint. So, that is what we will show, holds in you know provided this condition holds right. So, we will show that next.

So, what do we want to do? We want to be able to impose this constraint Q of x comma y is equal to dou F by dou y. So, let us just take a partial derivative throughout with respect to y. So, I have dou F by dou y is equal to dou by dou y of integral P dx plus d G y by G of y by dy, right because the partial derivative of G with respect to y is the same as the total derivative of G with respect to y.

So, rewriting it I can write it as d by dy of G is equal to dou by dou y of integral P of x P of x comma y dx minus dou F by dou y is the same as Q right. Because this is something that we already have it is an essential you know it is a condition which we already imposed. So, I have minus Q.

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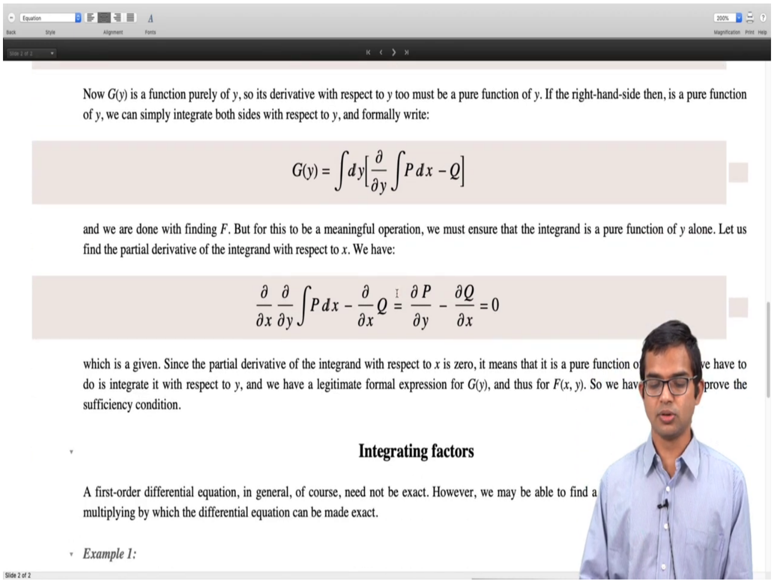

So, now you look at this equation carefully. So, on the left hand side we have that full derivative, the total derivative with respect to y of a function, which is a you know pure function of y. So, the derivative with respect to y of a function, which is purely a function of y must be another function of y it cannot have any x in it, but on the right hand side you have you know some complicated stuff.

You have a partial derivative with respect to y often a function P of x comma y which is integrated with respect to dx and then you have a minus Q of x comma y but, the only way that this would be you know a compatible equation if is if the right hand side turns out to be purely a function of y alone. In other words, there can be no dependence on x of this combination on the right hand side, right.

So, you know another way of stating this is if you can find the partial derivative of the right hand side with respect to x and show that it is equal to 0, right. So, let us find the partial derivative of the right hand side with respect to x, right. So, if that happens then of course, G of y can be just integrated, right.

If the right hand side is purely a function of y then we can just solve for G of y in terms of this integral you know with respect to y and then you are done because F is you know this integral, P is known and G is known, according to this prescription you have managed to work out a formal expression for this function F in terms of Ps and Qs. So, you are done right.

So, but there is only one block in the way and that block is that we must ensure that this right hand side must be purely a function of y. And that we will now argue is actually true in order to do that we will take the partial derivative of the right hand side with respect to x. We have dou by dou x of dou by dou y of this integral P of P dx minus dou by dou x cube, but then I can you know exchange these partial derivatives.

So, dou by dou x integral P dx will just be P and then I have dou P by dou y minus dou Q by dou x, but this is equal to 0 because, we are given that you know your original differential equation satisfies this condition dou P by dou y is equal to dou Q by dou x. So, basically we have managed to show this integrand, which goes into this you know computation of G of y is purely a function of y right.

Because it is partial derivative with respect to x is 0 and therefore, this is a completely legitimate operation to do and you will get a function you know in y and that you use to get your you know function F of x comma y. Therefore, you will be able to so we have explicitly shown how you know given this condition dou P by dou y is equal to dou Q by dou x.

It is going to imply that your original differential equation is exact. So, it works both ways. It is a necessary and sufficient condition for your differential equation to be exact, right.

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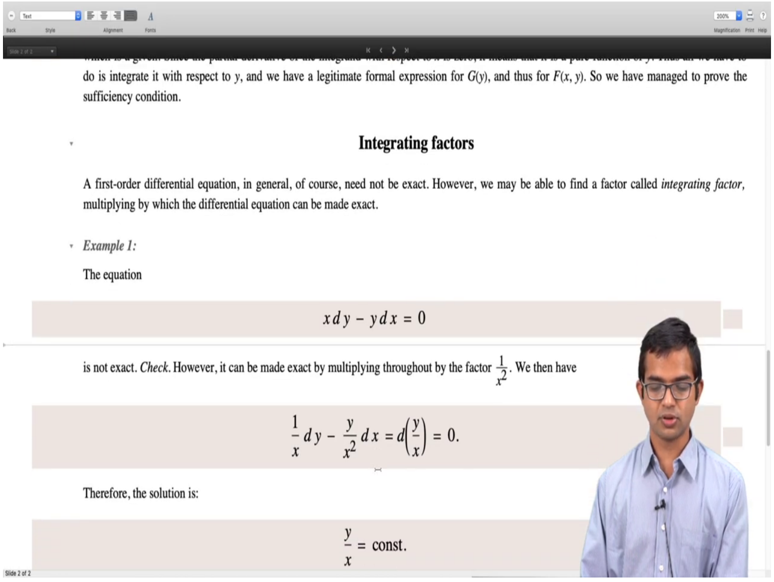

So, a quick word about what we were doing for the ODE, the first-order linear differential equation. It was actually that we were finding something called an integrating factor. So, not all differential equations are going to be exact right. So, in fact, you have to verify if it is exact or not using this technique we have to find whether dou P by dou y is equal to dou Q by dou x.

And so in fact, the vast majority of differential equations you might encounter will not be exact. But what we showed for first-order linear differential equations is that there is a prescription by which you can multiply throughout and then the differential equation becomes exact, right.

So, let us just quickly look at a few examples of what integrating factors are right. So, integrating factors are factors with which you multiply a differential equation and convert a non exact are inexact differential equation into an exact equation. So, let us look at this very simple example.

So, xdy minus ydx is equal to 0. So, this differential equation of course, you can solve it right, but that is not the point here. So, if you look at this differential equation it is not exact. Why is it not exact? If you take the yth derivative or you know you will get a you can take the yth derivative of this term and the x derivative of this you know one of them will be 1 and the other one is minus 1. So, it is not exact.

But, on the other hand if you multiply throughout with 1 over x squared it is going to become 1 by x dy minus y by x square dx. And you can explicitly write this down as the derivative of y by x and this is equal to 0. So, now it is you have managed to convert an in-exact differential equation into an exact differential equation by multiplying by this factor 1 over x squared right. So, therefore, the solution immediately follows, right.

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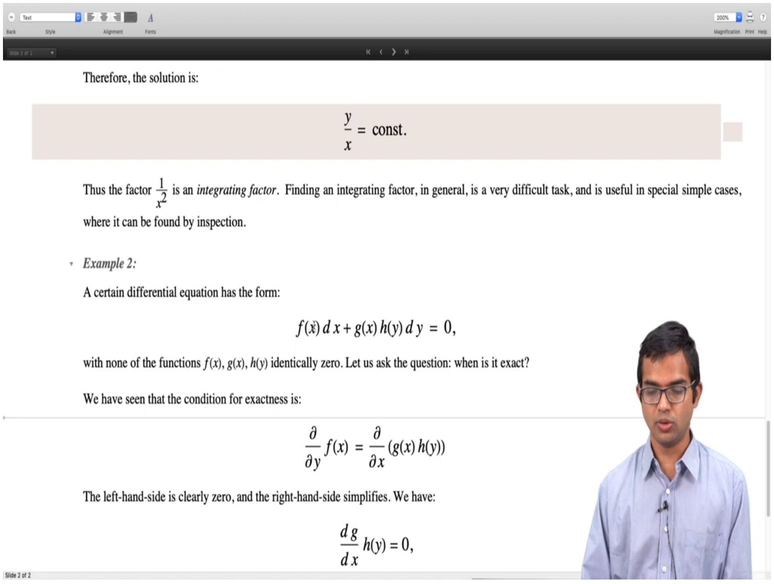

So, the factor 1 over x squared is called the integrating factor. So, let us look at another example, if you are given some differential equation and it has this form f of x dx plus g of x h of y dy equal to 0, with none of these functions identically 0. If you ask when this differential equation is going to be exact, right. So, the condition is this dou f by dou y must be equal to dou by dou x of g times h, but f, f is purely a function of x so that the left hand side will vanish.

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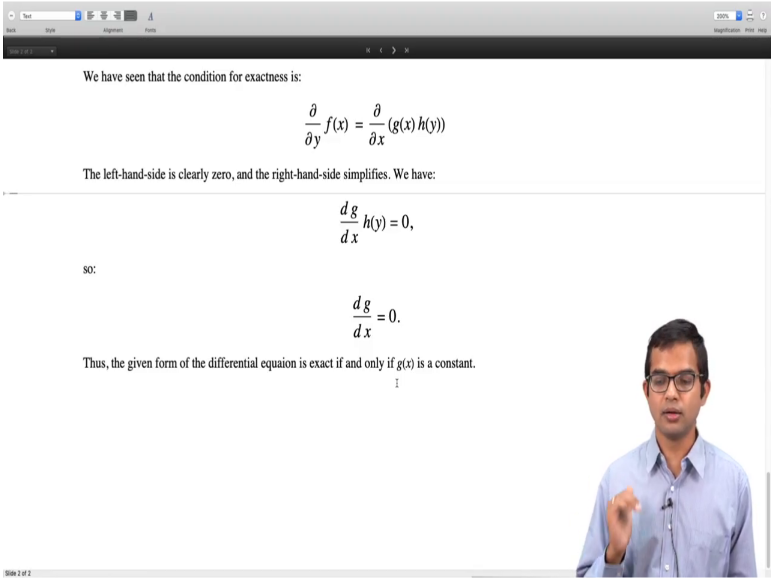

So, this implies that you know the right hand side must be equal to 0. But, the right hand side itself can be simplified; you can write it as dg by dx times h of y equal to 0. Now, h of y itself cannot be 0 because, that is given in the question and therefore, it implies that dg by dx equal to 0. So, this equation is exact only if g is a constant it cannot have any you know more non trivial dependence on x if it is a constant then this overall differential equation will be exact, right.

So in this lecture, you know the focus of this lecture was to define what exact differential equations are and to tell you what integrating factors are right.

So, and also just give you a hint that they already use integrating factors, in some clever way with first order ODE’s, linear ODE’s where we found this mysterious factor e to the i and we when we multiplied throughout with e to the i this the equation was actually going to become exact, but we will discuss a little more about this ahead, but that is all for this lecture.

Thank you.