Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 59
Ordinary Differential Equations

Ok, so we have seen how ordinary differential equations are classified broadly, and how we can solve linear first order differential equations, we have given out the full prescription for the same, right. So, there is this topic of orthogonal trajectories which is important right when we are dealing with families of curves, right.

So often, we saw for example, in the case of first order differential equations, you get a free parameter for the general solution right, and so that corresponds to the whole family of curves. So, while the family of curves is of interest you know there is another family of curves corresponding to a given family of curves which is going to be orthogonal to this which is also of interest.

So, in this lecture, we look at how to think about a family of orthogonal trajectories. And also how you know we can come up with a differential equation and work out this orthogonal trajectory given a family right. So, that is the topic for this lecture, ok.

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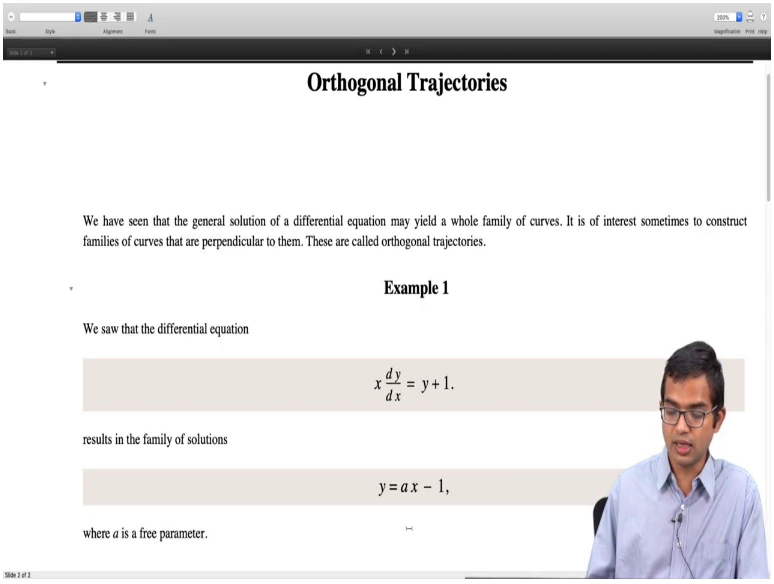

So, this is perhaps best understood with the help of examples, right. So, you have given some differential equation and you know suppose it yields a whole family of curves. Now, it is of interest to find a family of curves which are perpendicular to the first family, right. So, how would we do this? So, let us look at this example which we have already looked at.

So, if you take the differential equation x times dy by dx is equal to y plus 1. This gives us you know the general solution y equal to a x minus 1 right, which is actually a bunch of straight lines right. Where a you know corresponds to the slope of this straight line, and as you vary the slope of this straight line will vary, and you cover a whole you know family of such straight lines.

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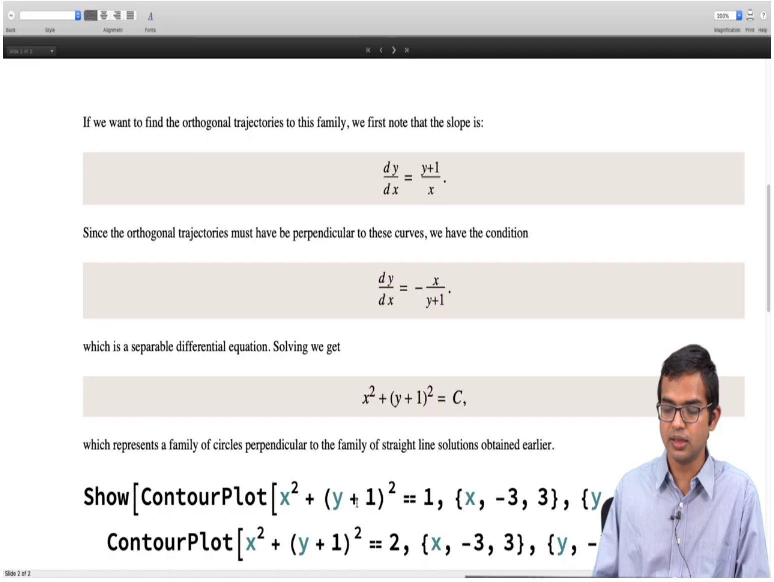

Now, if you want to find a family of curves which are orthogonal to all these straight lines right, we would note that the slope of this family, this orthogonal you know family which is orthogonal to this should be dy by dx should be equal to you know minus 1 over the slope of the first family. But for the first family, we already know what the slope is; it is dy by dx is equal to y plus 1 over x. It is given to us.

But if we want to find orthogonal trajectories their slope must be dy by dx. It should be equal to minus x divided by y plus 1, right. So, this is a familiar result from basic coordinate geometry. We know that if two curves intersect at right angles, then the product of their slopes must be equal to minus 1 right, so that is the that is being used here and. So, we have dy by dx is equal to minus x over y plus 1, and which is a separable differential equation very, very simple scenario that I have considered here, right. There are more complicated examples which you can perhaps work out as homework later on.

So, solving this, we get x squared plus y plus 1 the whole squared is equal to C right. So, this is a separable differential equation very straightforward to solve. And so now, once again we have a free constant C, which is you know which so now, we have another family of curves which is immediately seen to be a family of circles. And these circles are perpendicular to the straight line solutions obtained earlier, right.

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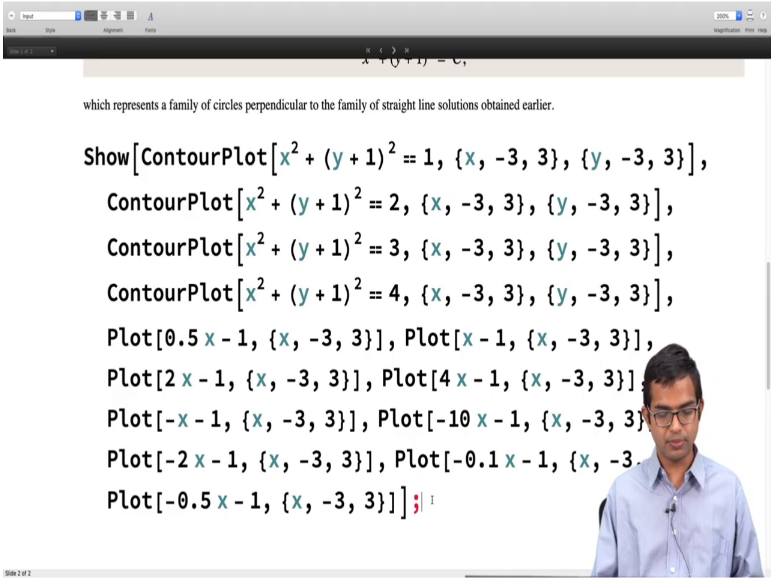

So, it is instructive to plot all of these curves. I have collected together a bunch of representative samples of you know members of each of these families and then I am plotting them all. So, and ok, so this is taking a while.

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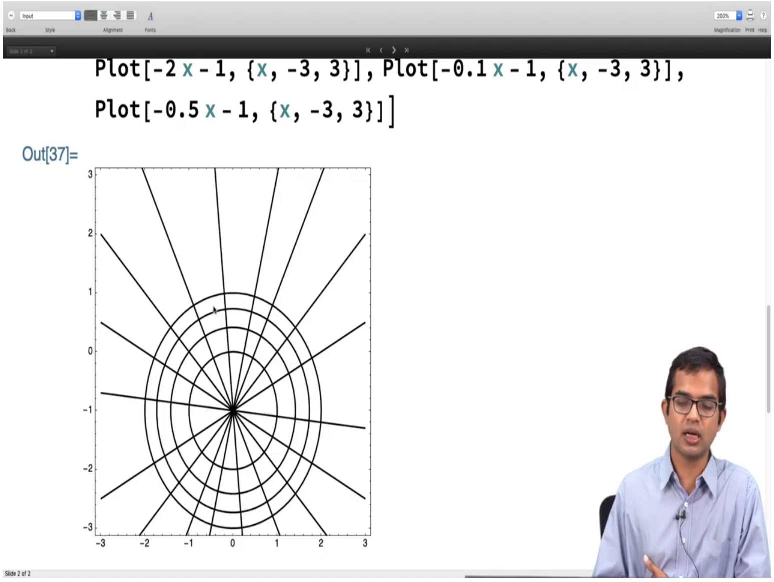

There you go. So, it looks something like this right. So, the family of circles are like this. And the center of all the circles is of course, this point 0 comma minus 1 right, 0 comma minus 1.

And all these straight lines also pass through this point 0 comma minus 1 right. If you put x equal to 0, y has to be equal to minus 1, no matter what the value of a is, right.

So, those are all these straight lines. So, all of these straight lines with different slopes pass through this point and these curves with different radii right, they are all circles, right. Orthogonal trajectories for this particular family set of family of curves is the family of circles, right. So, in general, this problem can be more difficult to solve.

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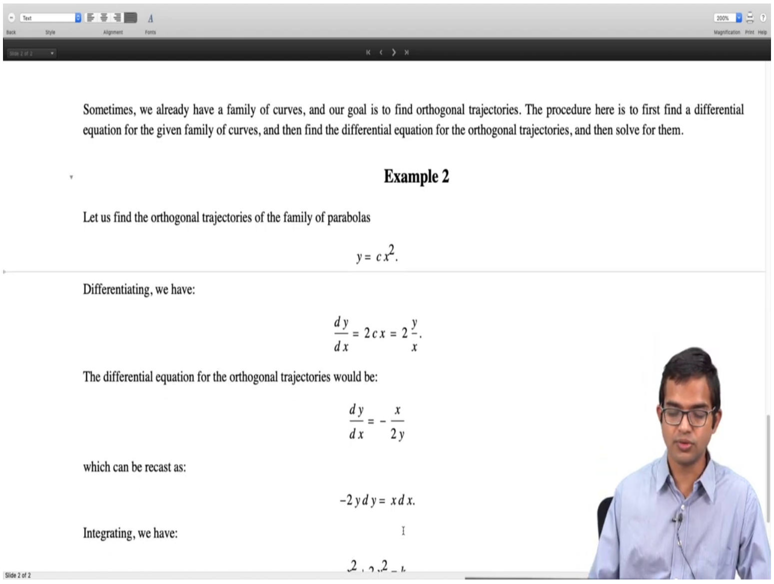

So, let us look at another example, where we are not given the differential equation. We are given a family and then we are asked to find a family of curves which is orthogonal to the first family. So, let us look at this example. So, suppose you are given this family of parabolas y equal to c x squared, right.

So, the technique here is to first find a differential equation for you know the first family, and then you know self-consistently eliminate the constant which is there in here. So, let us look at this example and then it will become clear. So, we have dy by dx is equal to 2 c x, right. So, I mean eventually we want to write another differential equation dy by dx is equal to minus 1 over the slope of this.

But this constant is not a convenient object to work with. We want to eliminate this constant itself. But what is this c? c is y over x squared. So, dy by dx is equal to 2 times y over x squared into x which is just 2 times y over x. So, we have managed to write down a differential equation where there is no constant, right. So, if you solve this differential equation of course, you have to get back the same solution and with this arbitrary constant, right.

So, now once you have brought your differential equation, your you got your differential equation for your given family of curves, and the next step is to write down the differential equation corresponding to the orthogonal trajectories which is just dy by dx is equal to minus 1 over the slope of the first family, so that is going to become minus 1 over 2 y by x which is minus x by 2 y, right. So, once you have this, this is straightforward again to solve. So, it is a separable differential equation. You have minus 2 y dy is equal to x times d x, so that gives you on integrating you get x squared plus 2 y squared is equal to a constant, right.

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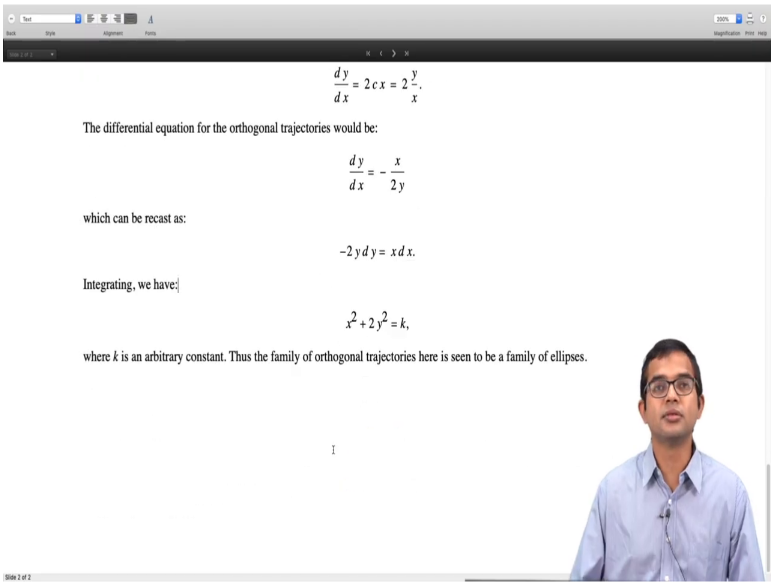

So, now you have a family of ellipses, right. So, this family of ellipses is going to be the orthogonal trajectories for your family of parabolas right. In the first example, we saw that for this family of straight lines we got a family of circles as the orthogonal trajectories. But, for this family of parabolas of this kind, we are going to get a family of ellipses like specified here, ok. So, that is all for this lecture.

Thank you.