Mathematical Methods 1

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Linear Algebra
Lecture - 06
The Cauchy-Schwartz inequality for Euclidean vectors

So, in this lecture we are going to derive the so-called Cauchy-Schwartz inequality for 3-d Euclidean vectors, right. So, in the final form it will look like something completely obvious, but we will use a rather elegant you know approach to get to this result, and the method we will see can be generalised to arbitrary you know vectors from a linear vector space, right. So, first we will start with 3d vectors and use a very nice elegant geometrical approach to obtain this result ok.

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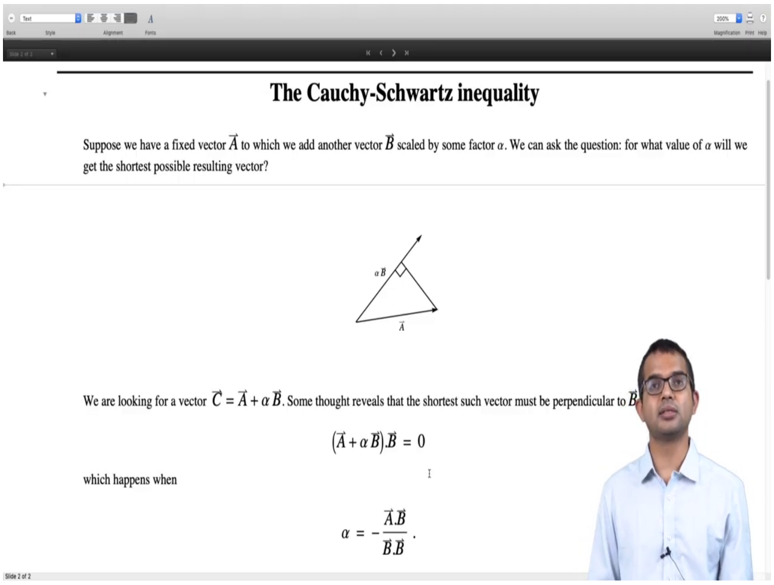

So, suppose you have some vector A right and you have some other vector B. Now, the vector A is fixed whereas, the vector B the direction of vector B is fixed, but it can be scaled. Let us say you have the freedom to stretch this vector B as much as you want, right. So, you think of some alpha times the vector B.

So, now, one can ask the question if I were to form a linear combination between this vector A and some scaled vector B. It is these two are two completely arbitrary vectors. So, in general you know we assume that they are not null vectors because they will give you those kinds of conditions and will give you some you know extreme limits of the problem.

So, let us for the moment let us assume that these are non-null vectors and well, I mean you can also consider cases where the two of them are pointing along the same direction and so on, but those are you know particular cases that you will extract later on.

For the moment let us think of a typical case like I have given this diagram here. There is a vector A which is pointing in a certain direction; there is another vector B which I am going to scale with a factor alpha. And, I am going to consider a third vector C that I can construct with the help of these two vectors and you know I can come up with a vector C which is A times A plus alpha times B.

Now, I asked the question: how would I choose alpha such that the resulting vector C is the smallest possible in magnitude? So, there is a whole locus of points that you know you can generate or a whole set of vectors that can come about because alpha is you know it is a factor that can take all real values, right.

So, geometrically you see that you know if I want what is A plus alpha B? It is this third vector I have if the third vector is going to be minimal, if I were to if I can somehow ensure that this third vector is going to be perpendicular to the direction of B, right. So, then you can I mean I will have to keep B to be exactly at this length, so that the third vector is going to be perpendicular to B. If I can arrange this then I am going to get you know C to be the smallest possible alright.

So, let me write down this condition: if I want to find alpha such that this A plus alpha times B is going to be perpendicular to B. So, how do I do this? I know how to do this now. I just take the dot product of A plus alpha B with B and then I just put it equal to 0. And, immediately this gives me alpha is equal to minus A dot B divided by B dot B, and if I can choose alpha to be this quantity then I am going to be able to get the smallest possible vector that I can form as a linear combination of A and B, right.

Now, here is the argument. So, you find the smallest possible vector and even the smallest possible vector, the magnitude of the smallest possible vector must be greater than or equal to 0, right. So, this is true for any vector. So, we are trying to get a you know I an inequality out of this right I mean this is a standard approach which is you know you find the smallest possible value of you know some set of objects and if you can; if you can bound that then you have a strong inequality which comes.

So, the smallest possible quantity among the entire set itself is going to be greater than or equal to 0, right. So, we will just simply invoke this which is a fact and then we will see that that will give us a very interesting inequality which is the famous inequality.

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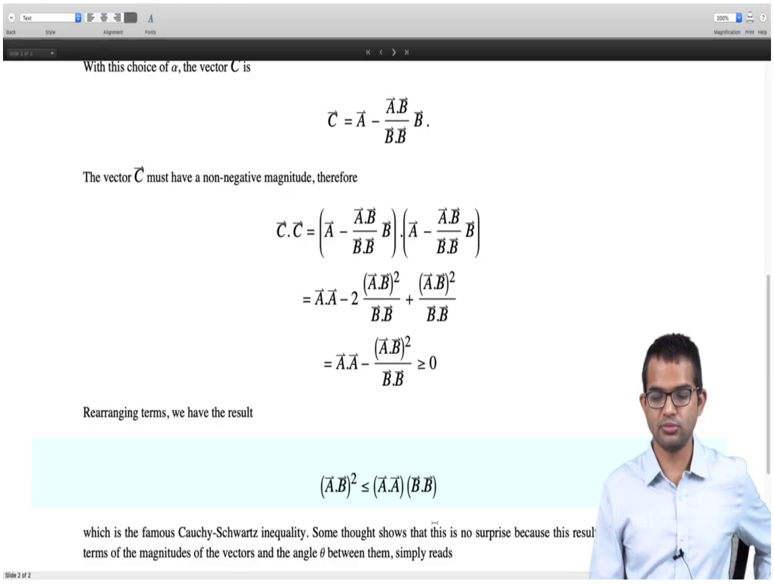

So, C so, I have written C as A minus A dot B divided by B dot B times B, right. So, in place of alpha I have put this quantity that I have just worked out and then the C dot C must be greater than or equal to 0. So, let me compute C dot C which is just A dot A minus you know I have A dot B divided by B dot B times B dot A.

So, then I have another you know when I multiply with this I get A dot B A dot B the whole square divided by B dot B I get 2 times of this and then when I take the square of this the final quantity this you know A dot B whole square divided by B dot B whole square times B dot B. So, all these B dot Bs will cancel and then again I will get back the same term. So, I have A plus A dot B the whole square divided by B dot B.

So, if I you know cancel out one of these terms from the second and third then I am just left with a dot A minus A dot B whole square divided by B dot B. Now, this quantity must be greater than or equal to 0 right.

So, if I rearrange these terms I have the famous Cauchy-Schwartz inequality which states that if I take the you know inner product of any two vectors squared then this necessarily must be less than or equal to A dot A times B dot B, right. This is the Cauchy-Schwartz inequality applied to the you know 3d vector case. But, a little bit of thought reviews here it seems like not at all a surprise, right.

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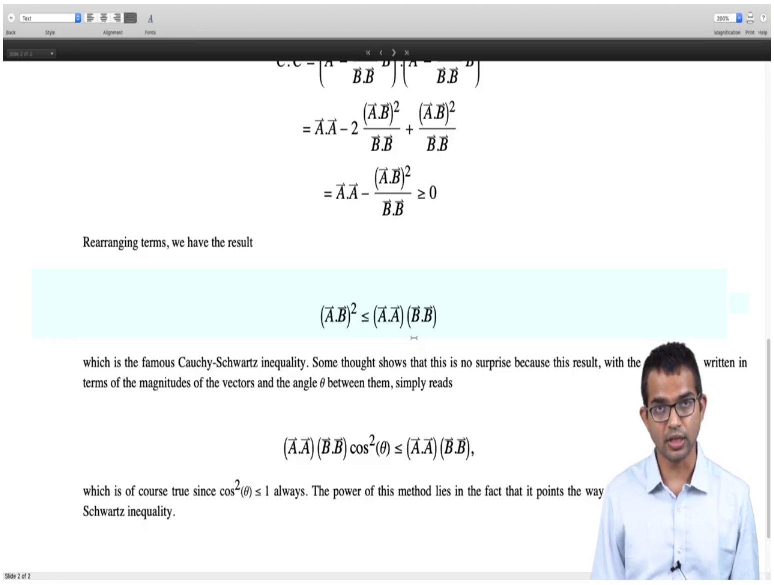

So, you know one way to think of this is just use the cosine formula we had. This dot product of two vectors A and B can be written in terms of the cosine of the angle between the two vectors and the magnitudes, right.

So, then of course, A dot B will just become the square of this is just A dot A times B dot B times cosine squared of theta, right. So, all we are saying is that A dot A times B dot B times cosine squared is less than or equal to A dot A times B dot B which is certainly true right because cosine squared theta is going to be less than or equal to 1, right. This is how cosine works, right.

So, in fact, this is no surprise at all right, but the power of this approach that we have just outlined is that it gives you a way to generalise; generalise this inequality. The Cauchy-Schwartz inequality is the you know beautiful and surprisingly powerful result and it has you know applications in all kinds of contexts and it has you know the it can be generalised to you know other kinds of vectors, right.

And, this method we will see in the next lecture we will carry through two arbitrary vectors in a linear vector space, ok. That is all for this lecture.

Thank you.